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WEIGHTED COMPOSITION OPERATORS FROM WEIGHTED BERGMAN SPACES TO STEVIĆ–TYPE SPACES

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Abstract. The boundedness, compactness and essential norm of weighted composition operators from weighted Bergman spaces with a double weight to Stević-type spaces on the unit disk are investigated in this paper.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane, and $H(\mathbb{D})$ the class of all functions analytic on $\mathbb{D}$. Assume that μ is a weight, that is, μ is a radial, positive and continuous function on $\mathbb{D}$. If $n \in \mathbb{N} \cup \{0\}$, the Stević-type space on $\mathbb{D}$, which he called the n-th weighted space, denoted by $W_\mu^{(n)}$, consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{W_\mu^{(n)}} := \sum_{k=0}^{n-1} |f^{(k)}(0)| + \sup_{z\in\mathbb{D}} \mu(z)|f^{(n)}(z)| < \infty.$$

The space $W_\mu^{(n)}$ was introduced by S. Stević in [28]. It is a Banach space with the norm $\|\cdot\|_{W_\mu^{(n)}}$. When $n = 0$, $W_\mu^{(n)}$ becomes the weighted-type space H_μ^∞. When $n = 1$ and $n = 2$, $W_\mu^{(n)}$ becomes the Bloch-type space $\mathscr{B}_\mu$ and the Zygmund-type space $\mathscr{Z}_\mu$, respectively. For some results on the spaces $H_\mu^\infty, \mathscr{B}_\mu, \mathscr{Z}_\mu$ with various weights μ, and operators acting from or to them, see, e.g., [1, 2, 3, 4, 6, 7, 9, 13, 14, 15, 16, 17, 18, 19, 29, 30, 32, 34, 42, 44, 45], and the related references therein.

The little Stević-type space, denoted by $W_{\mu,0}^{(n)}$, consists of all $f \in W_\mu^{(n)}$ such that

$$\lim_{|z|\to 1} \mu(z)|f^{(n)}(z)| = 0.$$

It is shown in a standard way that $W_{\mu,0}^{(n)}$ is a closed subspace of $W_\mu^{(n)}$.

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Suppose ω is an integrable weight on $(0,1)$. We say that ω is regular, and write it as $\omega \in \mathscr{R}$, if there is a constant $C>0$ determined by ω, such that

$$\frac{1}{C} < \frac{1}{(1-r)\omega(r)} \int_r^1 \omega(s)ds < C, \quad \text{when } 0<r<1.$$

We say that ω is rapidly increasing, and write it as $\omega \in \mathscr{I}$, if

$$\lim_{r\to 1} \frac{1}{(1-r)\omega(r)} \int_r^1 \omega(s)ds = \infty.$$

Let $v_{\alpha,\beta}(r) = (1-r)^\alpha \left(\log \frac{e}{1-r}\right)^\beta$ (such weights can be found in [29, 30]). By a calculation, we have the following typical examples of regular and rapidly increasing weights, see [24], for example.

(i) When $\alpha > -1$ and $\beta \in \mathbb{R}$, $v_{\alpha,\beta} \in \mathscr{R}$;

(ii) When $\alpha = -1$ and $\beta < -1$, $v_{\alpha,\beta} \in \mathscr{I}$ and $\left|\sin\left(\log \frac{1}{1-r}\right)\right| v_{\alpha,\beta}(r) + 1 \in \mathscr{I}$.

Suppose ω is an integrable weight on $(0,1)$. If there is a constant $C>0$ such that

$$\int_r^1 \omega(s)ds < C \int_{\frac{1+r}{2}}^1 \omega(s)ds, \quad \text{when } 0<r<1,$$

we say that ω is a double weight, and write it as $\omega \in \hat{\mathscr{D}}$. From [24, 25], we see that $\mathscr{I} \cup \mathscr{R} \subset \hat{\mathscr{D}}$. See [24, 25] for more details about $\mathscr{I}, \mathscr{R}$ and $\hat{\mathscr{D}}$.

Let $0<p<\infty$ and $\omega \in \hat{\mathscr{D}}$. The weighted Bergman space A^p_ω is the space of $f \in H(\mathbb{D})$ for which

$$\|f\|^p_{A^p_\omega} := \int_{\mathbb{D}} |f(z)|^p \omega(z) dA(z) < \infty,$$

where $dA(z) = \frac{1}{\pi} dxdy$ is the normalized Lebesgue area measure on $\mathbb{D}$. When $p \geqslant 1$, A^p_ω is a Banach space. When $\omega(t) = (1-t)^\alpha$ $(\alpha > -1)$, the space A^p_ω becomes the classical weighted Bergman space A^p_α. In [24, 25] there are plenty of results which show that the Bergman space A^p_ω induced by a rapidly increasing weight lie "closer" to the Hardy space H^p than any A^p_α.

Let φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. The weighted composition operator, denoted by uC_φ, is defined on $H(\mathbb{D})$ by

$$(uC_\varphi f)(z) = u(z) f(\varphi(z)), \qquad f \in H(\mathbb{D}).$$

It is important to give function theoretic descriptions of when u and φ induce a bounded or compact weighted composition operator on various function spaces. Recently, there has been a great interest in studying weighted composition operators on analytic function spaces on the unit disk, see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 20, 21, 22, 27, 31, 33, 35, 36, 37, 38, 40, 41, 43, 44, 45].

In [28], Stević studied the boundedness and compactness of the composition operator from A^p_α to $W^{(n)}_\mu$ on $\mathbb{D}$. In [35], Stević studied the boundedness and compactness of the weighted differentiation composition operators from H^∞ and the Bloch space to $W^{(n)}_\mu$ on $\mathbb{D}$. In [41], Zhang and Zeng generalized the results in [28] to the case of weighted differentiation composition operators. For some very general results on the essential norm of generalized composition operators between Stević-type spaces see [37].

Motivated by [28], we study the boundedness and compactness of uC_φ from weighted Bergman spaces A^p_ω to $W^{(n)}_\mu$ and $W^{(n)}_{\mu,0}$ in this paper. Moreover, we give some estimates of the essential norm of uC_φ from A^p_ω to $W^{(n)}_\mu$ and $W^{(n)}_{\mu,0}$.

Let X and Y be Banach spaces. Recall that the essential norm of linear operator $T : X \to Y$ is defined by

$$\|T\|_{e,X\to Y} = \inf\{\|T-K\|_{X\to Y} : K \text{ is compact from } X \text{ to } Y\}.$$

Obviously $T : X \to Y$ is compact if and only if $\|T\|_{e,X\to Y} = 0$.

Throughout this paper, the letter C will denote constants and may differ from one occurrence to the other. The notation $A \lesssim B$ means that there is a positive constant C such that $A \leqslant CB$. The notation $A \approx B$ means $A \lesssim B$ and $B \lesssim A$.

2. Auxiliary results

LEMMA 1. *Assume that $\omega \in \hat{\mathscr{D}}$, $r \in [0,1]$ and $\omega_*(r) = \int_r^1 s\omega(s)\log\frac{s}{r}ds$. Then the following statements hold.*

(i) $\omega_* \in \mathscr{R}$ *and* $\omega_*(r) \approx (1-r)\int_r^1 \omega(t)dt$*;*

(ii) There are $0 < a < b < +\infty$ *and* $\delta \in [0,1)$, *such that*

$$\frac{\omega_*(r)}{(1-r)^a} \text{ is decreasing on } [\delta,1) \text{ and } \lim_{r\to1}\frac{\omega_*(r)}{(1-r)^a} = 0;$$

$$\frac{\omega_*(r)}{(1-r)^b} \text{ is increasing on } [\delta,1) \text{ and } \lim_{r\to1}\frac{\omega_*(r)}{(1-r)^b} = \infty;$$

(iii) $\omega_*(r)$ *is decreasing on* $[\delta,1)$ *and* $\lim_{r\to1}\omega_*(r) = 0$.

Proof. By [26, Lemmas A and 9], (*i*) holds. By (1.19) in [24], (*ii*) holds. From (*ii*) and the fact that $\omega_*(r) = \frac{\omega_*(r)}{(1-r)^a}(1-r)^a$, we see that (*iii*) holds. □

REMARK 1. Without loss of generality, we can assume δ related to ω_* in Lemma 1 is 0. We assume that ω_* is radial, that is, $\omega_*(z) = \omega_*(|z|)$ for all $z \in \mathbb{D}$.

Let $\gamma_0 > 0$ be one of the admissible constants in [24, Lemma 2.3]. It follows by Lemma 1, [23, Lemma 3.1] and [24, Lemma 2.4] we have the following result.

LEMMA 2. *Let $\omega \in \hat{\mathscr{D}}$, $p>0$ and $k \in \mathbb{N}\cup\{0\}$. Set*

$$g_{a,k}(z) = \left(\frac{1-|a|^2}{1-\overline{a}z}\right)^{k+\gamma_0} \frac{1}{\omega_*^{1/p}(a)}, \quad z \in \mathbb{D}.$$

Then

$$\|g_{a,k}\|_{A^p_\omega} \approx 1, \ \textit{when} \ \ a \in \mathbb{D},$$

and

$$\lim_{|a|\to 1} \sup_{|z|\leqslant r} |g_{a,k}(z)| = 0, \ \textit{when} \ \ r \in (0,1).$$

For $f \in H(\mathbb{D})$ and $0<r<1$, set

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p dt\right)^{1/p}, \ \ 0<p<\infty,$$

and

$$M_\infty(r,f) = \sup_{|z|=r} |f(z)|.$$

Then we have the following lemma.

LEMMA 3. *Suppose $\omega \in \hat{\mathscr{D}}$, $0<p<\infty$ and $N \in \mathbb{N}\cup\{0\}$. Then there exists $C = C(p,\omega,N)$ such that*

$$(1-|z|)^k \omega_*^{1/p}(z)|f^{(k)}(z)| \leqslant C\|f\|_{A^p_\omega}, \tag{1}$$

for all $f \in H(\mathbb{D})$ and $k = 0,1,\cdots,N+1$.

Proof. Let $s_k(r) = 1-\frac{1-r}{2^k}$. Then $\frac{1+s_k(r)}{2} = s_{k+1}(r)$. By well-known estimates, there is a $C_1 = C(p) < \infty$, such that

$$M_\infty(r,f) \leqslant C_1 \frac{M_p(\frac{1+r}{2},f)}{(1-r)^{1/p}}, \ \text{ and } \ M_p(r,f') \leqslant C_1 \frac{M_p(\frac{1+r}{2},f)}{1-r}.$$

Hence,

$$M_\infty(r,f^{(k)}) \leqslant C_1 \frac{M_p(s_1(r),f^{(k)})}{(1-r)^{1/p}} \leqslant 2^{\frac{k(k+1)}{2}} C_1^{k+1} \frac{M_p(s_{k+1}(r),f)}{(1-r)^{k+1/p}}.$$

Then

$$\begin{aligned}(1-r)^{pk+1} M_\infty^p(r,f^{(k)}) \int_{s_{k+1}(r)}^1 \omega(t)dt &\leqslant 2^{\frac{pk(k+1)}{2}} C_1^{p(k+1)} M_p^p(s_{k+1}(r),f) \int_{s_{k+1}(r)}^1 2\omega(t)t\,dt \\ &\leqslant 2^{\frac{pk(k+1)}{2}+1} C_1^{p(k+1)} \|f\|^p_{A^p_\omega}.\end{aligned}$$

By Lemma 1, there exist $C_2 = C(\omega) > 0$ and $b > 0$, such that $\frac{\omega_*(t)}{(1-t)^b}$ is increasing on $[0,1)$ and

$$\frac{1}{C_2}(1-t)\int_t^1 \omega(s)ds \leqslant \omega_*(t) \leqslant C_2(1-t)\int_t^1 \omega(s)ds,\ t \in [0,1].$$

Hence,

$$\frac{\omega_*(r)}{(1-r)^b} \leqslant \frac{\omega_*(s_{k+1}(r))}{(1-s_{k+1}(r))^b} = \frac{2^{(k+1)b}\omega_*(s_{k+1}(r))}{(1-r)^b}.$$

Therefore,

$$\begin{aligned} M_\infty^p(r, f^{(k)}) &\leqslant \frac{2^{\frac{pk(k+1)}{2}+1} C_1^{p(k+1)} C_2 (1-s_{k+1}(r)) \|f\|_{A_\omega^p}^p}{(1-r)^{kp+1}\omega_*(s_{k+1}(r))} \\ &\leqslant \frac{2^{\frac{pk(k+1)}{2}+(b-1)(k+1)+1} C_1^{p(k+1)} C_2 \|f\|_{A_\omega^p}^p}{(1-r)^{kp}\omega_*(r)}. \end{aligned}$$

So, there exists $C = C(p, \omega, N)$ such that (1) always holds. The proof is complete. $\square$

The next two lemmas can be found in [35].

LEMMA 4. [35] *Fix $n \in \mathbb{N} \cup \{0\}, a > 0$. Let matrix $D_{n+1}(a) = (\beta_{ij}(a))_{i,j=1,2,\cdots,n+1}$, where*

$$\beta_{ij}(a) = \begin{cases} 1, & i = 1, \\ \prod_{k=1}^{i-1}(a+k+j-2), & i = 2,3,\cdots,n+1. \end{cases}$$

Then

$$det(D_{n+1}(a)) = \begin{cases} 1, & n = 0, \\ \prod_{k=1}^{n} k!, & n \in \mathbb{N}. \end{cases}$$

Here $det(D_{n+1}(a))$ is the determinant of $D_{n+1}(a)$.

LEMMA 5. [35] *Suppose $n \in \mathbb{N} \cup \{0\}, u \in H(\mathbb{D})$, and φ is an analytic self-map of $\mathbb{D}$. For all $f \in H(\mathbb{D})$, we have*

$$(uC_\varphi f)^{(n)}(z) = \sum_{k=0}^{n} f^{(k)}(\varphi(z)) \sum_{l=k}^{n} C_n^l u^{(n-l)}(z) B_{l,k}(\varphi'(z), \varphi''(z), \cdots, \varphi^{(l-k+1)}(z)), \quad (2)$$

where

$$B_{l,k}(\varphi'(z), \varphi''(z), \cdots, \varphi^{(l-k+1)}(z)) = \sum_{k_1,\cdots,k_l} \frac{l!}{k_1!k_2!\cdots k_l!} \prod_{j=1}^{l} \left(\frac{\varphi^{(j)}(z)}{j!}\right)^{k_j}, \quad (3)$$

and the sum in (3) is over all non-negative integers $k_1, k_2, \cdots, k_l$ satisfying

$$k_1 + k_2 + \cdots + k_l = k \text{ and } k_1 + 2k_2 + \cdots + lk_l = l.$$

For brief, we will write $B_{l,k}(\varphi'(z), \varphi''(z), \cdots, \varphi^{(l-k+1)}(z))$ as $B_{l,k}(\varphi(z))$, that is

$$B_{l,k}(\varphi(z)) = B_{l,k}(\varphi'(z), \varphi''(z), \cdots, \varphi^{(l-k+1)}(z)).$$

To study the compactness, we need the following well known lemma.

LEMMA 6. [39, Lemma 2.10] *Suppose* $p \geqslant 1$, $\omega \in \hat{\mathscr{D}}$ *and* μ *is a weight. If* $T : A^p_\omega \to W^{(n)}_\mu (W^{(n)}_{\mu,0})$ *is linear and bounded, then* T *is compact if and only if whenever* $\{f_k\}$ *is bounded in* A^p_ω *and* $f_k \to 0$ *uniformly on compact subsets of* $\mathbb{D}$, $\lim_{k\to\infty} \|Tf_k\|_{W^{(n)}_\mu} = 0$.

3. Main results and proofs

THEOREM 1. *Assume that* $p \geqslant 1$, $n \in \mathbb{N} \cup \{0\}$, $u \in H(\mathbb{D})$, φ *is an analytic self-map of* $\mathbb{D}$, $\omega \in \hat{\mathscr{D}}$, *and* μ *is a weight. Then* $uC_\varphi : A^p_\omega \to W^{(n)}_\mu$ *is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)\left|\sum\limits_{l=k}^{n} C_n^l u^{(n-l)}(z) B_{l,k}(\varphi(z))\right|}{(1-|\varphi(z)|)^k \omega_*^{1/p}(\varphi(z))} < \infty, \quad k = 0, 1, \cdots, n. \tag{4}$$

Proof. Suppose that (4) holds. For any $f \in A^p_\omega$, from Lemmas 3 and 5, there exists $C = C(\omega, p, n)$, such that

$$\begin{aligned}\mu(z)\left|(uC_\varphi f)^{(n)}(z)\right| &= \mu(z)\left|\sum_{k=0}^{n} f^{(k)}(\varphi(z)) \sum_{l=k}^{n} C_n^l u^{(n-l)}(z) B_{l,k}(\varphi(z))\right| \\ &\leqslant C\|f\|_{A^p_\omega} \sum_{k=0}^{n} \frac{\mu(z)\left|\sum\limits_{l=k}^{n} C_n^l u^{(n-l)}(z) B_{l,k}(\varphi(z))\right|}{(1-|\varphi(z)|)^k \omega_*^{1/p}(\varphi(z))}.\end{aligned}$$

For $j = 0, 1, \cdots, n-1$, the following inequality holds obviously,

$$\begin{aligned}\left|(uC_\varphi f)^{(j)}(0)\right| &= \left|\sum_{k=0}^{j} f^{(k)}(\varphi(0)) \sum_{l=k}^{j} C_j^l u^{(j-l)}(0) B_{l,k}(\varphi(0))\right| \\ &\leqslant C\|f\|_{A^p_\omega} \sum_{k=0}^{j} \frac{\left|\sum\limits_{l=k}^{j} C_j^l u^{(j-l)}(0) B_{l,k}(\varphi(0))\right|}{(1-|\varphi(0)|)^k \omega_*^{1/p}(\varphi(0))}.\end{aligned}$$

So $uC_\varphi : A^p_\omega \to W^{(n)}_\mu$ is bounded.

Conversely, suppose that $uC_\varphi : A^p_\omega \to W^{(n)}_\mu$ is bounded. For $a \in \mathbb{D}$ and $\vec{c} = (c_1, c_2, \cdots, c_{n+1})^T$, set

$$g_a(z) = \sum_{j=1}^{n+1} c_j g_{a,j}(z), \tag{5}$$

where $g_{a,j}$ are defined in Lemma 2. We get

$$g^{(t)}_{a,j}(z) = \frac{(\overline{a}^t)(j+\gamma_0)(j+\gamma_0+1)\cdots(j+\gamma_0+t-1)}{(1-\overline{a}z)^{j+\gamma_0+t}} \frac{(1-|a|^2)^{j+\gamma_0}}{\omega_*^{1/p}(a)}.$$

So,

$$g_{a,j}^{(t)}(a) = \frac{(\overline{a}^t)(j+\gamma_0)(j+\gamma_0+1)\cdots(j+\gamma_0+t-1)}{(1-|a|^2)^t \omega_*^{1/p}(a)}.$$

Therefore,

$$\begin{aligned} g_a^{(t)}(a) &= \left(g_{a,1}^{(t)}(a), g_{a,2}^{(t)}(a), \cdots, g_{a,n}^{(t)}(a), g_{a,n+1}^{(t)}(a)\right) \circ \vec{c}, \\ &= \frac{\overline{a^t}}{(1-|a|^2)^t \omega_*^{1/p}(a)} \left(\prod_{i=1}^{t}(i+\gamma_0), \prod_{i=2}^{t+1}(i+\gamma_0), \cdots, \prod_{i=n}^{t+n-1}(i+\gamma_0), \prod_{i=n+1}^{t+n}(i+\gamma_0)\right) \circ \vec{c}. \end{aligned}$$

Fix $k = 0,1,\cdots,n$, we will choose $\vec{c}$, such that

$$g_a^{(t)}(a) = \begin{cases} \frac{(\overline{a^k})}{(1-|a|^2)^k \omega_*^{1/p}(a)}, & t = k, \\ 0, & t \neq k \text{ and } 0 \leqslant t \leqslant n. \end{cases} \tag{6}$$

That is to say,

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ \gamma_0+1 & \gamma_0+2 & \cdots & \gamma_0+n & \gamma_0+n+1 \\ (\gamma_0+1)(\gamma_0+2) & (\gamma_0+2)(\gamma_0+3) & \cdots & (\gamma_0+n)(\gamma_0+n+1) & (\gamma_0+n+1)(\gamma_0+2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \prod_{i=1}^{n}(\gamma_0+i) & \prod_{i=1}^{n}(\gamma_0+i+1) & \cdots & \prod_{i=1}^{n}(\gamma_0+n+i-1) & \prod_{i=1}^{n}(\gamma_0+n+i) \end{pmatrix} \vec{c} = A,$$

or

$$D_{n+1}(\gamma_0+1)\vec{c} = A,$$

where A is a column vector, in which the $k+1$-st element is 1 and the others are 0. By Lemma 4, $\vec{c}$ exists and depends on γ_0, k and n. By Lemma 2, there exists $C = C(\omega, p, k, n)$, such that $\|g_a\|_{A_\omega^p} \leqslant C$, for all $a \in \mathbb{D}$.

When $|\varphi(z)| \geqslant \frac{1}{2}$, by Lemma 5 and (6), we get

$$\begin{aligned} \frac{\mu(z)\left|\sum_{l=k}^{n} C_n^l u^{(n-l)}(z) B_{l,k}(\varphi(z))\right|}{(1-|\varphi(z)|)^k \omega_*^{1/p}(\varphi(z))} &= \frac{1}{|\varphi(z)|^k}\mu(z)|(uC_\varphi g_{\varphi(z)})^{(n)}(z)| \\ &\leqslant 2^k \|uC_\varphi\|_{A_\omega^p \to W_\mu^{(n)}} \|g_{\varphi(z)}\|_{A_\omega^p}. \end{aligned} \tag{7}$$

So, for $k = 0,1,\cdots,n$, there exists $C = C(\omega, p, n)$ such that

$$\sup_{|\varphi(z)| \geqslant \frac{1}{2}} \frac{\mu(z)\left|\sum_{l=k}^{n} C_n^l u^{(n-l)}(z) B_{l,k}(\varphi(z))\right|}{(1-|\varphi(z)|)^k \omega_*^{1/p}(\varphi(z))} \leqslant 2^k C \|uC_\varphi\|_{A_\omega^p \to W_\mu^{(n)}} < \infty.$$

When $|\varphi(z)| \leqslant \frac{1}{2}$, let $k = 0, 1, \cdots, n$. Define the test function $h_k(z) = z^k$, then $\|h_k\|^p_{A^p_\omega} \lesssim \omega(\mathbb{D})$. Here $\omega(\mathbb{D}) = \int_{\mathbb{D}} \omega(z)dA(z)$. Obviously, we have

$$\begin{aligned}\mu(z)\left|\sum_{l=0}^{n} C_n^l u^{(n-l)}(z) B_{l,0}(\varphi(z))\right| &= \mu(z)\left|(uC_\varphi h_0)^{(n)}(z)\right| \leqslant \|uC_\varphi h_0\|_{W_\mu^{(n)}} \\ &\leqslant \|uC_\varphi\|_{A^p_\omega \to W_\mu^{(n)}} \|h_0\|_{A^p_\omega} = \|uC_\varphi\|_{A^p_\omega \to W_\mu^{(n)}} \omega^{1/p}(\mathbb{D}).\end{aligned}$$

So, when $k = 0$, we have

$$\mu(z)\left|\sum_{l=0}^{n} C_n^l u^{(n-l)}(z) B_{l,k}(\varphi(z))\right| \leqslant \omega^{1/p}(\mathbb{D}) \|uC_\varphi\|_{A^p_\omega \to W_\mu^{(n)}}.$$

Suppose $k \geqslant 1$. Then we have

$$\begin{aligned}&k!\mu(z)\left|\sum_{l=k}^{n} C_n^l u^{(n-l)}(z) B_{l,k}(\varphi(z))\right| \\ =&\mu(z)\left|(uC_\varphi h_k)^{(n)}(z) - \sum_{i=0}^{k-1} h_k^{(i)}(\varphi(z)) \sum_{l=i}^{n} C_n^l u^{(n-l)}(z) B_{l,i}(\varphi(z))\right| \\ \leqslant&\mu(z)\left|(uC_\varphi h_k)^{(n)}(z)\right| + \mu(z)\sum_{i=0}^{k-1} \frac{k!}{(k-i)!}\left|\sum_{l=i}^{n} C_n^l u^{(n-l)}(z) B_{l,i}(\varphi(z))\right| \\ \leqslant&\|uC_\varphi\|_{A^p_\omega \to W_\mu^{(n)}} \|h_k\|_{A^p_\omega} + \sum_{i=0}^{k-1} \frac{k!}{(k-i)!}\mu(z)\left|\sum_{l=i}^{n} C_n^l u^{(n-l)}(z) B_{l,i}(\varphi(z))\right|.\end{aligned}$$

So, we have

$$\begin{aligned}&\mu(z)\left|\sum_{l=k}^{n} C_n^l u^{(n-l)}(z) B_{l,k}(\varphi(z))\right| \\ \leqslant&\frac{1}{k!}\omega^{1/p}(\mathbb{D}) \|uC_\varphi\|_{A^p_\omega \to W_\mu^{(n)}} + \sum_{i=0}^{k-1} \frac{1}{(k-i)!}\mu(z)\left|\sum_{l=i}^{n} C_n^l u^{(n-l)}(z) B_{l,i}(\varphi(z))\right|.\end{aligned}$$

Using the last inequality $k = 1, 2, \cdots, n$ repeatedly, we can find a $C > 0$, such that

$$\mu(z)\left|\sum_{l=k}^{n} C_n^l u^{(n-l)}(z) B_{l,k}(\varphi(z))\right| \leqslant C\omega^{1/p}(\mathbb{D}) \|uC_\varphi\|_{A^p_\omega \to W_\mu^{(n)}}.$$

Since $\sup\limits_{0 \leqslant r \leqslant \frac{1}{2}} \frac{1}{(1-r)^{pk}\omega_*(r)} < \infty$, for $k = 0, 1, \cdots, n$, we get

$$\sup_{|\varphi(z)| \leqslant \frac{1}{2}} \frac{\mu(z)\left|\sum\limits_{l=k}^{n} C_n^l u^{(n-l)}(z) B_{l,k}(\varphi(z))\right|}{(1-|\varphi(z)|)^k \omega_*^{1/p}(\varphi(z))} < \infty.$$

Therefore (4) holds. The proof is complete. $\square$

THEOREM 2. *Assume that* $p \geqslant 1$, $n \in \mathbb{N} \cup \{0\}$, $u \in H(\mathbb{D})$, φ *is an analytic self-map of* $\mathbb{D}$, $\omega \in \hat{\mathscr{D}}$, *and* μ *is a weight. If* $uC_\varphi : A^p_\omega \to W^{(n)}_\mu$ *is bounded, then*

$$\|uC_\varphi\|_{e,A^p_\omega \to W^{(n)}_\mu} \approx \sum_{k=0}^{n} \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)\left|\sum\limits_{l=k}^{n} C_n^l u^{(n-l)}(z) B_{l,k}(\varphi(z))\right|}{(1-|\varphi(z)|)^k \omega_*^{1/p}(\varphi(z))}.$$

Proof. The lower estimate of $\|uC_\varphi\|_{e,A^p_\omega \to W^{(n)}_\mu}$.

Suppose $\{w_m\}_{m=1}^\infty \subset \mathbb{D}$ such that $\lim\limits_{m\to\infty} |\varphi(w_m)| = 1$ and $K : A^p_\omega \to W^{(n)}_\mu$ is compact. For a given $k = 0, 1, \cdots, n$, let $g_{\varphi(w_m)}$ be defined as in (5) and satisfy (6). By Lemma 2, there exists $C = C(\omega, p, n)$, such that $\|g_{\varphi(w_m)}\|_{A^p_\omega} \leqslant C$ and $\{g_{\varphi(w_m)}\}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$. By Lemma 6, $\lim\limits_{m\to\infty} \|Kg_{\varphi(w_m)}\|_{W^{(n)}_\mu} = 0$. Therefore,

$$\begin{aligned}
\|uC_\varphi - K\|_{A^p_\omega \to W^{(n)}_\mu} &\gtrsim \limsup_{m\to\infty} \|(uC_\varphi - K) g_{\varphi(w_m)}\|_{A^p_\omega \to W^{(n)}_\mu} \\
&\geqslant \limsup_{m\to\infty} \left(\|uC_\varphi g_{\varphi(w_m)}(w_m)\|_{W^{(n)}_\mu} - \|Kg_{\varphi(w_m)}\|_{A^p_\omega \to W^{(n)}_\mu} \right) \\
&\gtrsim \limsup_{m\to\infty} \frac{\mu(w_m)\left|\sum\limits_{l=k}^{n} C_n^l u^{(n-l)}(w_m) B_{l,k}(\varphi(w_m))\right|}{(1-|\varphi(w_m)|)^k \omega_*^{1/p}(\varphi(w_m))}.
\end{aligned}$$

Since k, K and $\{w_m\}$ are arbitrary, we have

$$\|uC_\varphi\|_{e,A^p_\omega \to W^{(n)}_\mu} \gtrsim \sum_{k=0}^{n} \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)\left|\sum\limits_{l=k}^{n} C_n^l u^{(n-l)}(z) B_{l,k}(\varphi(z))\right|}{(1-|\varphi(z)|)^k (\omega_*(\varphi(z)))^{1/p}}.$$

The upper estimate of $\|uC_\varphi\|_{e,A^p_\omega \to W^{(n)}_\mu}$.

Suppose $\frac{1}{2} < \rho < 1$ and $0 < r < 1$. For all $f \in A^p_\omega$, let $f_\rho(z) = f(\rho z)$. Then

$$\|uC_{\rho\varphi} f\|_{W^{(n)}_\mu} = \|uC_\varphi f_\rho\|_{W^{(n)}_\mu} \leqslant \|uC_\varphi\|_{A^p_\omega \to W^{(n)}_\mu} \|f_\rho\|_{A^p_\omega} \leqslant \|uC_\varphi\|_{A^p_\omega \to W^{(n)}_\mu} \|f\|_{A^p_\omega}.$$

So, $uC_{\rho\varphi}$ is bounded. By Lemma 6, (4), as well as the Cauchy estimate for the derivatives of analytic functions on compacts, we see that $uC_{\rho\varphi}$ is compact.

When $|\varphi(z)| \leqslant r < 1$ and $k = 0, 1, \cdots, n$, by Lemma 3 we have

$$|f^{(k)}(\rho(\varphi(z)))| \leqslant \frac{C\|f\|_{A^p_\omega}}{(1-(\rho|\varphi(z)|)^2)^k \omega_*^{1/p}(\rho|\varphi(z)|)} \leqslant \frac{C\|f\|_{A^p_\omega}}{(1-|\varphi(z)|^2)^k \omega_*^{1/p}(\rho\varphi(z))}.$$

By Remark 1, we assume that $\omega_*(t)$ is decreasing on $[0, 1)$. So

$$\omega_*(\rho\varphi(z)) \geqslant \omega_*(\varphi(z)).$$

Hence,

$$|f^{(k)}(\rho(\varphi(z)))| \leqslant \frac{C\|f\|_{A^p_\omega}}{(1-|\varphi(z)|)^k \omega_*^{1/p}(\varphi(z))}.$$

Therefore, there exists $C = C(\omega, p, n)$ such that

$$\begin{aligned}
&|f^{(k)}(\varphi(z)) - \rho^k f^{(k)}(\rho\varphi(z))| \\
\leqslant\ & |f^{(k)}(\varphi(z)) - f^{(k)}(\rho\varphi(z))| + (1-\rho^k)|f^{(k)}(\rho\varphi(z))| \\
\leqslant\ & \left|\int_{\rho\varphi(z)}^{\varphi(z)} f^{(k+1)}(\eta)d\eta\right| + \frac{kC(1-\rho)\|f\|_{A^p_\omega}}{(1-|\varphi(z)|)^k \omega_*^{1/p}(\varphi(z))} \\
\leqslant\ & \frac{C(1-\rho)|\varphi(z)|\|f\|_{A^p_\omega}}{(1-|\varphi(z)|)^{(k+1)} \omega_*^{1/p}(\varphi(z))} + \frac{kC(1-\rho)\|f\|_{A^p_\omega}}{(1-|\varphi(z)|)^k \omega_*^{1/p}(\varphi(z))}.
\end{aligned}$$

By Lemma 5,

$$\begin{aligned}
&\sup_{|\varphi(z)|\leqslant r} \mu(z)|(uC_\varphi f - uC_{\rho\varphi} f)^{(n)}(z)| \\
\leqslant\ & \sup_{|\varphi(z)|\leqslant r} \mu(z) \sum_{k=0}^{n} |f^{(k)}(\varphi(z)) - \rho^k f^{(k)}(\rho\varphi(z))| \cdot \left|\sum_{l=k}^{n} C_n^l u^{(n-l)}(z) B_{l,k}(\varphi(z))\right| \\
\leqslant\ & C(1-\rho)\|f\|_{A^p_\omega} \sum_{k=0}^{n} \left(\frac{1}{1-r} + k\right) \frac{\mu(z)\left|\sum_{l=k}^{n} C_n^l u^{(n-l)}(z) B_{l,k}(\varphi(z))\right|}{(1-|\varphi(z)|)^k \omega_*^{1/p}(\varphi(z))}.
\end{aligned}$$

Since $uC_\varphi : A^p_\omega \to W^{(n)}_\mu$ is bounded, (4) holds. So,

$$\lim_{\rho\to 1} \sup_{|\varphi(z)|\leqslant r} \sup_{\|f\|_{A^p_\omega}\leqslant 1} \mu(z)|(uC_\varphi f - uC_{\rho\varphi} f)^{(n)}(z)| = 0. \tag{8}$$

In a similar way, we have

$$\lim_{\rho\to 1} \sup_{\|f\|_{A^p_\omega}\leqslant 1} |(uC_\varphi f - uC_{\rho\varphi} f)^{(k)}(0)| = 0,\ k = 0, 1, \cdots, n-1. \tag{9}$$

When $r < |\varphi(z)| < 1$ and $k = 0, 1, \cdots, n$, by Lemma 3 we have

$$\sup_{|\eta|=|\varphi(z)|} |f^{(k)}(\eta) - \rho^k f^{(k)}(\rho\eta)| \leqslant 2 \sup_{|\eta|=|\varphi(z)|} |f^{(k)}(\eta)| \leqslant \frac{C\|f\|_{A^p_\omega}}{(1-|\varphi(z)|)^k \omega_*^{1/p}(\varphi(z))}.$$

By Lemma 5,

$$\begin{aligned}
&\sup_{r<|\varphi(z)|<1} \sup_{\|f\|_{A^p_\omega}\leqslant 1} \mu(z)|(uC_\varphi f - uC_{\rho\varphi} f)^{(n)}(z)| \\
\leqslant\ & C \sum_{k=0}^{n} \sup_{r<|\varphi(z)|<1} \frac{\mu(z)\left|\sum_{l=k}^{n} C_n^l u^{(n-l)}(z) B_{l,k}(\varphi(z))\right|}{(1-|\varphi(z)|)^k \omega_*^{1/p}(\varphi(z))}.
\end{aligned} \tag{10}$$

From (8), (9) and (10), we have

$$\limsup_{\rho\to 1}\|uC_\varphi - uC_{\rho\varphi}\|_{A^p_\omega\to W^{(n)}_\mu} \lesssim \sum_{k=0}^{n}\sup_{r<|\varphi(z)|<1}\frac{\mu(z)\left|\sum_{l=k}^{n}C_n^l u^{(n-l)}(z)B_{l,k}(\varphi(z))\right|}{(1-|\varphi(z)|)^k\omega_*^{1/p}(\varphi(z))}. \tag{11}$$

Since $uC_{\rho\varphi}$ is compact, by letting $r\to 1$, we have

$$\|uC_\varphi\|_{e,A^p_\omega\to W^{(n)}_\mu} \lesssim \sum_{k=0}^{n}\limsup_{|\varphi(z)|\to 1}\frac{\mu(z)\left|\sum_{l=k}^{n}C_n^l u^{(n-l)}(z)B_{l,k}(\varphi(z))\right|}{(1-|\varphi(z)|)^k\omega_*^{1/p}(\varphi(z))}.$$

The proof is complete. $\square$

THEOREM 3. *Assume that $p\geqslant 1$, $n\in\mathbb{N}\cup\{0\}$, $u\in H(\mathbb{D})$, φ is an analytic self-map of $\mathbb{D}$, $\omega\in\hat{\mathscr{D}}$, and μ is a weight. Then $uC_\varphi : A^p_\omega\to W^{(n)}_{\mu,0}$ is bounded if and only if $uC_\varphi : A^p_\omega\to W^{(n)}_\mu$ is bounded and*

$$\lim_{|z|\to 1}\mu(z)\sum_{l=k}^{n}C_n^l u^{(n-l)}(z)B_{l,k}(\varphi(z))=0,\quad k=0,1,\cdots,n. \tag{12}$$

Proof. Suppose that $uC_\varphi : A^p_\omega\to W^{(n)}_\mu$ is bounded and (12) holds. Let $P(z)$ be a polynomial. Then there is constant $C=C(P)$ such that $\sup_{z\in\mathbb{D}}|P^{(k)}(z)|\leqslant C$ when $k=0,1,\cdots,n$. Therefore,

$$\lim_{|z|\to 1}\mu(z)|(uC_\varphi P)^{(n)}(z)|\leqslant \lim_{|z|\to 1}\mu(z)\sum_{k=0}^{n}\left|P^{(k)}(\varphi(z))\sum_{l=k}^{n}C_n^l u^{(n-l)}(z)B_{l,k}(\varphi(z))\right|=0.$$

Thus $uC_\varphi P\in W^{(n)}_{\mu,0}$. By [24, p. 16], the set of polynomials is dense in A^p_ω. Then for any $f\in A^p_\omega$, there is a sequence of polynomials p_n such that $\lim_{n\to\infty}\|f-p_n\|_{A^p_\omega}=0$. Therefore

$$\lim_{n\to\infty}uC_\varphi p_n = uC_\varphi f.$$

Since $W^{(n)}_{\mu,0}$ is closed in $W^{(n)}_\mu$, $uC_\varphi f\in W^{(n)}_{\mu,0}$. So, $uC_\varphi : A^p_\omega\to W^{(n)}_{\mu,0}$ is bounded.

Conversely, suppose that $uC_\varphi : A^p_\omega\to W^{(n)}_{\mu,0}$ is bounded. Obviously, $uC_\varphi : A^p_\omega\to W^{(n)}_\mu$ is bounded. Suppose $h_s(z)=z^s (s\in\mathbb{N}\cup\{0\})$. Then $uC_\varphi h_s\in W^{(n)}_{\mu,0}$, that is

$$\lim_{|z|\to 1}\mu(z)|(uC_\varphi h_s)^{(n)}(z)|=0.$$

Let $s=0,1,\cdots,n$ in order. By Lemma 5 and triangle inequality, (12) holds. The proof is complete. $\square$

THEOREM 4. *Assume $p \geqslant 1$, $n \in \mathbb{N} \cup \{0\}$, $u \in H(\mathbb{D})$, φ is an analytic self-map of $\mathbb{D}$, $\omega \in \hat{\mathscr{D}}$, and μ is a weight. If $uC_\varphi : A^p_\omega \to W^{(n)}_{\mu,0}$ is bounded, then*

$$\|uC_\varphi\|_{e,A^p_\omega \to W^{(n)}_{\mu,0}} \approx \sum_{k=0}^{n} \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)\left|\sum\limits_{l=k}^{n} C_n^l u^{(n-l)}(z) B_{l,k}(\varphi(z))\right|}{(1-|\varphi(z)|)^k \omega_*^{1/p}(\varphi(z))}$$
$$\approx \sum_{k=0}^{n} \limsup_{|z|\to 1} \frac{\mu(z)\left|\sum\limits_{l=k}^{n} C_n^l u^{(n-l)}(z) B_{l,k}(\varphi(z))\right|}{(1-|\varphi(z)|)^k \omega_*^{1/p}(\varphi(z))}.$$

Proof. The lower estimate of $\|uC_\varphi\|_{e,A^p_\omega \to W^{(n)}_{\mu,0}}$.

Let $k = 0,1,\cdots,n$. There is $\{z_j\} \subset \mathbb{D}$ satisfying $\lim\limits_{j\to\infty} |z_j| = 1$, $\lim\limits_{j\to\infty} |\varphi(z_j)| = a$ and

$$\limsup_{|z|\to 1} \frac{\mu(z)\left|\sum\limits_{l=k}^{n} C_n^l u^{(n-l)}(z) B_{l,k}(\varphi(z))\right|}{(1-|\varphi(z)|)^k \omega_*^{1/p}(\varphi(z))} = \lim_{j\to\infty} \frac{\mu(z_j)\left|\sum\limits_{l=k}^{n} C_n^l u^{(n-l)}(z_j) B_{l,k}(\varphi(z_j))\right|}{(1-|\varphi(z_j)|)^k \omega_*^{1/p}(\varphi(z_j))}. \tag{13}$$

If $a<1$, suppose $|\varphi(z_j)| < \frac{1+a}{2}$ holds for all j. Then

$$\frac{\mu(z_j)\left|\sum\limits_{l=k}^{n} C_n^l u^{(n-l)}(z_j) B_{l,k}(\varphi(z_j))\right|}{(1-|\varphi(z_j)|)^k \omega_*^{1/p}(\varphi(z_j))}$$
$$\leqslant \sup_{0\leqslant r\leqslant \frac{1+a}{2}} \left(\frac{1}{(1-r)^k(\omega_*(r))^{1/p}}\right) \mu(z_j)\left|\sum_{l=k}^{n} C_n^l u^{(n-l)}(z_j) B_{l,k}(\varphi(z_j))\right|.$$

By Theorem 3 and (13), we have

$$0 = \limsup_{|z|\to 1} \frac{\mu(z)\left|\sum\limits_{l=k}^{n} C_n^l u^{(n-l)}(z) B_{l,k}(\varphi(z))\right|}{(1-|\varphi(z)|)^k \omega_*^{1/p}(\varphi(z))} \lesssim \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)\left|\sum\limits_{l=k}^{n} C_n^l u^{(n-l)}(z) B_{l,k}(\varphi(z))\right|}{(1-|\varphi(z)|)^k \omega_*^{1/p}(\varphi(z))}.$$

If $a=1$, by (13) and $|\varphi(z_j)| \to a$, we have

$$\limsup_{|z|\to 1} \frac{\mu(z)\left|\sum\limits_{l=k}^{n} C_n^l u^{(n-l)}(z) B_{l,k}(\varphi(z))\right|}{(1-|\varphi(z)|)^k \omega_*^{1/p}(\varphi(z))} \lesssim \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)\left|\sum\limits_{l=k}^{n} C_n^l u^{(n-l)}(z) B_{l,k}(\varphi(z))\right|}{(1-|\varphi(z)|)^k \omega_*^{1/p}(\varphi(z))}.$$

Since $\|uC_\varphi\|_{e,A^p_\omega\to W^{(n)}_{\mu,0}} \geqslant \|uC_\varphi\|_{e,A^p_\omega\to W^{(n)}_{\mu}}$, by Theorem 2 we get

$$\begin{aligned}\|uC_\varphi\|_{e,A^p_\omega\to W^{(n)}_{\mu,0}} &\gtrsim \sum_{k=0}^{n}\limsup_{|\varphi(z)|\to 1}\frac{\mu(z)\left|\sum\limits_{l=k}^{n} C_n^l u^{(n-l)}(z)B_{l,k}(\varphi(z))\right|}{(1-|\varphi(z)|)^k\omega_*^{1/p}(\varphi(z))}\\ &\geqslant \sum_{k=0}^{n}\limsup_{|z|\to 1}\frac{\mu(z)\left|\sum\limits_{l=k}^{n} C_n^l u^{(n-l)}(z)B_{l,k}(\varphi(z))\right|}{(1-|\varphi(z)|)^k\omega_*^{1/p}(\varphi(z))}.\end{aligned}$$

The upper estimate of $\|uC_\varphi\|_{e,A^p_\omega\to W^{(n)}_{\mu,0}}$.

For $0<\rho<0$, let $f_\rho(z)=f(\rho z)$. Then $uC_{\rho\varphi}f=uC_\varphi f_\rho$. Therefore $uC_{\rho\varphi}$ is an operator from A^p_ω to $W^{(n)}_{\mu,0}$. By the proof of Theorem 2, $uC_{\rho\varphi}:A^p_\omega\to W^{(n)}_{\mu,0}$ is compact. Therefore, (11) holds. By Theorem 2 and

$$\|uC_\varphi-uC_{\rho\varphi}\|_{A^p_\omega\to W^{(n)}_{\mu}}=\|uC_\varphi-uC_{\rho\varphi}\|_{A^p_\omega\to W^{(n)}_{\mu,0}},$$

we have

$$\begin{aligned}\|uC_\varphi\|_{e,A^p_\omega\to W^{(n)}_{\mu,0}} &\leqslant \limsup_{\rho\to 1}\|uC_\varphi-uC_{\rho\varphi}\|_{A^p_\omega\to W^{(n)}_{\mu}}\\ &\lesssim \sum_{k=0}^{n}\sup_{r<|\varphi(z)|<1}\frac{\mu(z)\left|\sum\limits_{l=k}^{n} C_n^l u^{(n-l)}(z)B_{l,k}(\varphi(z))\right|}{(1-|\varphi(z)|)^k\omega_*^{1/p}(\varphi(z))}.\end{aligned}$$

By letting $r\to 1$, we have

$$\begin{aligned}\|uC_\varphi\|_{e,A^p_\omega\to W^{(n)}_{\mu,0}} &\lesssim \sum_{k=0}^{n}\limsup_{|\varphi(z)|\to 1}\frac{\mu(z)\left|\sum\limits_{l=k}^{n} C_n^l u^{(n-l)}(z)B_{l,k}(\varphi(z))\right|}{(1-|\varphi(z)|)^k\omega_*^{1/p}(\varphi(z))}\\ &\lesssim \sum_{k=0}^{n}\limsup_{|z|\to 1}\frac{\mu(z)\left|\sum\limits_{l=k}^{n} C_n^l u^{(n-l)}(z)B_{l,k}(\varphi(z))\right|}{(1-|\varphi(z)|)^k\omega_*^{1/p}(\varphi(z))}.\end{aligned}$$

The proof is complete. □

From Theorems 2 and 4, we can easily get the following two corollaries.

COROLLARY 5. *Assume that* $p\geqslant 1$, $n\in\mathbb{N}\cup\{0\}$, $u\in H(\mathbb{D})$, φ *is an analytic self-map of* $\mathbb{D}$, $\omega\in\hat{\mathscr{D}}$, *and* μ *is a weight. If* $uC_\varphi:A^p_\omega\to W^{(n)}_\mu$ *is bounded, then* uC_φ *is compact if and only if*

$$\limsup_{|\varphi(z)|\to 1}\frac{\mu(z)\left|\sum\limits_{l=k}^{n} C_n^l u^{(n-l)}(z)B_{l,k}(\varphi(z))\right|}{(1-|\varphi(z)|)^k\omega_*^{1/p}(\varphi(z))}=0,\ \ for\ \ k=0,1,\cdots,n.$$

COROLLARY 6. *Assume that $p \geqslant 1$, $n \in \mathbb{N} \cup \{0\}$, $u \in H(\mathbb{D})$, φ is an analytic self-map of $\mathbb{D}$, $\omega \in \hat{\mathscr{D}}$, and μ is a weight. If $uC_\varphi : A^p_\omega \to W^{(n)}_{\mu,0}$ is bounded, then the following statements are equivalent.*

(i) $uC_\varphi : A^p_\omega \to W^{(n)}_{\mu,0}$ *is compact.*

(ii) *For* $k = 0, 1, \cdots, n$,

$$\limsup_{|z|\to 1} \frac{\mu(z)\left|\sum_{l=k}^{n} C_n^l u^{(n-l)}(z) B_{l,k}(\varphi(z))\right|}{(1-|\varphi(z)|)^k \omega_*^{1/p}(\varphi(z))} = 0.$$

(iii) *For* $k = 0, 1, \cdots, n$,

$$\limsup_{|\varphi(z)|\to 1} \frac{\mu(z)\left|\sum_{l=k}^{n} C_n^l u^{(n-l)}(z) B_{l,k}(\varphi(z))\right|}{(1-|\varphi(z)|)^k \omega_*^{1/p}(\varphi(z))} = 0.$$

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