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THE OSTROWSKI TYPE INEQUALITIES WITH THE APPLICATION TO THE THREE POINT INTEGRAL FORMULA

SANJA KOVAČ, JOSIP PEČARIĆ AND SANJA TIPURIĆ-SPUŽEVIĆ

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Abstract. The generalization of the integral formula with three nodes is introduced, and some sharp and the best possible inequalities for the functions whose higher order derivatives belong to L_p spaces are given. We establish non-weighted version of the three point integral formula. From the general non-weighted formula we shall get the famous Simpson, dual Simpson and Maclaurin formulae. Some new errors of approximation in these integral formulae are obtained.

1. Introduction

The most elementary quadrature rules with three nodes are Simpson's rule, based on the Simpson's formula [3]

$$\int_a^b f(t)\mathrm{d}t = \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{(b-a)^5}{2880}f^{(4)}(\eta), \tag{1}$$

where $\eta \in [a,b]$, the dual Simpson's rule based on the following three-point formula [3]

$$\int_a^b f(t)\mathrm{d}t = \frac{b-a}{3}\left[2f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{a+3b}{4}\right)\right]+\frac{7(b-a)^5}{23040}f^{(4)}(\xi), \tag{2}$$

where $\xi \in [a,b]$, and Maclaurin's rule based on the Maclaurin's formula

$$\int_a^b f(t)\mathrm{d}t = \frac{b-a}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]+\frac{7(b-a)^5}{51840}f^{(4)}(\theta), \tag{3}$$

where $\theta \in [a,b]$. These formulae are valid for any function f with continuous forth derivative $f^{(4)}$ on $[a,b]$.

In this paper we will use the concept of the harmonic polynomials which was considered by J. Pečarić and S. Varošanec [13]. Namely, let

$$\sigma = \{a = x_0 < x_1 < \cdots < x_m = b\}$$

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be a subdivision of the interval $[a,b]$. Set

$$S_n(t,\sigma)=\begin{cases} P_{1n}(t), & t\in[a,x_1] \\ P_{2n}(t), & t\in(x_1,x_2] \\ \vdots & \\ P_{mn}(t), & t\in(x_{m-1},x_m], \end{cases}$$

where $\{P_{jn}\}_n$ are the sequences of harmonic polynomials, i.e. $P'_{jk}(t)=P_{j,k-1}(t)$. By succesive integration by parts they have proved whenever the integrals exist that

$$(-1)^n\int_a^b S_n(t,\sigma)\mathrm{d}f^{(n-1)}(t)=\int_a^b f(t)\mathrm{d}t+\sum_{k=1}^{n}(-1)^k\Big[P_{mk}(b)f^{(k-1)}(b) + \sum_{j=1}^{m-1}(P_{jk}(x_j)-P_{j+1,k}(x_j))f^{(k-1)}(x_j)-P_{1k}(a)f^{(k-1)}(a)\Big]. \tag{4}$$

Using formula (4) we will obtain the general quadrature formula with three nodes x, $\frac{a+b}{2}$ and $a+b-x$, for some $x\in[a,\frac{a+b}{2}]$. Such formulae will include the values of the higher order derivatives of function f in nodes $x, \frac{a+b}{2}$ and $a+b-x$. Further, we observe functions f whose higher ordered derivatives belong to L_p spaces and establish sharp and the best possible constants for such inequalities. Finally, for appropriate choices of x we will give the generalizations ot the well-known Simpson's (1), dual Simpson's (2) and Maclaurin's formula (3). Also, the Legendre-Gauss quadrature formula with two nodes

$$\int_a^b f(t)\mathrm{d}t=\frac{b-a}{2}\left(f\left(\frac{a+b}{2}-\frac{b-a}{2\sqrt{3}}\right)+f\left(\frac{a+b}{2}+\frac{b-a}{2\sqrt{3}}\right)\right)+\frac{(b-a)^5}{4320}f^{(4)}(\eta) \tag{5}$$

will appear as special case, since its exactness is the same as the exactness of the above mentioned three-point Newton-Cotes formulae.

For $c_0\in[a,b]$ set $slD(c_0)$ which stands for the class of all continuous functions $f:[a,b]\to\mathbb{R}$ differentiable on $\langle a,c_0\rangle\cup\langle c_0,b\rangle$ such that

$$M_l:=\sup_{t\in\langle a,c_0\rangle}|f'(t)|<\infty \text{ and } M_r:=\sup_{t\in\langle c_0,b\rangle}|f'(t)|<\infty.$$

If $c_0=a$ we set $M_l=0$ and if $c_0=b$ we set $M_r=0$.

In [5] the generalization of the M. Niezgoda result ([12]) is obtained for the weighted integral formula.

DEFINITION 1. Let $w:[a,b]\to[0,\infty)$ be an integrable weight function and $w_k:[a,b]\to\mathbb{R}$ are differentiable functions for $k\in\mathbb{N}$. We say that $\{w_k\}_{k\in\mathbb{N}}$ is *w-harmonic sequence of functions* if for $k\geqslant 2$, $w'_k(t)=w_{k-1}(t)$ and $w'_1(t)=w(t)$, for $t\in[a,b]$.

Given a subdivision $\sigma=\{a=x_0<x_1<\ldots<x_{m-1}<x_m=b\}$ of the interval $[a,b]$, let us consider different w-harmonic sequences of functions $\{w_{jk}\}_{k\in\mathbb{N}}$ on each

interval $[x_{j-1},x_j]$, $j\in\{1,2,\dots,m\}$. Define

$$W_{n,w}(t,\sigma)=\begin{cases} w_{1n}(t), & t\in[a,x_1] \\ w_{2n}(t), & t\in(x_1,x_2] \\ \vdots \\ w_{mn}(t), & t\in(x_{m-1},b]. \end{cases} \tag{6}$$

Then for every function $f:[a,b]\to\mathbb{R}$, such that $f^{(n)}$ is piecewise continuous on $[a,b]$, it is proved in [6] that

$$\begin{aligned} \int_a^b w(t)f(t)\mathrm{d}t = & \sum_{k=1}^{n}(-1)^{k-1}\Big[w_{mk}(b)f^{(k-1)}(b) \\ & +\sum_{j=1}^{m-1}\left[w_{jk}(x_j)-w_{j+1,k}(x_j)\right]f^{(k-1)}(x_j)-w_{1k}(a)f^{(k-1)}(a)\Big] \\ & +(-1)^n\int_a^b W_{n,w}(t,\sigma)f^{(n)}(t)\mathrm{d}t. \end{aligned} \tag{7}$$

The identity (7) is called weighted integral identity.

Here and hereafter the symbol $L^p_{[a,b]}$ $(p\geqslant 1)$ denotes the space of p-power integrable functions on the interval $[a,b]$ equipped with the norm

$$\|f\|_p=\left(\int_a^b |f(t)|^p\,dt\right)^{\frac{1}{p}}$$

and $L^\infty_{[a,b]}$ denotes the space of essentially bounded functions on $[a,b]$ with the norm

$$\|f\|_\infty = ess\sup_{t\in[a,b]}|f(t)|.$$

Let us denote

$$\begin{aligned} Tf(x_0,x_1,x_2,\dots,x_m) := & \int_a^b w(t)f(t)\mathrm{d}t-\sum_{k=1}^{n}(-1)^{k-1}\Big[w_{mk}(b)f^{(k-1)}(b) \\ & +\sum_{j=1}^{m-1}\left[w_{jk}(x_j)-w_{j+1,k}(x_j)\right]f^{(k-1)}(x_j)-w_{1k}(a)f^{(k-1)}(a)\Big]. \end{aligned}$$

The following three theorems have been established in [5] . For the reader convience we give a proof of the first theorem here :

THEOREM 1. *Let $f:[a,b]\to\mathbb{R}$ be a function such that $f^{(n-1)}\in D(c_0)$, for some $n\in\mathbb{N}$ and*

$$M_l=\sup_{t\in\langle a,c_0\rangle}\left|f^{(n)}(t)\right|<\infty \quad and\ \ M_r=\sup_{t\in\langle c_0,b\rangle}\left|f^{(n)}(t)\right|<\infty.$$

Then the following inequality holds:

$$|Tf(x_0,x_1,x_2,\dots,x_m)| \leqslant \begin{cases} \left[M_l^p(c_0-a)+M_r^p(b-c_0)\right]^{1/p}\cdot\|W_{n,w}(\cdot,\sigma)\|_q, & 1\leqslant p<\infty \\ \max\{M_l,M_r\}\cdot\|W_{n,w}(\cdot,\sigma)\|_1, & p=\infty. \end{cases}$$

Proof. From the identity (7) we have

$$\begin{aligned} Tf(x_0,x_2,\dots,x_m) &= (-1)^n\int_a^b W_{n,w}(t,\sigma)f^{(n)}(t)dt \\ &= (-1)^n\int_a^{c_0} W_{n,w}(t,\sigma)f^{(n)}(t)dt+(-1)^n\int_{c_0}^b W_{n,w}(t,\sigma)f^{(n)}(t)dt. \end{aligned}$$

By taking the absolute value and applying Hölder inequality we get

$$\begin{aligned} &|Tf(x_0,x_1,\dots,x_m)| \\ &\leqslant \left|\int_a^{c_0} W_{n,w}(t,\sigma)f^{(n)}(t)dt\right|+\left|\int_{c_0}^b W_{n,w}(t,\sigma)f^{(n)}(t)dt\right| \\ &\leqslant \|W_{n,w}(\cdot,\sigma)\|_{q,[a,c_0]}\cdot\|f^{(n)}\|_{p,[a,c_0]}+\|W_{n,w}(\cdot,\sigma)\|_{q,[c_0,b]}\cdot\|f^{(n)}\|_{p,[c_0,b]} \\ &\leqslant M_l\cdot(c_0-a)^{1/p}\cdot\|W_{n,w}(\cdot,\sigma)\|_{q,[a,c_0]}+M_r\cdot(b-c_0)^{1/p}\cdot\|W_{n,w}(\cdot,\sigma)\|_{q,[c_0,b]}. \end{aligned}$$

For $1\leqslant p<\infty$ we apply discrete Hölder inequality and get

$$|Tf(x_0,x_1,\dots,x_m)| \leqslant \left[M_l^p(c_0-a)+M_r^p(b-c_0)\right]^{1/p}\cdot\|W_{n,w}(\cdot,\sigma)\|_q,$$

while for $p=\infty$ we have

$$|Tf(x_0,x_1,\dots,x_m)| \leqslant \max\{M_l,M_r\}\cdot\|W_{n,w}(\cdot,\sigma)\|_1$$

so the proof is complete. □

In [5] following lemma is given:

LEMMA 1. *Let* $1\leqslant p\leqslant\infty$ *and* $f:[a,b]\to\mathbb{R}$ *be a function such that* $f^{(n)}\in D(c_0)$, *for some* $n\in\mathbb{N}$ *and*

$$M_l=\sup_{t\in\langle a,c_0\rangle}\left|f^{(n+1)}(t)\right|<\infty \quad \text{and} \quad M_r=\sup_{t\in\langle c_0,b\rangle}\left|f^{(n+1)}(t)\right|<\infty.$$

Then the following inequalities holds:

$$\begin{aligned} \|f^{(n)}(\cdot)-f^{(n)}(c_0)\|_{p,[a,c_0]}^p &\leqslant \frac{M_l^p(c_0-a)^{p+1}}{p+1}, \quad 1\leqslant p<\infty \\ \|f^{(n)}(\cdot)-f^{(n)}(c_0)\|_{\infty,[a,c_0]} &\leqslant M_l(c_0-a), \quad p=\infty \\ \|f^{(n)}(\cdot)-f^{(n)}(c_0)\|_{p,[c_0,b]}^p &\leqslant \frac{M_r^p(b-c_0)^{p+1}}{p+1}, \quad 1\leqslant p<\infty \\ \|f^{(n)}(\cdot)-f^{(n)}(c_0)\|_{\infty,[c_0,b]} &\leqslant M_r(b-c_0), \quad p=\infty. \end{aligned}$$

THEOREM 2. *Let* $f:[a,b]\to\mathbb{R}$ *be a function such that* $f^{(n)}\in D(c_0)$, *for some* $n\in\mathbb{N}$ *and*

$$M_l=\sup_{t\in\langle a,c_0\rangle}\left|f^{(n+1)}(t)\right|<\infty \quad \text{and} \ \ M_r=\sup_{t\in\langle c_0,b\rangle}\left|f^{(n+1)}(t)\right|<\infty.$$

Assume $1\leqslant p,q\leqslant\infty$ *are conjugate exponents* ($\frac{1}{p}+\frac{1}{q}=1$) *and* $W_{n,w}(\cdot,\sigma)$ *defined by (6). Then the following inequality holds:*

$$\left|Tf(x_0,x_1,x_2,\ldots,x_m)-(-1)^n\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\int_a^b W_{n,w}(t,\sigma)dt\right|$$
$$\leqslant\begin{cases}\frac{1}{(p+1)^{1/p}}\left[M_l^p(c_0-a)^{p+1}+M_r^p(b-c_0)^{p+1}\right]^{1/p}\cdot\|W_{n,w}(\cdot,\sigma)-\eta\|_{q,[a,b]}, & 1\leqslant p<\infty\\ \max\{M_l(c_0-a),M_r(b-c_0)\}\cdot\|W_{n,w}(\cdot,\sigma)-\eta\|_{1,[a,b]}, & p=\infty,\ q=1,\end{cases}$$

where $\eta=\frac{1}{b-a}\int_a^b W_{n,w}(t,\sigma)dt$.

THEOREM 3. *Let* $f:[a,b]\to\mathbb{R}$ *be a function such that* $f^{(n)}\in D(c_0)$, *for some* $n\in\mathbb{N}$ *and*

$$M_l=\sup_{t\in\langle a,c_0\rangle}\left|f^{(n+1)}(t)\right|<\infty \quad \text{and} \ \ M_r=\sup_{t\in\langle c_0,b\rangle}\left|f^{(n+1)}(t)\right|<\infty.$$

Assume $1\leqslant p,q\leqslant\infty$ *are conjugate exponents* ($\frac{1}{p}+\frac{1}{q}=1$) *and* $W_{n,w}(\cdot,\sigma)$ *defined by (6). Then the following inequality holds:*

$$\left|Tf(x_0,x_1,x_2,\ldots,x_m)-(-1)^n f^{(n)}(c_0)\cdot\int_a^b W_{n,w}(t,\sigma)dt\right|$$
$$\leqslant\begin{cases}\frac{1}{(p+1)^{1/p}}\left[M_l^p(c_0-a)^{p+1}+M_r^p(b-c_0)^{p+1}\right]^{1/p}\cdot\|W_{n,w}(\cdot,\sigma)\|_{q,[a,b]}, & 1\leqslant p<\infty\\ \max\{M_l(c_0-a),M_r(b-c_0)\}\cdot\|W_{n,w}(\cdot,\sigma)\|_{1,[a,b]}, & p=\infty,\ q=1.\end{cases}$$

The aim of this paper is to derive the analogues error bound for the three-point version of integral formula with $w(t)=\frac{1}{b-a}$ obtained with some special cases for $x\in[a,\frac{a+b}{2}]$.

2. The general three-point quadrature formula

For $x \in [a, \frac{a+b}{2})$ let us consider the following subdivision of the segment $[a,b]$:

$$\sigma := \{x_0 < x_1 < x_2 < x_3 < x_4\},$$

where $x_0 = a,\ x_1 = x,\ x_2 = \frac{a+b}{2},\ x_3 = a+b-x$ and $x_4 = b$. Let $Q_n(t)$ be some monic polynomial of degree n, for some $n \in \mathbf{N}$. Set

$$S_n(t,x) = \begin{cases} P_{1n}(t) = \frac{(t-a)^n}{n!}, & t \in [a,x] \\ P_{2n}(t) = \frac{Q_n(t)}{n!}, & t \in (x, \frac{a+b}{2}] \\ P_{3n}(t) = (-1)^n \frac{Q_n(a+b-t)}{n!}, & t \in (\frac{a+b}{2}, a+b-x] \\ P_{4n}(t) = \frac{(t-b)^n}{n!}, & t \in (a+b-x, b]. \end{cases} \tag{8}$$

Further, for $k = 0,1,\ldots,n-1$ we define

$$P_{1k}(t) = \frac{(t-a)^k}{k!}, \quad P_{2k}(t) = \frac{Q_n^{(n-k)}(t)}{n!}$$

$$P_{3k}(t) = \frac{(-1)^k Q_n^{(n-k)}(a+b-t)}{n!}, \quad P_{4k}(t) = \frac{(t-b)^k}{k!}.$$

REMARK 1. The sequences of the polynomials $\{P_{jk}\}_{k=0,1,\ldots,n}$ are harmonic, for $j = 1,2,3,4$, i.e. $P'_{jk}(t) = P_{j,k-1}(t)$, for $k = 1,\ldots,n$ and $P_{j0}(t) = 1$, for $j = 1,2,3,4$.

REMARK 2. If we put

$$Q_k(t) := \frac{k! Q_n^{(n-k)}(t)}{n!}, \quad k = 0,1,\ldots,n-1,$$

then we have

$$P_{2k}(t) = \frac{Q_k(t)}{k!}, \quad \text{and} \quad P_{3k}(t) = \frac{(-1)^k Q_k(a+b-t)}{k!}.$$

Further, polynomials Q_k satisfy $Q'_k(t) = Q_{k-1}(t)$.

REMARK 3. The following symmetry conditions are valid:

$$P_{1k}(t) = (-1)^k P_{4k}(a+b-t), \forall t \in [a,x]$$

and

$$P_{2k}(t) = (-1)^k P_{3k}(a+b-t), \forall t \in \left(x, \frac{a+b}{2}\right).$$

Now we can state the general three point formula:

THEOREM 4. *Let* $f:[a,b]\to\mathbf{R}$ *be a function with a piecewise continuous* n*-th derivative, for some* $n\in\mathbf{N}$ *and* $x\in[a,\frac{a+b}{2})$. *Further, let* $Q_n(t)$ *be some monic polynomial of degree* n *and* $S_n(t,x)$ *be defined by relation (8). Then the following formula holds*

$$\int_a^b f(t)\mathrm{d}t=\sum_{k=1}^{n}A_k(x)\left(f^{(k-1)}(x)+(-1)^{k-1}f^{(k-1)}(a+b-x)\right)\qquad(9)$$

$$+\sum_{\substack{k=1\\ odd}}^{n}B_k(x)f^{(k-1)}\left(\frac{a+b}{2}\right)+(-1)^n\int_a^b S_n(t,x)f^{(n)}(t)\mathrm{d}t,$$

where

$$A_k(x)=\frac{(-1)^{k-1}}{k!}\left((x-a)^k-Q_k(x)\right),\quad k\geqslant 1,$$

$$B_k(x)=\frac{2Q_k(\frac{a+b}{2})}{k!},\quad \textit{for odd}\ \ k\geqslant 1$$

and

$$B_k(x)=P_{2k}(x)-P_{3k}(x)=0,\quad \textit{for even}\ \ k\geqslant 1.$$

Proof. We consider subdivision $x_0=a$, $x_1=x$, $x_2=\frac{a+b}{2}$, $x_3=a+b-x$ and $x_4=b$ of the interval $[a,b]$ and apply formula (4) with $m=4$. We have

$$P_{1k}(a)=P_{4k}(b)=0,\quad \forall k=1,\dots,n.$$

Further, imposing polynomials (8) in (4) we get the coefficient by $f^{(k-1)}(x)$ and $(-1)^{k-1}f^{(k-1)}(a+b-x)$ equals to

$$A_k(x)=(-1)^{k-1}\left[P_{1k}(x)-P_{2k}(x)\right]=\frac{(-1)^{k-1}}{k!}\left((x-a)^k-Q_k(x)\right)$$

and coefficient by $f^{(k-1)}(\frac{a+b}{2})$ for odd k equals to

$$B_k(x)=(-1)^{k-1}\left[P_{2k}\left(\frac{a+b}{2}\right)-P_{3k}\left(\frac{a+b}{2}\right)\right]=\frac{2Q_k(\frac{a+b}{2})}{k!}.\quad\square$$

REMARK 4. Analogue results for the general two-point formula with nodes x and $a+b-x$ were considered in [4] and [9].

Now we will state L_p inequalities for the general three point integral formula.

THEOREM 5. *Let $f:[a,b]\to\mathbf{R}$ be a function with a piecewise continuous n-th derivative and $f^{(n)}\in L_p[a,b]$ for some $n\in\mathbf{N}$ and some $1\leqslant p\leqslant\infty$. Then we have the following inequality*

$$\begin{aligned}&\Big|\int_a^b f(t)\mathrm{d}t-\sum_{k=1}^{n}A_k(x)\left(f^{(k-1)}(x)+(-1)^{k-1}f^{(k-1)}(a+b-x)\right)\\&-\sum_{\substack{k=1\\ odd}}^{n}B_k(x)f^{(k-1)}\left(\frac{a+b}{2}\right)\Big|\\ \leqslant\ & C(n,p,x)\cdot\|f^{(n)}\|_p,\end{aligned} \tag{10}$$

where

$$C(n,p,x)=\begin{cases}\frac{2^{1/q}}{n!}\left[\frac{(x-a)^{nq+1}}{nq+1}+\int_x^{\frac{a+b}{2}}|Q_n(t)|^q\,\mathrm{d}t\right]^{\frac{1}{q}}, & \frac{1}{p}+\frac{1}{q}=1,\quad 1<p\leqslant\infty\\ \frac{1}{n!}\max\left\{(x-a)^n,\sup_{t\in[x,\frac{a+b}{2}]}|Q_n(t)|\right\}, & p=1.\end{cases} \tag{11}$$

The inequality is sharp for $1<p\leqslant\infty$ and the best possible for $p=1$. Equality is attained for the function $f_*:[a,b]\to\mathbf{R}$ defined by

$$f_*(t)=\frac{1}{(n-1)!}\int_a^t(t-s)^{n-1}|S_n(s,x)|^{\frac{1}{p-1}}\,sgnS_n(s,x)\mathrm{d}s \tag{12}$$

for $1<p<\infty$, while for $p=\infty$

$$f_*(t)=\frac{1}{(n-1)!}\int_a^t(t-s)^{n-1}sgnS_n(s,x)\mathrm{d}s \tag{13}$$

Proof. The first part of the theorem follows from the Hölder inequality to the identity (9). For $1\leqslant q<\infty$ we have

$$\|S_n(\cdot,x)\|_q=\frac{2^{\frac{1}{q}}}{n!}\cdot\left[\frac{(x-a)^{nq+1}}{nq+1}+\int_x^{\frac{a+b}{2}}|Q_n(t)|^q\mathrm{d}t\right]^{\frac{1}{q}},$$

and

$$\|S_n(\cdot,x)\|_\infty=\frac{1}{n!}\max\left\{(x-a)^n,\sup_{t\in[x,\frac{a+b}{2}]}|Q_n(t)|\right\}.$$

For the proof of the sharpness, we need to find function f such that

$$\left|\int_a^b S_n(t,x)f^{(n)}(t)\mathrm{d}t\right|=\|f^{(n)}\|_p\cdot\|S_n(\cdot,x)\|_q,$$

for $1<p\leqslant\infty$ and $1\leqslant q<\infty$ such that $\frac{1}{p}+\frac{1}{q}=1$. The function f_* defined by (12) and (13) is $n-1$ times continuously differentiable and it has a piecewise existing n-th derivative. Further, f_* is a solution of the differentiable equation

$$S_n(t,x)f^{(n)}(t)=|S_n(t,x)|^q,$$

so the assertion follows.

For $p=1$ we shall prove that

$$\Big|\int_a^b S_n(t,x)f^{(n)}(t)\mathrm{d}t\Big| \leqslant \max_{t\in[a,\frac{a+b}{2}]}|S_n(t,x)|\cdot\int_a^b|f^{(n)}(t)|\mathrm{d}t \tag{14}$$

is the best possible inequality. Suppose that $|S_n(t,x)|$ attains its maximum at point $t_0\in[a,b]$. First, let us assume that $S_n(t_0,x)>0$. For ε small enough define $f_\varepsilon^{(n-1)}(t)$ by

$$f_\varepsilon^{(n-1)}(t)=\begin{cases}0, & t\leqslant t_0\\ \frac{t-t_0}{\varepsilon}, & t\in[t_0,t_0+\varepsilon]\\ 1, & t\geqslant t_0+\varepsilon,\end{cases}$$

if $t_0=x$ and $\max_{t\in[a,\frac{a+b}{2}]}|S_n(t,x)|=Q_n(x)$. Then, for ε small enough,

$$\Big|\int_a^b S_n(t,x)f_\varepsilon^{(n)}(t)\mathrm{d}t\Big|=\Big|\int_{t_0}^{t_0+\varepsilon}S_n(t,x)\frac{1}{\varepsilon}\mathrm{d}t\Big|=\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}Q_n(t)\mathrm{d}t.$$

Now, relation (14) implies

$$\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}Q_n(t)\mathrm{d}t\leqslant\frac{1}{\varepsilon}Q_n(t_0)\int_{t_0}^{t_0+\varepsilon}\mathrm{d}t=Q_n(t_0).$$

Since

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}Q_n(t)\mathrm{d}t=Q_n(t_0),$$

the statement follows. On the other hand, if $\max_{t\in[a,\frac{a+b}{2}]}|S_n(t,x)|\neq|Q_n(x)|$, we define

$$f_\varepsilon^{(n-1)}(t)=\begin{cases}1, & t\leqslant t_0-\varepsilon\\ \frac{t_0-t}{\varepsilon}, & t\in[t_0-\varepsilon,t_0]\\ 0, & t\geqslant t_0.\end{cases}$$

Then, for ε small enough,

$$\Big|\int_a^b S_n(t,x)f_\varepsilon^{(n)}(t)\mathrm{d}t\Big|=\Big|\int_{t_0-\varepsilon}^{t_0}S_n(t,x)\frac{1}{\varepsilon}\mathrm{d}t\Big|=\frac{1}{\varepsilon}\int_{t_0-\varepsilon}^{t_0}S_n(t,x)\mathrm{d}t.$$

Now, relation (14) implies

$$\frac{1}{\varepsilon}\int_{t_0-\varepsilon}^{t_0}S_n(t,x)\mathrm{d}t\leqslant\frac{1}{\varepsilon}S_n(t_0,x)\int_{t_0-\varepsilon}^{t_0}\mathrm{d}t=S_n(t_0,x).$$

Since

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{t_0-\varepsilon}^{t_0} S_n(t,x)\mathrm{d}t = S_n(t_0,x),$$

the statement follows.

For the case $S_n(t_0,x)<0$, the proof is similar. □

Let us apply upper results to the following example of monic polynomial Q_n:

$$\begin{aligned} Q_{n,x}(t) :=\; & (t-x)^n + n\Big(x-a-\frac{(b-a)^3}{6(2x-a-b)^2}\Big)(t-x)^{n-1} \\ & + \binom{n}{2}(x-a)^2(t-x)^{n-2} + \binom{n}{3}(x-a)^3(t-x)^{n-3} \\ & + \binom{n}{4}(x-a)^4(t-x)^{n-4}, \quad t\in\Big[x,\frac{a+b}{2}\Big]. \end{aligned} \tag{15}$$

After some calculation from Theorem 4 we get

$$A_1(x) = \frac{(b-a)^3}{6(2x-a-b)^2}, \qquad B_1(x) = b-a-\frac{(b-a)^3}{3(2x-a-b)^2},$$

$$A_2(x) = A_3(x) = A_4(x) = B_3(x) = 0.$$

Now we have

COROLLARY 1. *Let $f:[a,b]\to\mathbf{R}$ be a function with a piecewise continuous n-th derivative, for some $n\in\mathbf{N}$ and $x\in[a,\frac{a+b}{2})$. Further, let $Q_{n,x}(t)$ be defined by relation (15) and $S_n(t,x)$ be defined by relation (8). Then the following formula holds*

$$\int_a^b f(t)\mathrm{d}t = D(f,x) + T_n(f,x) + (-1)^n\int_a^b S_n(t,x)f^{(n)}\mathrm{d}t, \tag{16}$$

where

$$D(f,x) = \frac{(b-a)^3}{6(2x-a-b)^2}\big(f(x)+f(a+b-x)\big) + \Big(b-a-\frac{(b-a)^3}{3(2x-a-b)^2}\Big) f\Big(\frac{a+b}{2}\Big), \tag{17}$$

and

$$\begin{aligned} T_n(f,x) =\; & \sum_{k=5}^{n} A_k(x)\Big(f^{(k-1)}(x) + (-1)^{k-1}f^{(k-1)}(a+b-x)\Big) \\ & + \sum_{\substack{k=5\\ odd}}^{n} B_k(x) f^{(k-1)}\Big(\frac{a+b}{2}\Big). \end{aligned} \tag{18}$$

Proof. The proof follows from the Theorem 4 for the polynomial $Q_{n,x}(t)$ defined by relation (15). □

LEMMA 2. *For $x \in [\frac{5a+b}{6}, \frac{a+b}{2})$ we have $Q_{n,x}(t) > 0$, for $t \in (x, \frac{a+b}{2})$, when $n \geqslant 3$. Further, we have $Q_{n,a}(t) \leqslant 0$, for $t \in (a, \frac{a+b}{2})$, when $n \geqslant 3$.*

Proof. We use mathematical induction by n. For $n = 3$ we have

$$Q_{3,x}(t) = (t-a)^3 - \frac{(b-a)^3}{2(2x-a-b)^2}(t-x)^2.$$

The zeros are $t_1 = \frac{a+b}{2}$,

$$t_{2,3} = \frac{a+b}{2} + \frac{b-a}{4(2x-a-b)^2} \cdot [(b-a)^2 - 3(2x-a-b)^2 \pm \sqrt{-3(2x-a-b)^4 - 8(b-a)(2x-a-b)^3 - 6(b-a)^2(2x-a-b)^2 + (b-a)^4}].$$

By some calculation, we check that $t_2 < x$, and for $x \in [\frac{5a+b}{6}, \frac{a+b}{2})$, $t_3 > \frac{a+b}{2}$, i.e. $t_{1,2,3} \notin (x, \frac{a+b}{2})$. So, $Q_{3,x}(t) > 0$, for $t \in (x, \frac{a+b}{2})$, since $Q_{3,x}(x) > 0$. For $n = 4$ we have $Q_{4,x}(x) > 0$. Since $Q_3(t,x) > 0$ on $(x, \frac{a+b}{2})$ and by remark 2 we conclude that $Q_{4,x}(t)$ is monotone increasing on $(x, \frac{a+b}{2})$, for $x \in [\frac{5a+b}{6}, \frac{a+b}{2})$ so $Q_{4,x}(t) > 0$. For $n > 5$ we know from the definition of the $Q_{n,x}$ that $Q_{n,x}(x) = 0$. Now, let us assume that $Q_{n,x}(t) > 0$ for some $n > 3$. Relation $Q'_{n+1,x}(t) = (n+1)Q_{n,x}(t) > 0$ implies that $Q_{n+1,x}$ is monotone increasing on $((x, \frac{a+b}{2})$. So, since $Q_{n+1,x}(x) = 0$, we conclude $Q_{n+1,x}(t) > 0$, for $t \in (x, \frac{a+b}{2})$, when $x \in [\frac{5a+b}{6}, \frac{a+b}{2})$.

For the case $x = a$ we have

$$Q_{n,a}(t) = (t-a)^n - \frac{n(b-a)}{6}(t-a)^{n-1} = (t-a)^{n-1}\left[t - a - \frac{n(b-a)}{6}\right],$$

so obviously for $n \geqslant 3$ we have $Q_{n,a}(t) \leqslant 0$, when $t \in (a, \frac{a+b}{2})$. □

THEOREM 6. *For $x \in \{a\} \cup [\frac{5a+b}{6}, \frac{a+b}{2})$ and $f^{(2n)}$ continuous function on $[a,b]$ for some $n \geqslant 2$, we have*

$$\int_a^b f(t)\mathrm{d}t = D(f,x) + T_{2n}(f,x) + f^{(2n)}(\eta) \cdot C(2n,\infty,x), \quad \textit{for some } \eta \in (a,b),$$

where $D(f,x)$, $T_{2n}(f,x)$ and $C(2n,\infty,x)$ are defined by relations (17), (18) and (11) respectively.

Proof. The proof follows from the integral mean value theorem. □

Specially, for $n = 2$ we have

$$\int_a^b f(t)\mathrm{d}t = D(f,x) + \frac{(b-a)^3}{384}\left[\frac{(b-a)^2}{5} - \frac{(a+b-2x)^2}{3}\right] f^{(4)}(\eta),$$

so for $x = a, \frac{5a+b}{6}, \frac{3a+b}{4}, \frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}$ we get the Simpson's, Maclaurin's, dual Simpson's nad Gauss-Legendre's two-point quadrature formula, respectively.

3. Ostrowski type inequalities for threepoint formula

Let us denote

$$Tf(a,x,a+b-x,b)=\frac{1}{b-a}\left(\int_a^b f(t)dt-D(f,x)-T_n(f,x)\right).$$

At first,we shall give the upper bound for $Tf(a,x,a+b-x,b)$ for functions f such that $f^{(n-1)}\in D(c_0)$.

COROLLARY 2. *Let $f:[a,b]\to\mathbb{R}$ be a function such that $f^{(n-1)}\in D(c_0)$, for some $n\in\mathbb{N}$ and*

$$M_l=\sup_{t\in\langle a,c_0\rangle}\left|f^{(n)}(t)\right|<\infty \quad and \;\; M_r=\sup_{t\in\langle c_0,b\rangle}\left|f^{(n)}(t)\right|<\infty.$$

Then the following inequality holds:

$$\begin{aligned}&|Tf(a,x,a+b-x,b)| \\ &\leqslant\begin{cases}\left[M_l^p(c_0-a)+M_r^p(b-c_0)\right]^{1/p}\cdot\frac{2^{1/q}}{n!}\left[\frac{(x-a)^{nq+1}}{nq+1}+\int_x^{\frac{a+b}{2}}|Q_{n,x}(t)|^q\,\mathrm{d}t\right]^{\frac{1}{q}}, \\ \frac{1}{p}+\frac{1}{q}=1,\quad 1<p<\infty. \\ \max\{M_l,M_r\}\cdot\frac{2}{n!}\left[\frac{(x-a)^{n+1}}{n+1}+\int_x^{\frac{a+b}{2}}|Q_{n,x}(t)|\,\mathrm{d}t\right], \\ p=\infty,\ q=1. \\ \left[M_l(c_0-a)+M_r(b-c_0)\right]\cdot\frac{1}{n!}\max\left\{(x-a)^n,\sup_{t\in[x,\frac{a+b}{2}]}|Q_{n,x}(t)|\right\} \\ p=1,\ q=\infty.\end{cases}\end{aligned} \tag{19}$$

Proof. The proof follows from Theorem 1 and Theorem 5. □

Now we shall give the upper bound for

$$Tf(a,x,a+b-x,b)-\frac{(-1)^n}{b-a}\int_a^b S_n(t,x)dt\cdot\int_a^b f^{(n)}(t)dt. \tag{20}$$

Let η denote mean

$$\eta=\frac{1}{b-a}\int_a^b S_n(t,x)dt=\frac{2}{n!\,(b-a)}\left(\frac{(x-a)^{n+1}}{n+1}+\int_x^{\frac{a+b}{2}}Q_{n,x}(t)\mathrm{d}t\right).$$

COROLLARY 3. *Let $f:[a,b]\to\mathbb{R}$ be a function such that $f^{(n)}\in D(c_0)$, for some $n\in\mathbb{N}$ and*

$$M_l=\sup_{t\in\langle a,c_0\rangle}\left|f^{(n+1)}(t)\right|<\infty \quad and \;\; M_r=\sup_{t\in\langle c_0,b\rangle}\left|f^{(n+1)}(t)\right|<\infty.$$

Assume $1 \leqslant p,q \leqslant \infty$ *are conjugate exponents* ($\frac{1}{p}+\frac{1}{q}=1$) *and* $S_n(\cdot,x)$ *defined by* (8). *Then the following inequality holds:*

$$\left|Tf(a,x,a+b-x,b)-(-1)^n\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\int_a^b S_n(t,x)dt\right|$$
$$\leqslant \begin{cases} \frac{1}{(p+1)^{1/p}}\left[M_l^p(c_0-a)^{p+1}+M_r^p(b-c_0)^{p+1}\right]^{1/p}\cdot\|S_n(\cdot,x)-\eta\|_{q,[a,b]}, & 1\leqslant p<\infty \\ \max\{M_l(c_0-a),M_r(b-c_0)\}\cdot\|S_n(\cdot,x)-\eta\|_{1,[a,b]}, & p=\infty,\ q=1. \end{cases}$$

Proof. The proof follows from Theorem 2 ,Theorem 5 and Lema 1. □

Now, we shall consider the upper bound for

$$Tf(a,x,a+b-x,b)-(-1)^n f^{(n)}(c_0)\int_a^b S_n(t,x)dt. \tag{21}$$

COROLLARY 4. *Let* $f:[a,b]\to\mathbb{R}$ *be a function such that* $f^{(n)}\in D(c_0)$, *for some* $n\in\mathbb{N}$ *and*

$$M_l=\sup_{t\in\langle a,c_0\rangle}\left|f^{(n+1)}(t)\right|<\infty \quad \text{and} \quad M_r=\sup_{t\in\langle c_0,b\rangle}\left|f^{(n+1)}(t)\right|<\infty.$$

Assume $1 \leqslant p,q \leqslant \infty$ *are conjugate exponents* ($\frac{1}{p}+\frac{1}{q}=1$) *and* $S_n(\cdot,x)$ *defined by* (8). *Then the following inequality holds:*

$$\left|Tf(a,x,a+b-x,b)-(-1)^n f^{(n)}(c_0)\int_a^b S_n(t,x)dt\right| \tag{22}$$
$$\leqslant \begin{cases} \frac{1}{(p+1)^{1/p}}\left[M_l^p(c_0-a)^{p+1}+M_r^p(b-c_0)^{p+1}\right]^{1/p}\cdot\frac{2^{1/q}}{n!}\left[\frac{(x-a)^{nq+1}}{nq+1}+\int_x^{\frac{a+b}{2}}|Q_{n,x}(t)|^q\,\mathrm{d}t\right], \\ 1<p<\infty. \\ \frac{1}{2}\left[M_l(c_0-a)^2+M_r(b-c_0)^2\right]\cdot\frac{1}{n!}\max\left\{(x-a)^n,\sup_{t\in[x,\frac{a+b}{2}]}|Q_{n,x}(t)|\right\}, \\ p=1,q=\infty. \\ \max\{M_l(c_0-a),M_r(b-c_0)\}\cdot\frac{2}{n!}\left[\frac{(x-a)^{n+1}}{n+1}+\int_x^{\frac{a+b}{2}}|Q_{n,x}(t)|\,\mathrm{d}t\right], \\ p=\infty,q=1. \end{cases}$$

Proof. The proof follows from Theorem 3 and Theorem 5. □

Now let us see some special cases:

Case 1. $x=a$, $n=1$

For this case we get

$$Q_{1,a}(t)=t-a-\frac{b-a}{6}$$

and

$$2^{1/q}\left[\int_a^{\frac{a+b}{2}}|Q_{1,a}(t)|^q\,\mathrm{d}t\right]^{1/q}=\frac{b-a}{6}\left[\frac{(b-a)}{3(q+1)}\left(2^{q+1}-(-1)^{q+1}\right)\right]^{1/q}.$$

By Corollary 2, we get

$$\left|\frac{1}{b-a}\left[\int_a^b f(t)\,dt-\frac{b-a}{6}(f(a)+f(b))-\frac{2}{3}(b-a)f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leqslant\begin{cases}\left[M_l^p(c_0-a)+M_r^p(b-c_0)\right]^{1/p}\cdot\frac{b-a}{6}\left[\frac{(b-a)}{3(q+1)}\left(2^{q+1}-(-1)^{q+1}\right)\right]^{1/q},\\ \frac{1}{p}+\frac{1}{q}=1,\ 1<p<\infty.\\ \frac{(b-a)^2}{12}\max\{M_l,M_r\},\ \ p=\infty,\ q=1.\\ \frac{b-a}{6}\cdot[M_l(c_0-a)+M_r(b-c_0)],\ \ p=1,\ q=\infty\end{cases}$$

where

$$M_l=\sup_{t\in\langle a,c_0\rangle}\left|f'(t)\right|<\infty\quad\text{and}\quad M_r=\sup_{t\in\langle c_0,b\rangle}\left|f'(t)\right|<\infty$$

and $f\in D(c_0)$.

Case 2. $x=\frac{3a+b}{4}$, $n=1$

For this case we get

$$Q_{1,\frac{3a+b}{4}}(t)=t-a-\frac{2(b-a)}{3}$$

and

$$2^{1/q}\left[\frac{(x-a)^{q+1}}{q+1}+\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left|Q_{1,\frac{3a+b}{4}}(t)\right|^q\mathrm{d}t\right]^{\frac{1}{q}}=\frac{b-a}{2}\left[\frac{(b-a)}{(q+1)}\left(\frac{1}{2^{q+1}}+\frac{5^{q+1}}{6^{q+1}}-\frac{1}{3^{q+1}}\right)\right]^{1/q}.$$

By Corollary 2, we get

$$\left|\frac{1}{b-a}\left[\int_a^b f(t)\,dt-\frac{2(b-a)}{3}\left(f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right)-\frac{b-a}{3}f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leqslant\begin{cases}\left[M_l^p(c_0-a)+M_r^p(b-c_0)\right]^{1/p}\cdot\frac{b-a}{2}\left[\frac{(b-a)}{(q+1)}\left(\frac{1}{2^{q+1}}+\frac{5^{q+1}}{6^{q+1}}-\frac{1}{3^{q+1}}\right)\right]^{1/q},\\ \frac{1}{p}+\frac{1}{q}=1,\ \ 1<p<\infty\\ \frac{5(b-a)^2}{24}\max\{M_l,M_r\},\ \ p=\infty,\ q=1.\\ \frac{5(b-a)}{12}\cdot[M_l(c_0-a)+M_r(b-c_0)],\ \ p=1,\ q=\infty\end{cases}$$

where

$$M_l=\sup_{t\in\langle a,c_0\rangle}\left|f'(t)\right|<\infty\quad\text{and}\quad M_r=\sup_{t\in\langle c_0,b\rangle}\left|f'(t)\right|<\infty$$

and $f\in D(c_0)$.

Case 3. $x = \frac{5a+b}{6}$, $n = 1$

For this case we get

$$Q_{1,\frac{5a+b}{6}}(t) = t - a - \frac{2(b-a)}{3}$$

and

$$2^{1/q}\left[\frac{(x-a)^{q+1}}{q+1} + \int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} \left|Q_{1,\frac{5a+b}{6}}(t)\right|^q \mathrm{d}t\right]^{\frac{1}{q}} = \frac{b-a}{2}\left[\frac{(b-a)}{(q+1)}\left(\frac{1}{3^{q+1}} + \frac{5^{q+1}}{12^{q+1}} + \frac{1}{8^{q+1}}\right)\right]^{1/q}.$$

By Corollary 2, we get

$$\left|\frac{1}{b-a}\left[\int_a^b f(t)\,dt - \frac{3(b-a)}{8}\left(f\left(\frac{5a+b}{6}\right) + f\left(\frac{a+5b}{6}\right)\right) - \frac{b-a}{4} f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leqslant \begin{cases} \left[M_l^p(c_0-a) + M_r^p(b-c_0)\right]^{1/p} \cdot \frac{b-a}{2}\left[\frac{(b-a)}{(q+1)}\left(\frac{1}{3^{q+1}} + \frac{5^{q+1}}{12^{q+1}} + \frac{1}{8^{q+1}}\right)\right]^{1/q}, \\ \frac{1}{p} + \frac{1}{q} = 1,\ 1 < p < \infty. \\ \frac{25(b-a)^2}{288} \max\{M_l, M_r\}, \quad p = \infty,\ q = 1. \\ \frac{5(b-a)}{24} \cdot \left[M_l(c_0-a) + M_r(b-c_0)\right], \quad p = 1,\ q = \infty \end{cases}$$

where

$$M_l = \sup_{t \in \langle a, c_0\rangle} \left|f'(t)\right| < \infty \quad \text{and} \quad M_r = \sup_{t \in \langle c_0, b\rangle} \left|f'(t)\right| < \infty$$

and $f \in D(c_0)$.

4. Some examples of the three-point integral formula

In this section we will apply results from the previous chapter to some special cases of $x \in [a, \frac{a+b}{2}]$.

4.1. $x = a$

For this case we get the generalization of the famous Simpson's formula. Using Theorem 4 we get $A_1(a) = \frac{b-a}{6}$ and $A_k(a) = 0$ for $k > 1$, $B_k(a) = \frac{(b-a)^k}{2^{k-1}(k-1)!}\left[\frac{1}{k} - \frac{1}{3}\right]$ for odd k and $B_{2k}(a) = 0$, so the generalization of the Simpson's formula states

$$\int_a^b f(t)\mathrm{d}t = \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right) + \sum_{\substack{k=5 \\ odd}}^{n} B_k(a) f^{(k-1)}\left(\frac{a+b}{2}\right) + (-1)^n \int_a^b S_n(t,a) f^{(n)}(t)\mathrm{d}t. \tag{23}$$

For $f : [a,b] \to \mathbf{R}$ such that $f^{(4)}$ is continuous, we get the well-known Simpson rule (1).

REMARK 5. This formula and related inequalities were obtained in [10] and [13].

4.2. $x = \frac{3a+b}{4}$

For this case we get the generalization of the dual Simpson's formula. Function $S_n(t, \frac{3a+b}{4})$ is determined by (8) and polynomial

$$Q_{n,\frac{3a+b}{4}}(t) := \left(t - \frac{3a+b}{4}\right)^n - \frac{5n(b-a)}{12}\left(t - \frac{3a+b}{4}\right)^{n-1} + \binom{n}{2}\frac{(b-a)^2}{4^2}\left(t - \frac{3a+b}{4}\right)^{n-2} + \binom{n}{3}\frac{(b-a)^3}{4^3}\left(t - \frac{3a+b}{4}\right)^{n-3} + \binom{n}{4}\frac{(b-a)^4}{4^4}\left(t - \frac{3a+b}{4}\right)^{n-4}, \quad t \in \left[\frac{3a+b}{4}, \frac{a+b}{2}\right]. \tag{24}$$

Further, from Theorem 4 we have $A_1(\frac{3a+b}{4}) = \frac{2(b-a)}{3}$, $A_k(\frac{3a+b}{4}) = 0$, for $k = 2,3,4$ and $A_k(\frac{3a+b}{4}) = \frac{(-1)^{k-1}(b-a)^k}{4^k k!}$, for $k \geqslant 5$. $B_1(\frac{5a+b}{6}) = \frac{b-a}{4}$, $B_k(\frac{5a+b}{6}) = 0$, for $k = 2,3,4$ $B_k(\frac{3a+b}{4}) = \frac{(b-a)^k}{2^{2k-1}k!}\left[1 - \frac{5k}{3} + \binom{k}{2} + \binom{k}{3} + \binom{k}{4}\right]$, for odd $k \geqslant 5$, and $B_{2k}(\frac{3a+b}{4}) = 0$. For $f : [a,b] \to \mathbf{R}$ with a piecewise continuous n-th derivative we have by Corollary 1

$$\int_a^b f(t)\mathrm{d}t = D\left(f, \frac{3a+b}{4}\right) + T_n\left(f, \frac{3a+b}{4}\right) + (-1)^n \int_a^b S_n\left(t, \frac{3a+b}{4}\right) f^{(n)}(t)\mathrm{d}t \tag{25}$$

where

$$D\left(f, \frac{3a+b}{4}\right) = \frac{b-a}{3}\left(2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right)\right).$$

Further, if $f^{(n)} \in L_p[a,b]$, then the following inequality holds:

$$\left|\int_a^b f(t)\mathrm{d}t - D\left(f, \frac{3a+b}{4}\right) - T_n\left(f, \frac{3a+b}{4}\right)\right| \leqslant C\left(n, p, \frac{3a+b}{4}\right) \cdot \|f^{(n)}\|_p.$$

Specially,

$$C\left(1, \infty, \frac{3a+b}{4}\right) = \frac{5(b-a)^2}{24}, \quad C\left(1, 1, \frac{3a+b}{4}\right) = \frac{5(b-a)}{12}$$

$$C\left(2, \infty, \frac{3a+b}{4}\right) = \frac{5(b-a)^3}{324}, \quad C\left(2, 1, \frac{3a+b}{4}\right) = \frac{(b-a)^2}{24}$$

$$C\left(3, \infty, \frac{3a+b}{4}\right) = \frac{(b-a)^4}{576}, \quad C\left(3, 1, \frac{3a+b}{4}\right) = \frac{5(b-a)^3}{1296}$$

$$C\left(4, \infty, \frac{3a+b}{4}\right) = \frac{7(b-a)^5}{23040}, \quad C\left(4, 1, \frac{3a+b}{4}\right) = \frac{(b-a)^4}{1152}.$$

REMARK 6. The same constants were obtained in [1].

If $f:[a,b]\to\mathbf{R}$ is such that $f^{(2n)}$ is continuous for some $n\in\mathbf{N}$, then we have $\int_a^b f(t)\mathrm{d}t = D(f,\frac{3a+b}{4}) + T_{2n}(f,\frac{3a+b}{4}) + C(2n,\infty,\frac{3a+b}{4})f^{(2n)}(\eta)$, for some

$$\eta\in(a,b). \tag{26}$$

Specially, for $n=2$ we get (2).

4.3. $x=\frac{5a+b}{6}$

For this case we get the generalization of the Maclaurin formula. Function $S_n(t,\frac{5a+b}{6})$ is determined by (8) and polynomial

$$\begin{aligned} Q_n\Big(t,\frac{5a+b}{6}\Big) := {} & \Big(t-\frac{5a+b}{6}\Big)^n - \frac{5n(b-a)}{24}\Big(t-\frac{5a+b}{6}\Big)^{n-1} \\ & + \binom{n}{2}\frac{(b-a)^2}{6^2}\Big(t-\frac{5a+b}{6}\Big)^{n-2} + \binom{n}{3}\frac{(b-a)^3}{6^3}\Big(t-\frac{5a+b}{6}\Big)^{n-3} \\ & + \binom{n}{4}\frac{(b-a)^4}{6^4}\Big(t-\frac{5a+b}{6}\Big)^{n-4}, \quad t\in\Big[\frac{5a+b}{6},\frac{a+b}{2}\Big]. \end{aligned} \tag{27}$$

Further, from Theorem 4 we have $A_1(\frac{5a+b}{6}) = \frac{3(b-a)}{8}$, $A_k(\frac{5a+b}{6}) = 0$, for $k=2,3,4$ and $A_k(\frac{5a+b}{6}) = \frac{(-1)^{k-1}(b-a)^k}{6^k k!}$, for $k\geqslant 5$. Further,

$$B_k\Big(\frac{5a+b}{6}\Big) = \frac{2(b-a)^k}{3^k k!}\left[1-\frac{5k}{8}+\frac{k(k-1)}{8}+\frac{k(k-1)(k-2)}{48}+\frac{k(k-1)(k-2)(k-3)}{384}\right],$$

for odd k, and $B_{2k}(\frac{5a+b}{6}) = 0$. For $f:[a,b]\to\mathbf{R}$ with a piecewise continuous n-th derivative we have by Corollary 1

$$\int_a^b f(t)\mathrm{d}t = D\Big(f,\frac{5a+b}{6}\Big) + T_n\Big(f,\frac{5a+b}{6}\Big) + (-1)^n\int_a^b S_n\Big(t,\frac{5a+b}{6}\Big)f^{(n)}(t)\mathrm{d}t, \tag{28}$$

where

$$D\Big(f,\frac{5a+b}{6}\Big) = \frac{(b-a)}{8}\left(3f\Big(\frac{5a+b}{6}\Big)+2f\Big(\frac{a+b}{2}\Big)+3f\Big(\frac{a+5b}{6}\Big)\right).$$

Further, if $f^{(n)}\in L_p[a,b]$, then the following inequality holds:

$$\Big|\int_a^b f(t)\mathrm{d}t - D\Big(f,\frac{5a+b}{6}\Big) - T_n\Big(f,\frac{5a+b}{6}\Big)\Big| \leqslant C\Big(n,p,\frac{5a+b}{6}\Big)\cdot\|f^{(n)}\|_p.$$

Specially,

$$C\Big(1,\infty,\frac{5a+b}{6}\Big) = \frac{25(b-a)^2}{288}, \quad C\Big(1,1,\frac{5a+b}{6}\Big) = \frac{5(b-a)}{24}$$

$$C\Big(2,\infty,\frac{5a+b}{6}\Big) = \frac{(b-a)^3}{192}, \quad C\Big(2,1,\frac{5a+b}{6}\Big) = \frac{(b-a)^2}{72}$$

$$C\left(3,\infty,\frac{5a+b}{6}\right)=\frac{(b-a)^4}{1728},\quad C\left(3,1,\frac{5a+b}{6}\right)=\frac{(b-a)^3}{768}$$
$$C\left(4,\infty,\frac{5a+b}{6}\right)=\frac{7(b-a)^5}{51840},\quad C\left(4,1,\frac{5a+b}{6}\right)=\frac{(b-a)^4}{3456}.$$

REMARK 7. The same constants are also obtained in [2].

If $f:[a,b]\to\mathbf{R}$ is such that $f^{(2n)}$ is continuous for some $n\in\mathbf{N}$, then we have $\int_a^b f(t)\mathrm{d}t = D\left(f,\frac{5a+6}{b}\right)+T_{2n}\left(f,\frac{5a+b}{6}\right)+C\left(2n,\infty,\frac{5a+b}{6}\right)f^{(2n)}(\eta),$ for some

$$\eta\in(a,b). \tag{29}$$

Specially, for $n=2$ we get (3).

4.4. Legendre-Gauss two-point formula

We consider case where the term $f(\frac{a+b}{2})$ doesn't appear. If we put in relation (9) condition $B_1(x)=0$, then we get $A_1(x_G)=\frac{b-a}{2}$ and $x_G=\frac{a+b}{2}-\frac{b-a}{2\sqrt{3}}$. Function $S_n(t,x_G)$ is determined by (8) and polynomial

$$\begin{aligned} Q_{n,x_G}(t) := {} & (t-x_G)^n - \frac{n(b-a)}{2\sqrt{3}}(t-x_G)^{n-1} \\ & + \binom{n}{2}\frac{(b-a)^2(2-\sqrt{3})}{6}(t-x_G)^{n-2} \\ & + \binom{n}{3}\left(\frac{(b-a)}{2\sqrt{3}}\right)^3(6\sqrt{3}-10)(t-x_G)^{n-3} \\ & + \binom{n}{4}\left(\frac{(b-a)}{2\sqrt{3}}\right)^4(28-16\sqrt{3})(t-x_G)^{n-4}, \quad t\in[x_G,\frac{a+b}{2}]. \end{aligned} \tag{30}$$

Further, from Theorem 4 we have $A_k(x_G)=0$, for $k=2,3,4$ and $A_k(x_G)=\frac{(b-a)^k}{2^k k!}(1-\frac{1}{\sqrt{3}})^k$, for $k\geqslant 5$. Further,

$$B_k(x_G)=\frac{2(b-a)^k}{(2\sqrt{3})^k k!}\left[1-k+\binom{k}{2}(4-2\sqrt{3})+\binom{k}{3}(6\sqrt{3}-10)+\binom{k}{4}(28-16\sqrt{3})\right],$$

for odd $k\geqslant 5$, and $B_k(x_G)=0$ otherwise. For $f:[a,b]\to\mathbf{R}$ with a piecewise continuous n-th derivative, we have by Corollary 1 the following formula

$$\int_a^b f(t)\mathrm{d}t = D(f,x_G)+T_n(f,x_G)+(-1)^n\int_a^b S_n(t,x_G)f^{(n)}(t)\mathrm{d}t, \tag{31}$$

where

$$D(f,x_G)=\frac{b-a}{2}\left(f(x_G)+f(a+b-x_G)\right).$$

Further, if $f^{(n)} \in L_p[a,b]$, then the following inequality holds:

$$\left| \int_a^b f(t)\mathrm{d}t - D(f,x_G) - T_n(f,x_G) \right| \leqslant C(n,p,x_{G2}) \cdot \|f^{(n)}\|_p.$$

Specially,

$$C(1,\infty,x_G) = \frac{(5-2\sqrt{3})(b-a)^2}{12}, \quad C(1,1,x_G) = \frac{(3-\sqrt{3})(b-a)}{6}$$

$$C(2,\infty,x_G) = \frac{\sqrt{26\sqrt{3}-45}(b-a)^3}{18}, \quad C(2,1,x_G) = \frac{(2-\sqrt{3})(b-a)^2}{12}$$

$$C(3,\infty,x_G) = \frac{(9-4\sqrt{3})(b-a)^4}{1728}, \quad C(3,1,x_G) = \frac{(2-\sqrt{3})\sqrt{2\sqrt{3}-3}(b-a)^3}{72}$$

$$C(4,\infty,x_G) = \frac{(b-a)^5}{4320}, \quad C(4,1,x_G) = \frac{(9-4\sqrt{3})(b-a)^4}{3456}.$$

If $f:[a,b] \to \mathbf{R}$ is such that $f^{(2n)}$ is continuous for some $n \in \mathbf{N}$, then we have

$$\int_a^b f(t)\mathrm{d}t = D(f,x_G) + T_{2n}(f,x_G) + C(2n,\infty,x_G)f^{(2n)}(\eta), \quad \text{for some } \eta \in (a,b). \tag{32}$$

Specially, for $n=2$ we get (5).

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REFERENCES

[1] LJ. DEDIĆ, M. MATIĆ, J. PEČARIĆ, *On dual Euler-Simpson formulae*, Bull. Belg. Math. Soc. **l8** (2001), 479–504.

[2] LJ. DEDIĆ, M. MATIĆ, J. PEČARIĆ, *Euler-Maclaurin formulae*, Mathematical Inequalities and Applications **6**, 2 2003, 247–275.

[3] P. J. DAVIS, P. RABINOWITZ, *Methods of numerical integration*, Academic Press, New York-San Francisco-London, 1975.

[4] A. GUESSAB, G. SCHMEISSER, *Sharp integral inequalities of the Hermite-Hadamard type*, Journal of Approximation Theory **115**, 2 (2002), 260–288.

[5] S. KOVAČ, J. PEČARIĆ, S. TIPURIĆ-SPUŽEVIĆ, *Weighted Ostrowski Type Inequalities with Application to Onepoint Integral Formula*, Mediterranean Journal of Mathematics **11** (2014), 13–30.

[6] S. KOVAČ, J. PEČARIĆ, *Weighted version of general integral formula*, Mathematical Inequalities& Applications, **13**, 3 (2010), 579–599.

[7] S. KOVAČ, J. PEČARIĆ, *Generalization of an Integral Formula of Guessab and Schmeisser*, Banach Journal of Mathematical Analysis **5**, 1 (2011), 1–18.

[8] S. KOVAČ, J. PEČARIĆ, *Weighted version of general integral formula of the Euler type*, to appear in Math. Inequal. Appl.

[9] S. KOVAČ, J. PEČARIĆ, A. VUKELIĆ, *A generalization of general two-point formula with applications in numerical integration*, Nonlinear Analysis: Theory, Methods & Applications **68** (2008), 2445–2463.

[10] Z. LIU, *An inequality of Simpson type*, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences **461** (2005), 2155–2158.

[11] D. S. MITRINOVIĆ, J. E. PEČARIĆ, AND A. M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.
[12] M. NIEZGODA, *Grüss and Ostrowski Type Inequalities*, Applied Mathematics and Computation **217** 23 (2011), 9779–9789.
[13] J. PEČARIĆ, S. VAROŠANEC, *Harmonic Polynomials and Generalization of Ostrowski Inequality with Applications in Numerical Integration*, Nonlinear Analysis: Theory, Methods & Applications **47** (2001), 2365–2374.

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Sanja Kovač
Faculty of Geotehnical Engineering
University of Zagreb
Hallerova aleja 7, 42000 Varaždin, Croatia
e-mail: sanja.kovac@gfv.hr

Josip Pečarić
RUDN University
Miklukho-Maklaya str. 6, 117 198 Moscow, Russia
e-mail: pecaric@hazu.hr

Sanja Tipurić-Spužević
Faculty of Science and Education
University of Mostar
Matice hrvatske bb, 88 000 Mostar, Bosnia and Herzegovina
e-mail: sanja.spuzevic@gmail.com