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LANDAU–KOLMOGOROV TYPE INEQUALITIES FOR CURVES ON RIEMANNIAN MANIFOLDS

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Abstract. We obtain Landau-Kolmogorov type inequalities for mappings defined on the whole real axis and taking values in Riemannian manifolds. In terms of an auxiliary convex function, we find conditions under which the boundedness of covariant derivative along the curve under consideration ensures the boundedness of the corresponding tangent vector field. We use the square of the distance function as the auxiliary one to establish counterparts of the Landau – Hadamard and the Landau-Kolmogorov inequalities where the norms of higher order derivatives of mapping are replaced, respectively, by the Chebyshev radius of curve and the corresponding iterates of covariant derivative along the curve.

1. Introducton

Let $I \subseteq \mathbb{R}$ be an interval and let for some $n \in \mathbb{N}$ a function $f(\cdot) \in \mathrm{C}^n(I \mapsto \mathbb{R})$ satisfies the inequalities

$$\|f(\cdot)\|_{\infty,I} := \sup_{t\in I} |f(t)| < \infty, \qquad \left\|f^{(n)}(\cdot)\right\|_{\infty,I} < \infty.$$

There is a vast literature concerning inequalities between $\left\|f^{(k)}(\cdot)\right\|_{I,\infty}$ $(1 \leqslant k < n)$, $\|f(\cdot)\|_{I,\infty}$ and $\left\|f^{(n)}(\cdot)\right\|_{I,\infty}$ (see, e.g., [6, 15] and references therein). The classical Landau – Hadamard inequality reads

$$\left\|f'(\cdot)\right\|_{\infty,I} \leqslant C_{2,1}(I)\sqrt{\|f(\cdot)\|_{\infty,I}\,\|f''(\cdot)\|_{\infty,I}}$$

with the best possible constants $C_{2,1}(\mathbb{R}_+) = 2$ [14] and $C_{2,1}(\mathbb{R}) = \sqrt{2}$ [8]. A. N. Kolmogorov [12] determined the best constants $C_{n,k}(\mathbb{R})$ for inequalities

$$\left\|f^{(k)}(\cdot)\right\|_{\infty,I} \leqslant C_{n,k}(I)\,\|f(\cdot)\|_{\infty,I}^{1-k/n}\left\|f^{(n)}(\cdot)\right\|_{\infty,I}^{k/n}.$$

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Further results on Landau-Kolmogorov type inequalities and their generalizations, including inequalities for vector space valued functions, can be found in [1, 2, 3, 4, 5, 9, 11, 13, 18, 19, 20].

In the present paper, we aim to obtain Landau-Kolmogorov type inequalities for mappings taking values in Riemannian manifolds. In what follows, we deal only with the case where $I=\mathbb{R}$ and use the simplified notation $\|\cdot\|_\infty$ instead of $\|\cdot\|_{\infty,\mathbb{R}}$.

Let $(\mathfrak{M},\mathfrak{g}=\langle\cdot,\cdot\rangle)$ be a smooth complete Riemannian manifold with the metric tensor $\mathfrak{g}$, and let ∇ be the Levi-Civita connection with respect to $\mathfrak{g}$. We denote by $\rho(\cdot,\cdot):\mathfrak{M}\times\mathfrak{M}\mapsto\mathbb{R}_+$ the corresponding distance function, and by $\|\cdot\|$ the norm associated with the inner product $\langle\cdot,\cdot\rangle$ on tangent spaces $T_x\mathfrak{M}$, $x\in\mathfrak{M}$.

For a given smooth mapping $x(\cdot):I\mapsto\mathfrak{M}$ and for a smooth vector field $\xi(\cdot):I\mapsto T\mathfrak{M}$ along $x(\cdot)$, denote by $\nabla_{\dot{x}}\xi(t)$ the covariant derivative of $\xi(\cdot)$ along the tangent vector $\dot{x}(t)\in T_{x(t)}\mathfrak{M}$, $t\in I$, and by $\nabla_{\dot{x}}^k$ the k-th iterate of $\nabla_{\dot{x}}$. Here $T\mathfrak{M}=\bigsqcup_{x\in\mathfrak{M}}T_x\mathfrak{M}$ stands for the total space of the tangent bundle with natural projection $\pi(\cdot):T\mathfrak{M}\mapsto\mathfrak{M}$.

Now we ask the question: is it true that the boundedness of $x(\mathbb{R})$ and $\|\nabla_{\dot{x}}\dot{x}(\cdot)\|_\infty$ yields the boundedness of $\|\dot{x}(\cdot)\|_\infty$? Such a question naturally arises, e.g., in studying the existence problem for bounded solutions of Newtonian equation

$$\nabla_{\dot{x}}\dot{x}=F(x,\dot{x})$$

on Riemnnian manifold [16]. It turns out that generally the answer is negative.

To see this, consider the following example.

EXAMPLE 1. Let J be the symplectic unit operator in Euclidean space $\mathbb{E}^{2m}=\left(\mathbb{R}^{2m},\langle\cdot,\cdot\rangle\right)$:

$$\langle Jx,y\rangle=-\langle x,Jy\rangle,\quad J^2x=-x\quad\forall\{x,y\}\subset\mathbb{E}^{2m}.$$

For any unit vector $x_0\in\mathbb{E}^{2n}$ define the curve $\mathbb{R}\ni t\mapsto x(t):=\mathrm{e}^{t^2J}x_0$ lying on the unite sphere $\mathbb{S}^{2m-1}:=\left\{x\in\mathbb{E}^{2m}:\langle x,x\rangle=1\right\}$. The sphere $\mathbb{S}^{2m-1}$ is endowed with induced Riemannian metric and Levi-Civita connection ∇. Let $\imath:\mathbb{S}^{2m-1}\mapsto\mathbb{E}^{2m}$ be the natural isometric embedding. In what follows it will not lead to confusion if we write x and $\dot{x}$ instead of $\imath(x)$ and $\imath_*(\dot{x})$, respectively for any $x\in\mathbb{S}^{2m-1}$, $\dot{x}\in T_x\mathbb{S}^{2m-1}$. Since $\langle\dot{x}(t),x(t)\rangle\equiv0$ and $\langle\dot{x}(t),\dot{x}(t)\rangle+\langle\ddot{x}(t),x(t)\rangle\equiv0$, then

$$\nabla_{\dot{x}}\dot{x}(t)=\ddot{x}(t)-\langle\ddot{x}(t),x(t)\rangle x(t)=\ddot{x}(t)+\langle\dot{x}(t),\dot{x}(t)\rangle x(t).$$

But

$$\dot{x}(t)=2tJ\mathrm{e}^{t^2J}x_0,\quad \ddot{x}(t)=2J\mathrm{e}^{t^2J}x_0-4t^2\mathrm{e}^{t^2J}x_0,$$
$$\langle\dot{x}(t),\dot{x}(t)\rangle=4t^2\left\langle J\mathrm{e}^{t^2J}x_0,J\mathrm{e}^{t^2J}x_0\right\rangle=4t^2,\quad \nabla_{\dot{x}}\dot{x}(t)=2J\mathrm{e}^{t^2J}x_0.$$

Hence, $\|x(t)\|\equiv1$, $\|\nabla_{\dot{x}}\dot{x}(t)\|\equiv2$, and $\|\dot{x}(t)\|\equiv2|t|\to\infty$ when $|t|\to\infty$.

Thus, one must impose additional conditions to ensure the boundedness of $\|\dot{x}(\cdot)\|_\infty$. One of the main goals of the present paper is to obtain the inequality

$$\|\dot{x}(\cdot)\|_\infty \leqslant K\sqrt{\inf_{x\in\mathfrak{M}} \|\rho(x,x(\cdot))\|_\infty \|\nabla_{\dot{x}}x(\cdot)\|_\infty}$$

and find the constant $K>0$. It is not hard to see that in the case where $x(\mathbb{R})$ is bounded, the function $\sup_{t\in\mathbb{R}}\rho(\cdot,x(t)) : \mathfrak{M}\mapsto\mathbb{R}_+$ is lower semi-continuous and attains its minimum at least at one point $x_*\in\mathfrak{M}$ (Chebyshev center of the curve $x(\cdot)$). Hence,

$$\min_{x\in\mathbb{S}^n}\sup_{t\in\mathbb{R}}\rho(x,x(t)) = \sup_{t\in\mathbb{R}}\rho(x_*,x(t)) := R[x(\cdot)].$$

The number $R[x(\cdot)]$ is called the Chebyshev radius of the curve $x(\cdot)$.

2. Landau type inequality and convex functions

For a smooth function $U(\cdot):\mathfrak{M}\mapsto\mathbb{R}$ denote by $\nabla U(x)\in T_x\mathfrak{M}$ and by $H_U(x): T_x\mathfrak{M}\mapsto T_x\mathfrak{M}$, respectively, the gradient vector and the Hesse form of $U(\cdot)$ at point x.[1] We obtain the following estimate for $\dot{x}(\cdot)$ in terms of an auxiliary function $U(\cdot)$ and the covariant derivative $\nabla_{\dot{x}}x(\cdot)$ (see also [17]).

THEOREM 1. *Let $x(\cdot):\mathbb{R}\mapsto\mathfrak{M}$ be a smooth mapping such that*

$$r_2 := \|\nabla_{\dot{x}}\dot{x}(\cdot)\|_\infty < \infty.$$

Suppose that there exists a smooth function $U(\cdot):\mathfrak{M}\mapsto\mathbb{R}$ satisfying the inequalities

$$\sup_{t\in\mathbb{R}} U\circ x(t) < \infty, \quad 0 < r_0 := \|\nabla U\circ x(\cdot)\|_\infty < \infty,$$

and

$$\lambda := \inf_{t\in\mathbb{R}}\min\left\{\langle [H_U\circ x(t)]\,\xi,\xi\rangle : \xi\in T_{x(t)}\mathfrak{M},\ \|\xi\|=1\right\} > 0.$$

Then

$$r_1 := \|\dot{x}(\cdot)\|_\infty \leqslant C\sqrt{r_0 r_2/\lambda}$$

where the constant C does not exceed the positive root of the polynomial $\zeta^3-3\zeta-1$. In particular, $C<1.87939$.

Proof. Introduce the notations

$$u(t) := U\circ x(t), \quad v(t) := \dot{u}(t) \equiv \langle \nabla U\circ x(t), \dot{x}(t)\rangle.$$

[1] Recall that by the definition, $\langle H_U(x)\xi,\eta\rangle = \langle \nabla_\xi \nabla U(x),\eta\rangle$ for any $x\in\mathfrak{M}$ and any $\xi,\eta\in T_x\mathfrak{M}$).

Since $|v(t)| \leqslant r_0 \|\dot{x}(t)\|$, then

$$\begin{aligned}\dot{v}(t) &= \langle [H_U \circ x(t)]\dot{x}(t), \dot{x}(t)\rangle + \langle \nabla U \circ x(t), \nabla_{\dot{x}}\dot{x}(t)\rangle \\ &\geqslant \lambda \|\dot{x}(t)\|^2 - r_0 r_2 \geqslant \frac{\lambda}{r_0^2} v^2(t) - r_0 r_2.\end{aligned}$$

Let us show that $v^2(t) \leqslant r_0^3 r_2/\lambda$ for all $t \in \mathbb{R}$. In fact, if there exists t_0 such that $v(t_0) > \sqrt{r_0^3 r_2/\lambda}$, then $v(t)$ increases for $t \geqslant t_0$, and $\dot{v}(t) \geqslant \lambda v^2(t_0)/r_0^2 - r_0 r_2 > 0$. Thus $v(t) \to +\infty$ and we arrive at contradiction: $u(t) \to +\infty$ as $t \to +\infty$. Now suppose that there exists t_0 such that $v(t_0) < -\sqrt{r_0^3 r_2/\lambda}$. Then $v(t)$ increases for $t \leqslant t_0$, and we obtain

$$v(t) \leqslant v(t_0) < 0, \quad \dot{v}(t) \geqslant \lambda v^2(t_0)/r_0^2 - r_0 r_2 > 0 \quad \forall t \leqslant t_0.$$

This yields

$$\int_t^{t_0} \dot{v}(s)\mathrm{d}s \geqslant \left[\lambda v^2(t_0)/r_0^2 - r_0 r_2\right](t_0 - t)$$

and, as a consequence,

$$\begin{aligned}v(t) &\leqslant v(t_0) + \left[v^2(t_0)/r_0^2 - r_0 r_2\right](t - t_0) \to -\infty, \quad t \to -\infty, \\ u(t) &= u(t_0) - \int_t^{t_0} v(s)\mathrm{d}s \geqslant u(t_0) - v(t_0)(t_0 - t) \to +\infty, \quad t \to -\infty.\end{aligned}$$

We again arrive at contradiction.

Observe that if for some $\varepsilon > 0$ there exists a segment $[t_1, t_2]$ where $\|\dot{x}(t)\|^2 \geqslant (r_0 r_2 + \varepsilon)/\lambda$, then $\dot{v}(t) > \varepsilon$, and the inequality $v(t_2) \geqslant v(t_1) + \varepsilon(t_2 - t_1)$ yields

$$t_2 - t_1 \leqslant \frac{2\sqrt{r_0^3 r_2/\lambda}}{\varepsilon}.$$

Hence, for any $\varepsilon > 0$ and any $T > 0$ there exists $t_\varepsilon < -T$ such that $\|\dot{x}(t_\varepsilon)\|^2 < (r_0 r_2 + \varepsilon)/\lambda$. Now it remains to estimate $\|\dot{x}(t)\|$ on the segment $[t_1, t_2]$ such that $\|\dot{x}(t_i)\|^2 = (r_0 r_2 + \varepsilon)/\lambda$ and $\|\dot{x}(t)\|^2 > (r_0 r_2 + \varepsilon)/\lambda$ for all $t \in (t_1, t_2)$.

On account of

$$2\|\dot{x}(t)\| \left|\frac{\mathrm{d}\|\dot{x}(t)\|}{\mathrm{d}t}\right| = \left|\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{x}(t)\|^2\right| = 2|\langle \dot{x}(t), \nabla_{\dot{x}}\dot{x}(t)\rangle| \leqslant 2r_2\|\dot{x}(t)\|$$

on any interval where $\dot{x}(t) \neq 0$, we obtain

$$\left|\frac{\mathrm{d}\|\dot{x}(t)\|}{\mathrm{d}t}\right| \leqslant r_2 \quad \forall t \in (t_1, t_2).$$

Set $z(t) := \|\dot{x}(t)\|$, $z_\varepsilon := \sqrt{(r_0 r_2 + \varepsilon)/\lambda}$. Then

$$\dot{v}(t) \geqslant \lambda \left[\|\dot{x}(t)\|^2 - r_0 r_2/\lambda \right] \geqslant \lambda \left[z^2(t) - z_\varepsilon^2 \right],$$

and thus,

$$\left| \left[z^2(t) - z_\varepsilon^2 \right] \frac{\mathrm{d}z(t)}{\mathrm{d}t} \right| \leqslant r_2 \left[z^2(t) - z_\varepsilon^2 \right] \leqslant \frac{r_2}{\lambda} \dot{v}(t) \quad \forall t \in (t_1, t_2). \tag{1}$$

If we define

$$I(z) := \frac{z^3}{3} - z_\varepsilon^2 z + \frac{2z_\varepsilon^3}{3},$$

then one can rewrite (1) in the form

$$-\frac{r_2}{\lambda} \dot{v}(t) \leqslant \frac{\mathrm{d}}{\mathrm{d}t} I(z(t)) \leqslant \frac{r_2}{\lambda} \dot{v}(t) \quad \forall t \in (t_1, t_2).$$

From this it follows that

$$\begin{aligned} 2z_\varepsilon^3 \geqslant 2\sqrt{r_0^3 r_2^3/\lambda^3} &\geqslant \frac{r_2}{\lambda} \left[v(t_2) - v(t_1) \right] \\ &= \frac{r_2}{\lambda} \int_{t_1}^{t_2} \dot{v}(s) \mathrm{d}s = \frac{r_2}{\lambda} \int_{t_1}^{t} \dot{v}(s) \mathrm{d}s + \frac{r_2}{\lambda} \int_{t}^{t_2} \dot{v}(s) \mathrm{d}s \\ &= \int_{t_1}^{t} \frac{\mathrm{d}}{\mathrm{d}s} I(z(s)) \mathrm{d}s - \int_{t}^{t_2} \frac{\mathrm{d}}{\mathrm{d}s} I(z(s)) \mathrm{d}s = 2I(z(t)). \end{aligned}$$

Hence,

$$\frac{z^3(t)}{3} - z_\varepsilon^2 z(t) + \frac{2z_\varepsilon^3}{3} \leqslant z_\varepsilon^3 \quad \forall t \in (t_1, t_2).$$

Introducing the new variable $\zeta = z/z_\varepsilon$, we obtain

$$\zeta^3(t) - 3\zeta(t) - 1 \leqslant 0 \quad \forall t \in (t_1, t_2),$$

and finally, by letting ε tend to zero,

$$z(t) \leqslant C\sqrt{r_0 r_2/\lambda} \quad \forall t \in (t_1, t_2). \quad \square$$

REMARK 1. If $\mathfrak{M} = \mathbb{E}^d := (\mathbb{R}^d, \langle \cdot, \cdot \rangle)$ and $U(x) := \|x\|^2/2$, then $\lambda = 1$ and Theorem 1 leads to the Landau inequality with the constant C somewhat greater then the best one $C_{2,1} = \sqrt{2}$ obtained for the case $d = 1$ in [8]. At the same time, observe that $C < 2$ and does not depend on d. Thus, in the case of Hilbert space, our approach makes it possible to obtain the Landau inequality with somewhat better constant then in [1, 2].

3. Landau type inequality for curves on the unit sphere

Let $\mathfrak{M} := \mathbb{S}^d$, $d \geqslant 2$, and let $\iota : \mathbb{S}^d \mapsto \mathbb{E}^{d+1}$ be the natural isometric embedding. Consider a curve $x(\cdot) \in \mathrm{C}^2(\mathbb{R} \mapsto \mathbb{S}^n)$. Introduce the coordinates $(x_1, \ldots, x_{d+1})$ in $\mathbb{E}^{d+1}$ in such a way that $x_* := (1, 0, \ldots, 0, 0) =: e_1$ stands for a Chebyshev center of $x(\cdot)$. (Recall that we have agreed to identify $x \in \mathbb{S}^d$ and $\iota(x) \in \mathbb{E}^{d+1}$.)

Define $U(x) := \rho^2(x_*, x)/2$. It is not hard to see that $\rho(x_*, x) = \arccos x_1\big|_{\mathbb{S}^d}$. Observe that if $F(\cdot) \in \mathrm{C}^1\left(\mathbb{E}^{d+1} \mapsto \mathbb{R}\right)$ then the gradient of restriction $F(\cdot)\big|_{\mathbb{S}^d}$ at $x \in \mathbb{S}^d$ is

$$\nabla F(x) = F'(x) - \langle F'(x), x \rangle x, \quad \left(F'(x) := \left(F'_{x_1}(x), \ldots, F'_{x_{d+1}}(x)\right)\right).$$

Hence, we can identify $\nabla \rho(x_*, x)$ with $\left(1 - x_1^2\right)^{-1/2}(x_1 x - e_1)$, $x_1 \neq \pm 1$. Obviously, $\|\nabla \rho(x_*, x)\| = 1$. In the same way, we identify $\nabla U(x)$ with $\arccos x_1 \left(1 - x_1^2\right)^{-1/2} \cdot (x_1 x - e_1)$ and find $\|\nabla U(x)\| = \rho(x_*, x)$. Observe that $\arccos^2 x_1$ is analytic at $x_1 = 1$, and thus, $U(\cdot)$ is smooth on $\mathbb{S}^d \setminus \{-e_1\}$.

Next, in order to calculate λ from Theorem 1, observe that for any $\xi \in T_x\mathbb{S}^d$, $\|\xi\| = 1$, we have

$$\langle H_U(x)\xi, \xi \rangle = \left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} U \circ \gamma(t; \xi)$$

where $\gamma(\cdot; \xi) : [0, 1] \mapsto \mathbb{S}^d$ is the naturally parametrized geodesic such that $\gamma(0, \xi) = x$, $\dot{\gamma}(0, \xi) = \xi$. One can identify $\gamma(\cdot, \xi)$ with the solution of the initial problem

$$\ddot{x} + \langle \dot{x}, \dot{x} \rangle x = 0, \quad x(0) = x, \quad \dot{x}(0) = \xi.$$

Since $\|\dot{\gamma}(t; \xi)\| \equiv 1$, then

$$\langle H_U(x)\xi, \xi \rangle = \left\langle U''(x)\xi, \xi \right\rangle + \left\langle U'(x), \ddot{\gamma}(0, \xi) \right\rangle = \left\langle U''(x)\xi, \xi \right\rangle - \left\langle U'(x), x \right\rangle,$$

or

$$\langle H_U(x)\xi, \xi \rangle = \frac{\sqrt{1 - x_1^2} - \arccos(x_1)x_1}{(1 - x_1^2)^{3/2}} \xi_1^2 + \frac{\arccos(x_1)x_1}{\sqrt{1 - x_1^2}} \geqslant \frac{\arccos(x_1)x_1}{\sqrt{1 - x_1^2}}.$$

The function in the right hand side monotonically icreases from 0 to 1 on $[0, 1]$. The above reasoning together with Theorem 1 proves the following assertion.

THEOREM 2. *Let $r \in (0, \pi/2)$. Denote by $\mathfrak{S}_r$ the class of mappings $x(\cdot) \in \mathrm{C}^2\left(\mathbb{R} \mapsto \mathbb{S}^d\right)$ whose Chebyshev radius $R[x(\cdot)]$ does not exceed r. Then*

$$\|\dot{x}(\cdot)\|_\infty \leqslant \frac{C}{\sqrt{r \cot r}} \sqrt{R[x(\cdot)]\, \|\nabla_{\dot{x}} x(\cdot)\|_\infty} \quad \forall x(\cdot) \in \mathfrak{S}_r$$

where C is defined in Theorem 1.

REMARK 2. Basing on Example1 one can show that $\mathrm{C}^2\left(\mathbb{R}\mapsto\mathbb{S}^d\right)\setminus\bigcup_{r\in(0,\pi/2)}\mathfrak{S}_r$ contains mappings for which boundedness of $\nabla_{\dot{x}}\dot{x}(\cdot)$ does not ensures the boundedness of $\dot{x}(\cdot)$.

4. Main theorems

Let now $\mathfrak{M}$ be an arbitrary smooth complete Riemannian manifold of dimension $d\geqslant 2$. Basing on the results of Section 2, we intend to construct an appropriate auxiliary function $U(\cdot)$ by means of the square of distance function. For this purpose, we need to recall a few facts from Riemannian geometry (see, e.g., [7, 10] for details). For a fixed point $x_*\in\mathfrak{M}$, denote by $\mathrm{ir}(x_*)$ its injectivity radius. Define the set

$$\mathscr{B}(x_*):=\exp_{x_*}\left(\{\xi\in T_{x_*}\mathfrak{M}:\|\xi\|<\mathrm{ir}(x_*)\}\right)$$

and the diffeomorphism

$$h(\cdot):=\exp_{x_*}^{-1}(\cdot):\mathscr{B}(x_*)\mapsto\{\xi\in T_{x_*}\mathfrak{M}:\|\xi\|<\mathrm{ir}(x_*)\}.$$

Then

$$[0,1]\ni s\mapsto\exp_{x_*}(sh(x))=:g(s,x_*,x)$$

is the shortest geodesic connecting x_* with x, and

$$\|h(x)\|=\rho(x_*,x)=:\sigma(x).$$

Next, let $\xi\in T_x\mathfrak{M}$, $\|\xi\|=1$, and let $\gamma(\cdot;x):(-\varepsilon,\varepsilon)$ be the naturally paramatrized geodesic such that $\gamma(0;x)=x$, $\dot{\gamma}(0;x)=\xi$, where $\dot{\gamma}(t;x):=\frac{\partial}{\partial t}\gamma(t,x)$. If we define the two parameter family of vector fields along the mapping $g(\cdot,x_*,\gamma(\cdot;x))$ by

$$X(s,t;x):=\frac{\partial}{\partial s}g\left(s,x_*,\gamma(t;x)\right)$$

and the Jacobi vector field $Y(\cdot)$ along $g(\cdot,x_*,x)$ satisfying $Y(0)=0$, $Y(1)=\xi$ by

$$Y(s):=\frac{\partial}{\partial t}\bigg|_{t=0}g(s,x_*,\gamma(t;x)),$$

then the Hesse form for the function $U(\cdot):=\sigma^2(\cdot)/2$ at $x\neq x_*$ can be computed in the standard way:

$$\begin{aligned}\langle H_U(x)\xi,\xi\rangle&=\frac{1}{2}\frac{\partial^2}{\partial t^2}\bigg|_{t=0}\int_0^1\|X(s,t;x)\|^2\,\mathrm{d}t\\&=\left\langle\xi,\nabla_\xi X(1,t;x)\right\rangle\Big|_{t=0}=I[Y(\cdot),Y(\cdot)](x).\end{aligned}$$

Here I is the index form of geodesic g. Observe that if t is the natural parameter and $X(1,0;x) \parallel \xi$, then

$$\sigma(\gamma(t;x)) = \rho(x_*,\gamma(t;x)) = \rho(x_*,x) + \rho(x,\gamma(t;x)) = \sigma(x) + t.$$

Besides, by the Gauss lemma we have (see [7, Sect. 5.2, Rem. (ii)] for details)

$$\nabla\sigma(\gamma(t;x)) = \dot\gamma(t;x).$$

Now it is not hard to see that

$$\nabla_\xi \nabla U(x) = \nabla_\xi [\sigma(x)\nabla\sigma(x)] = \xi.$$

Hence the pair $(1,\xi)$ represents eigenvalue and eigenvector for $H_U(x)$. All the other eigenvalues of $H_U(x)$ are expressed as $\lambda_i(x) = \sigma(x)\kappa_i(x)$ where $\kappa_1(x),\dots,\kappa_{d-1}(x)$ are the principal curvatures of the second fundamental form $l(\xi,\eta) := \langle \nabla_\xi \sigma(x), \eta\rangle$, $\xi,\eta \perp \nabla\sigma(x)$, for the "sphere" $\{y \in \mathfrak{M} : \sigma(y) = \sigma(x)\}$ at the point x. Finally, define

$$\lambda_*(x) = \min_{1\leqslant i\leqslant n} \{1,\lambda_1(x),\dots,\lambda_{d-1}(x)\}.$$

Note that since $U(\cdot)$ is smooth in $\mathscr{B}(x_*)$, then the value $\lambda_*(x_*)$ is correctly defined by the continuity. It is known [10, p. 203], that if the maximal sectional curvature in $\mathscr{B}(x_*)$ does not exceed $\varkappa \geqslant 0$ and $\sigma(x) < 0.5\pi/\sqrt{\varkappa}$, then

$$\lambda_*(x) \geqslant \sqrt{\varkappa}\rho(x)\cdot\cot(\sqrt{\varkappa}\rho(x)).$$

DEFINITION 1. Let $\lambda \in (0,1]$. A mapping $x(\cdot) \in C^2(\mathbb{R} \mapsto \mathfrak{M})$ is said to be of class $\mathfrak{C}_\lambda$ if the set $x(\mathbb{R})$ has a Chebyshev center x_* such that

$$x(\mathbb{R}) \subset \{x \in \mathscr{B}(x_*) : \lambda_*(x) \geqslant \lambda\}.$$

DEFINITION 2. Let $\varkappa \geqslant 0$, $0 < r < 0.5\pi/\sqrt{\varkappa}$. A mapping $x(\cdot) \in C^2(\mathbb{R} \mapsto \mathfrak{M})$ is said to be of class $\mathfrak{C}_{\varkappa,r}$ if the set $x(\mathbb{R})$ has a Chebyshev center x_* such that $R[x(\cdot)] \leqslant r \leqslant \mathrm{ir}(x_*)$ and the maximal sectional curvature in $\mathscr{B}(x_*)$ does no exceed $\varkappa$.

The foregoing facts together with Theorem 1 allow us to obtain the following generalization of Theorem 2.

THEOREM 3. *Let* $x(\cdot) \in \mathfrak{C}_\lambda \cup \mathfrak{C}_{\varkappa,r}$ *and* $\|\nabla_{\dot x} x(\cdot)\|_\infty < \infty$. Then

$$\|\dot x(\cdot)\|_\infty \leqslant \begin{cases} \frac{C}{\sqrt{\lambda}}\sqrt{R[x(\cdot)]\,\|\nabla_{\dot x} x(\cdot)\|_\infty} & \text{if } x(\cdot) \in \mathfrak{C}_\lambda, \\ \frac{C}{\sqrt{\sqrt{\varkappa}r\cot(\sqrt{\varkappa}r)}}\sqrt{R[x(\cdot)]\,\|\nabla_{\dot x} x(\cdot)\|_\infty} & \text{if } x(\cdot) \in \mathfrak{C}_{\varkappa,r} \end{cases}$$

where C *is defined in Theorem* 1.

Now we aim to obtain the counterpart of Landau – Kolmogorov inequalities involving iterates of covariant derivatives.

LEMMA 1. *Let $\xi(\cdot):\mathbb{R}\mapsto T\mathfrak{M}$ be a smooth vector field along a smooth mapping $x(\cdot):\mathbb{R}\mapsto\mathfrak{M}$ and let $n\geqslant 2$ be a natural number. Suppose that*

$$\|\xi(\cdot)\|_\infty<\infty,\quad \|\nabla_{\dot{x}}^n\xi(\cdot)\|_\infty<\infty.$$

Then for any natural $k<n$ there holds the inequality

$$\left\|\nabla_{\dot{x}}^k\xi(\cdot)\right\|_\infty\leqslant C_{n,k}\|\xi(\cdot)\|_\infty^{1-k/n}\|\nabla_{\dot{x}}^n\xi(\cdot)\|_\infty^{k/n}$$

where $C_{n,k}$ are the Kolmogorov constants for mappings $f(\cdot)\in\mathrm{C}^n\left(\mathbb{R}\mapsto\mathbb{R}^d\right)$.

Proof. Denote by $\Omega_s^t:T_{x(s)}\mathfrak{M}\mapsto T_{x(t)}\mathfrak{M}$ the cocycle of parallel transport along $x(\cdot)$, and define the mapping $f(\cdot):\mathbb{R}\mapsto T_{x(0)}\mathfrak{M}\simeq\mathbb{R}^d$ by $f(t):=\Omega_t^0\xi(t)$. Then

$$\begin{aligned}\nabla_{\dot{x}}\xi(t)&=\lim_{s\to 0}\frac{1}{s}\left[\Omega_{t+s}^t\xi(t+s)-\xi(t)\right]=\lim_{s\to 0}\frac{1}{s}\left[\Omega_{t+s}^t\Omega_0^{t+s}f(t+s)-\Omega_0^tf(t)\right]\\&=\Omega_0^t\lim_{s\to 0}\frac{1}{s}\left[f(t+s)-f(t)\right]=\Omega_0^tf'(t),\end{aligned}$$

and thus, $\nabla_{\dot{x}}^k\xi(t)=\Omega_0^tf^{(k)}(t)$. Now it remains only to observe that the parallel transport preserves the inner product. ☐

THEOREM 4. *Let $0\leqslant k<n$ and let $x(\cdot)\in(\mathfrak{C}_\lambda\cup\mathfrak{C}_{\varkappa,r})\cap\mathrm{C}^{n+1}(\mathbb{R}\mapsto\mathfrak{M})$. Define*

$$K_{1,0}:=\begin{cases}C/\sqrt{\lambda} & \text{if } x(\cdot)\in\mathfrak{C}_\lambda,\\ C/\sqrt{\sqrt{\varkappa}r\cot(\sqrt{\varkappa}r)} & \text{if } x(\cdot)\in\mathfrak{C}_{\varkappa,r}.\end{cases},$$

$$K_{n+1,k+1}:=C_{n,k}\left(K_{1,0}^2C_{n,1}\right)^{\frac{n-k}{n+1}}.$$

If $\|\nabla_{\dot{x}}^n\dot{x}(\cdot)\|_\infty<\infty$, and $\|\dot{x}(\cdot)\|_\infty<\infty$ or $\|\nabla_{\dot{x}}\dot{x}(\cdot)\|_\infty<\infty$, then

$$\left\|\nabla_{\dot{x}}^k\dot{x}(\cdot)\right\|_\infty\leqslant K_{n+1,k+1}\left(R[x(\cdot)]\right)^{1-\frac{k+1}{n+1}}\|\nabla_{\dot{x}}^n\dot{x}(\cdot)\|_\infty^{\frac{k+1}{n+1}}.$$

Proof. Set $l_0=\ln R[x(\cdot)]$, $l_1:=\ln\|\dot{x}(\cdot)\|_\infty$, $l_{k+1}=\ln\left\|\nabla_{\dot{x}}^k\dot{x}(\cdot)\right\|_\infty$, $c_{1,0}=\ln K_{1,0}$, $c_{n,k}=\ln C_{n,k}$ for $n>k\geqslant 1$, and $\xi(\cdot)=\dot{x}(\cdot)$. If $l_1<\infty$, then by Lemma 1

$$l_{k+1}\leqslant\left(1-\frac{k}{n}\right)l_1+\frac{k}{n}l_{n+1}+c_{n,k},\quad k\geqslant 1,\tag{2}$$

in particular, $l_2<\infty$, and now by Theorem 3

$$l_1\leqslant\frac{1}{2}l_0+\frac{1}{2}l_2+c_{1,0}.\tag{3}$$

If $l_2 < \infty$, then first we get (3), and then (2). In the both cases, the last two inequalities yield

$$l_1 \leqslant \frac{1}{2} l_0 + \frac{1}{2} \left[\left(1 - \frac{1}{n}\right) l_1 + \frac{1}{n} l_{n+1} + c_{n,1} \right] + c_{1,0},$$

and thus,

$$l_1 \leqslant \frac{n}{n+1} l_0 + \frac{1}{n+1} l_{n+1} + \frac{n\,(2c_{1,0} + c_{n,1})}{n+1}.$$

Finally, returning to (2), we arrive at the inequality

$$l_{k+1} \leqslant \left(1 - \frac{k+1}{n+1}\right) l_0 + \frac{k+1}{n+1} l_{n+1} + \frac{(n-k)\,(2c_{1,0} + c_{n,1})}{n+1} + c_{n,k}$$

which, after taking exponentials of both sides, completes the proof. □

Concluding Remark. There is still an open problem whether Theorem 4 remains true if we omit the boundedness requirements for $\|\dot{x}(\cdot)\|_\infty$ or $\|\nabla_{\dot{x}}\dot{x}(\cdot)\|_\infty$.

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