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OPTIMAL ESTIMATES FOR THE FRACTIONAL HARDY OPERATOR ON VARIABLE EXPONENT LEBESGUE SPACES

YOSHIHIRO MIZUTA, ALEŠ NEKVINDA AND TETSU SHIMOMURA

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Abstract. Let $A_\alpha f(x) = \frac{1}{|B(0,|x|)|^{\alpha/n}} \int_{B(0,|x|)} f(t)\, dt$ be the n-dimensional fractional Hardy operator, where $0 < \alpha \leqslant n$. We prove optimality results for the action of the operator A_α on variable exponent Lebesgue spaces $L^{p(\cdot)}$ and weighted variable exponent Lebesgue spaces, as an extension of [13, 14, 17].

1. Introduction

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space and Ω be an open subset of $\mathbb{R}^n$. For an integrable function u on a measurable set $E \subset \mathbb{R}^n$ of positive measure, we define the integral mean over E by

$$\fint_E u(x)\, dx = \frac{1}{|E|} \int_E u(x)\, dx,$$

where $|E|$ denotes the Lebesgue measure of E. We denote by $B(x,r)$ the open ball with center x and of radius $r > 0$, and by $|B(x,r)|$ its Lebesgue measure, i.e. $|B(x,r)| = \sigma_n r^n$, where σ_n is the volume of the unit ball in $\mathbb{R}^n$. For a locally integrable function f on Ω and $0 < \alpha \leqslant n$, we consider the fractional Hardy operator A_α, defined by

$$A_\alpha f(x) = \frac{1}{|B(0,|x|)|^{\alpha/n}} \int_{B(0,|x|)} f(t)\, dt,$$

the Hardy averaging operator A, defined by

$$Af(x) = \fint_{B(0,|x|)} f(t)\, dt$$

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and the centered Hardy-Littlewood maximal operator M, defined by

$$Mf(x) = \sup_{r>0} \fint_{B(x,r)} |f(y)|\, dy$$

by setting $f = 0$ outside Ω (for the fundamental properties of maximal functions, see Stein [19]). In the case $\alpha = n$, $A_\alpha f(x) = Af(x)$.

Let $1 < p < \infty$, $1/p + 1/p' = 1$ and

$$p_\alpha = \frac{np'}{\alpha p' - n} = \frac{np}{\alpha p - np + n}.$$

We know that A_α is bounded from L^p to L^{p_α} provided $n(1-1/p) < \alpha \leqslant n$. Clearly, $p_\alpha \geqslant p > 1$.

In the previous paper [14], we improved the result of Nekvinda and Pick [17] in the case when $\alpha = n = 1$ and Ω is a bounded interval, and that of the authors [13] within the framework of generalized Banach function spaces. Let $\hookrightarrow$ denote a continuous embedding and $\to$ denote a boundedness of an operator. Under the assumptions $A_\alpha : X \to Y$ and $M : Y \to Y$, we found the 'source' space $S_{\alpha,Y}$ and the 'target' space T_Y such that

(i) the fractional Hardy averaging operator A_α satisfies

$$A_\alpha : S_{\alpha,Y} \to T_Y;$$

(ii) this result improves the classical estimate

$$A_\alpha : X \to Y$$

in the sense that

$$X \hookrightarrow S_{\alpha,Y},\ T_Y \hookrightarrow Y;$$

(iii) this result cannot be improved any further, at least not within the environment of generalized Banach function spaces in the sense that whenever Z is a generalized Banach function space strictly larger than $S_{\alpha,Y}$,

$$A_\alpha : Z \not\to T_Y$$

and, likewise, when Z is a generalized Banach function space strictly smaller than T_Y, then

$$A_\alpha : S_{\alpha,Y} \not\to Z.$$

In this paper, we present applications of our results to variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$, as an extension of [13, 14, 17]. In Section 5, we prove optimality results for the action of the operator A_α on $L^{p(\cdot)}$ spaces. In Section 6, we prove optimality results for the action of the operator A_α on weighted variable exponent Lebesgue spaces.

In the last section, we show that the condition $T_Y = Y$ implies that the norm in Y is very similar to the norm in L_∞ in connection with Lang and Nekvinda [11] and Lang, Nekvinda and Rákosník [12] .

2. Preliminaries

Throughout this paper, let C denote various constants independent of the variables in question, and $C(a,b,\cdots)$ a constant that depends on $a,b,\cdots$.

Let $\mathscr{M}(\Omega)$ denote the space of measurable functions on an open set $\Omega \subset \mathbb{R}^n$ with values in $[-\infty,\infty]$. Denote by χ_E the characteristic function of E. Recall the frequently used definition of Banach function spaces which can be found for instance in [1].

DEFINITION 2.1. We say that a normed linear space $(X,\|.\|_X)$ is a Banach function space (BFS for short) if the following conditions are satisfied:

$$\|f\|_X \text{ is defined for all } f \in \mathscr{M}(\Omega), \text{ and } f \in X \text{ if and only if } \|f\|_X < \infty; \tag{1}$$

$$\|f\|_X = \|\,|f|\,\|_X \text{ for every } f \in \mathscr{M}(\Omega); \tag{2}$$

$$\text{if } 0 \leqslant f_n \nearrow f \text{ a.e. in } \Omega, \text{ then } \|f_n\|_X \nearrow \|f\|_X; \tag{3}$$

$$\text{if } E \subset \Omega \text{ is a measurable set of finite measure, then } \chi_E \in X; \tag{4}$$

$$\text{for every measurable set } E \subset \Omega \text{ of finite measure, there exists} \tag{5}$$

$$\text{a positive constant } C_E \text{ such that } \int_E |f(x)|dx \leqslant C_E \|f\|_X.$$

Denote by $\mathfrak{B} = \mathfrak{B}(\mathbb{R}^n)$ the class of all BFSs defined on Ω.

We will work with more general spaces where conditions (4) and (5) are omitted .

DEFINITION 2.2. We say that a normed linear space $(X,\|.\|_X)$ is a generalized Banach function space (shortly GBFS) if the following conditions are satisfied:

$$\|f\|_X \text{ is defined for all } f \in \mathscr{M}(\Omega), \text{ and } f \in X \text{ if and only if } \|f\|_X < \infty; \tag{6}$$

$$\|f\|_X = \|\,|f|\,\|_X \text{ for every } f \in \mathscr{M}(\Omega); \tag{7}$$

$$\text{if } 0 \leqslant f_n \nearrow f \text{ a.e. in } \mathbb{R}^n, \text{ then } \|f_n\|_X \nearrow \|f\|_X; \tag{8}$$

Denote by $\mathfrak{G} = \mathfrak{G}(\Omega)$ the class of all GBFSs defined on $\mathbb{R}^n$.

Recall that condition (8) immediately yields the following property:

$$\text{if } 0 \leqslant f \leqslant g, \text{ then } \|f\|_X \leqslant \|g\|_X. \tag{9}$$

To see this it suffices to set $f_1 = f, f_n = g$ for $n \geqslant 2$ in (8). It is well-known that each BFS is complete and so, it is a Banach space (see [1, Theorem 1.6]). We know that each GBFS is complete (see [13]).

Let X,Y be Banach spaces (not necessarily generalized Banach function spaces). Say that $X \hookrightarrow Y$ if $X \subset Y$ and there is $C > 0$ such that $\|f\|_Y \leqslant C\|f\|_X$ for all $f \in X$. Recall well-known theorems on Banach function spaces (see [1, Theorem 1.8]) which assert the implication

$$(\|f\|_X < \infty \Rightarrow \|f\|_Y < \infty) \Longrightarrow X \hookrightarrow Y.$$

In what follows we need a generalization of this remark as in [13].

DEFINITION 2.3. Let $(X,\|.\|_X)$ be a GBFS. Say that a mapping $T:(X,\|.\|_X)\to\mathscr{M}(\Omega)$ is a sublinear nondecreasing operator if the following conditions are satisfied for all $\alpha\in\mathbb{R}, f,g\in(X,\|.\|_X)$:

(i) $T(\alpha f)=\alpha T(f)$, $T(f+g)\leqslant T(f)+T(g)$ almost everywhere;

(ii) if $0\leqslant f\leqslant g$ almost everywhere implies $0\leqslant Tf\leqslant Tg$ almost everywhere.

LEMMA 2.4. ([13, Lemma 2.7]) *Let $(X,\|.\|_X),(Y,\|.\|_Y)$ be GBFSs and T a sublinear nondecreasing operator on $\mathscr{M}(\Omega)$. Then the following two conditions are equivalent:*

(i) $\|f\|_X<\infty\Rightarrow\|Tf\|_Y<\infty$;

(ii) *there is $C>0$ such that $\|Tf\|_Y\leqslant C\|f\|_X$ for all $f\in X$.*

Given a measurable function f on Ω set

$$\widetilde{f}(x)=\operatorname*{ess\,sup}_{\{t\in\Omega:|t|\geqslant|x|\}}|f(t)|.$$

If x is a Lebesgue point of f, then

$$|f(x)|\leqslant\widetilde{f}(x),$$

so that

$$|f(x)|\leqslant\widetilde{f}(x)\text{ a.e. on }\Omega. \tag{10}$$

DEFINITION 2.5. Let Y be a GiBFS and let f be a measurable function on Ω. Set

$$\|f\|_{T_Y}=\|\widetilde{f}\,\|_Y$$

and define the corresponding space

$$T_Y=\{f:\widetilde{f}\in Y\}.$$

Remark that T_Y is a GBFS ([13, Lemma 3.2]).

DEFINITION 2.6. Let Y be a GBFS and let f be a measurable function on Ω. Say that f has an absolutely continuous norm if $\lim_{|E_n|\to0}\|f\chi_{E_n}\|_Y=0$ for measurable sets $E_n\subset\mathbb{R}^n$.

Say that f has a continuous norm if $\lim_{r\to0_+}\|f\chi_{B(x,r)}\|_Y=0$ for every $x\in\Omega$ and $\lim_{R\to\infty}\|f\chi_{\Omega\setminus B(x,R)}\|_Y=0$.

Denote by Y_a the set of all functions with an absolutely continuous norm and by Y_c the set of all functions with a continuous norm. Say that Y has an absolutely continuous norm if $Y=Y_a$ and Y has a continuous norm if $Y=Y_c$.

LEMMA 2.7. ([14, Lemma 3.2]) *Let Y be a GBFS and $Y\neq0$. Then the embedding $T_Y\hookrightarrow Y$ holds and $T_Y\subsetneqq Y$ holds provided Y has an absolutely continuous norm.*

Commonly with the definition of the space T_Y a question appears when $T_Y = Y$. Clearly $T_Y = Y$ for $Y = L_\infty$. Remark that it is possible to adopt the proof of the previous lemma under the assumption Y has a continuous norm if $Y = Y_c$. The property $Y = Y_c$ is really weaker than $Y = Y_a$. Indeed, in [11] there is a space Y such that $\{0\} \subsetneqq Y_a \subsetneqq Y_c \subsetneqq Y$ and in [12] there is even a Y with $\{0\} = Y_a \subsetneqq Y_c = Y$. Nevertheless, we show in the last section that the condition $T_Y = Y$ implies that the norm in Y is very similar to the norm in L_∞.

LEMMA 2.8. ([14, Lemma 3.4]) *Let X, Y be GBFSs and suppose that*

$$A_\alpha : X \to Y, \qquad M : Y \to Y. \tag{11}$$

Then

$$A_\alpha : X \to T_Y.$$

DEFINITION 2.9. Let $Y \in \mathfrak{G}(\Omega)$ and let f be a measurable function on Ω. Set

$$\|f\|_{S_{\alpha,Y}} = \|A_\alpha |f|\,\|_{T_Y}$$

and the corresponding space

$$S_{\alpha,Y} = \{f : \widetilde{A_\alpha |f|} \in Y\}.$$

Remark that $S_{\alpha,Y}$ is a GBFS. Indeed, we can prove the fact as in the proof of [13, Lemma 3.6].

By Lemma 2.8 and [14, Lemma 3.6], we readily have the following result.

LEMMA 2.10. ([14, Lemma 3.7]) *Let X, Y be GBFSs and $A_\alpha : X \to Y$, $M : Y \to Y$. Then $A_\alpha : S_{\alpha,Y} \to T_Y$ and $X \hookrightarrow S_{\alpha,Y}$.*

3. Boundedness of A_α

We will frequently use the notation $\mathbf{B}$ for the unit ball $B(0,1)$.

Let $1 \leqslant p < \infty$ and $1 \leqslant p_\infty < \infty$. In this section, we consider continuous exponents $p(\cdot)$ on $\mathbb{R}^n$ such that

(P1) $1 \leqslant p^- \equiv \inf_{x \in \mathbb{R}^n} p(x) \leqslant \sup_{x \in \mathbb{R}^n} p(x) \equiv p^+ < \infty$;

(P2) $|p(x) - p| \leqslant C/\log(e + 1/|x|)$ whenever $x \in \mathbb{R}^n$;

(P3) $|p(x) - p_\infty| \leqslant C/\log(e + |x|)$ whenever $x \in \mathbb{R}^n$.

If p satisfies (P2), then p is said to satisfy the weak-Lipschitz condition at zero with respect to p. Moreover, we say that $p(\cdot)$ is weak-Lipschitz or log-Hölder if

(P4) $|p(x) - p(y)| \leqslant C/\log(e + 1/|x - y|)$ whenever $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$.

DEFINITION 3.1. Let Ω be an open subset of $\mathbb{R}^n$. Let us consider the family $L^{p(\cdot)}(\Omega)$ of all measurable functions f on Ω satisfying

$$\int_\Omega \left|\frac{f(y)}{\lambda}\right|^{p(y)} dy < \infty$$

for some $\lambda > 0$. We define the norm on this space by

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{f(y)}{\lambda}\right|^{p(y)} dy \leqslant 1\right\}.$$

Remark that $L^{p(\cdot)}(\Omega)$ is a BFS ([4]).

In the next we will use a little more general concept of Banach function spaces than in Definition 2.1. The last two axioms are weakened and so, we will call these spaces by weak Banach function spaces.

DEFINITION 3.2. We say that a normed linear space $(X, \|.\|_X)$ is called a weak Banach function space (WBFS for short) if the following conditions are satisfied:

the norm $\|f\|_X$ is defined for all $f \in \mathscr{M}(\Omega)$ and $f \in X$ if and only if $\|f\|_X < \infty$; (12)

$\|f\|_X = \|\,|f|\,\|_X$ for every $f \in \mathscr{M}(\Omega)$; (13)

if $0 \leqslant f_n \nearrow f$ a.e. in Ω then $\|f_n\|_X \nearrow \|f\|_X$; (14)

if E is a compact subset of Ω, then $\chi_E \in X$; (15)

for every compact set $E \subset \Omega$, there exists a positive constant C_E such that $\int_E |f(x)|dx \leqslant C_E\|f\|_X$. (16)

Note that each WBFS is complete and consequently, it is a Banach space ([13, Theorem 6.2]).

DEFINITION 3.3. Let Ω be an open subset of $\mathbb{R}^n$. Let us consider the family $T_{p(\cdot)}(\Omega)$ of all measurable functions f on Ω satisfying

$$\int_\Omega \left(\operatorname*{ess\,sup}_{\{t\in\Omega:|t|\geqslant|x|\}} \left|\frac{f(t)}{\lambda}\right|\right)^{p(x)} dx < \infty$$

for some $\lambda > 0$. We define the norm on this space by

$$\|f\|_{T_{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \left(\operatorname*{ess\,sup}_{\{t\in\Omega:|t|\geqslant|x|\}} \left|\frac{f(t)}{\lambda}\right|\right)^{p(x)} dx \leqslant 1\right\}.$$

If $p(\cdot)$ is a constant p, then we write $T_p(\Omega)$ and $\|f\|_{T_p(\Omega)}$ for $T_{p(\cdot)}(\Omega)$ and $\|f\|_{T_{p(\cdot)}(\Omega)}$, respectively.

Note that $T_{p(\cdot)}(\Omega) = T_X$ for $X = L^{p(\cdot)}(\Omega)$. Remark that any $T_{p(\cdot)}(\Omega)$ is a WBFS ([13, Lemma 7.2]).

THEOREM 3.4. ([13, Theorem 7.3 and Corollary 7.4]) *Suppose that $p(\cdot)$ satisfies* (P1) *and* (P2). *Then the norms in $T_{p(\cdot)}(\mathbf{B})$ and $T_p(\mathbf{B})$ are equivalent. Moreover,*

$$T_p(\mathbf{B}) \hookrightarrow L^{p(\cdot)}(\mathbf{B}).$$

THEOREM 3.5. ([13, Theorem 7.5 and Corollary 7.6]) *Suppose that $p(\cdot)$ satisfies* (P1) *and* (P3). *Then the norms in $T_{p(\cdot)}(\mathbb{R}^n \setminus \mathbf{B})$ and $T_{p_\infty}(\mathbb{R}^n \setminus \mathbf{B})$ are equivalent. Moreover,*

$$T_{p_\infty}(\mathbb{R}^n \setminus \mathbf{B}) \hookrightarrow L^{p(\cdot)}(\mathbb{R}^n \setminus \mathbf{B}).$$

Here we consider the following condition:

(P1′) $1 < p^- \leqslant p^+ < \infty$.

Remark that (P1′) and (P3) imply $1 < p_\infty < \infty$.

We know the boundedness of maximal operators in $L^{p(\cdot)}(\mathbb{R}^n)$, due to [3].

LEMMA 3.6. *Suppose that $p(\cdot)$ satisfies* (P1′), (P3) *and* (P4). *Then there exists a positive constant C such that*

$$\|Mf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

for all measurable functions $f \in L^{p(\cdot)}(\mathbb{R}^n)$.

Let

$$1/p_\alpha(x) = 1/p(x) - (n-\alpha)/n.$$

For $0 < \alpha < n$, we define the Riesz potential of order α for a locally integrable function f on $\mathbb{R}^n$ by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y)\, dy.$$

We define the fractional maximal operator for a locally integrable function f on $\mathbb{R}^n$ by

$$M_\alpha f(x) = \sup_{r>0} \frac{1}{|B(x,r)|^{1-\frac{\alpha}{n}}} \int_{B(x,r)} |f(y)|\, dy.$$

We know the following due to [2].

LEMMA 3.7. *Suppose that $p(\cdot)$ satisfies* (P1′), (P3) *and* (P4). *Then there exists a positive constant C such that*

$$\|I_{n-\alpha} f\|_{L^{p_\alpha(\cdot)}(\mathbb{R}^n)} \leqslant C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$.

LEMMA 3.8. *Suppose that* $p(\cdot)$ *satisfies* (P1′), (P3) *and* (P4). *Let* $n(1-1/p_+) < \alpha \leqslant n$. *Then* $A_\alpha : L^{p(\cdot)}(\mathbb{R}^n) \to L^{p_\alpha(\cdot)}(\mathbb{R}^n)$.

Proof. Since $|A_\alpha f(x)| \leqslant CM_{n-\alpha} f(x) \leqslant CI_{n-\alpha} f(x)$, we obtain the lemma by Lemma 3.7. □

LEMMA 3.9. *Suppose that* $p(\cdot)$ *satisfies* (P1′), (P3) *and* (P4). *Let* $n(1-1/p_+) < \alpha \leqslant n$. *Then* $A_\alpha : L^{p(\cdot)}(\mathbb{R}^n) \to T_{p_\alpha(\cdot)}(\mathbb{R}^n)$.

Proof. This lemma follows immediately from Lemmas 3.6, 3.8 and 2.8. □

THEOREM 3.10. (cf. [13, Theorem 7.12]) *Let* $1 < p < \infty$. *Suppose that* $p(\cdot)$ *satisfies* (P1′) *and* (P2). *Then* $A_\alpha : L^{p(\cdot)}(\mathbf{B}) \to T_{p_\alpha}(\mathbf{B})$.

Proof. By our assumption,

$$|p(x) - p| \leqslant C/\log(e + 1/|x|) \qquad \text{whenever } x \in \mathbf{B}.$$

We set $d = \inf_{x \in \mathbf{B}} p(x)$ and

$$q(x) = \max\left\{d, p - \frac{C}{\log(e + 1/|x|)}\right\}.$$

Then $q(x) \leqslant p(x)$ for $x \in \mathbf{B}$ and $q(\cdot)$ satisfies (P1′). Hence $L^{p(\cdot)}(\mathbf{B}) \hookrightarrow L^{q(\cdot)}(\mathbf{B})$ (see e.g. [9]). Hence $L^{p(\cdot)} \hookrightarrow L^{q(\cdot)}$ (see e.g. [9]). Next, by [13, Lemmas 7.10 and 7.11], q satisfies (P4). Thus, by Lemma 3.9, $A_\alpha : L^{q(\cdot)}(\mathbf{B}) \to T_{q_\alpha(\cdot)}(\mathbf{B})$ holds. Finally, in view of Theorem 3.4, $T_{q_\alpha(\cdot)}(\mathbf{B}) \hookrightarrow T_{p_\alpha}(\mathbf{B})$. Altogether,

$$\|A_\alpha f\|_{T_{p_\alpha}(\mathbf{B})} \leqslant C\|A_\alpha f\|_{T_{q_\alpha(\cdot)}(\mathbf{B})} \leqslant C\|f\|_{L^{q(\cdot)}(\mathbf{B})} \leqslant C\|f\|_{L^{p(\cdot)}(\mathbf{B})}. \quad \square$$

DEFINITION 3.11. Let us consider the family $S_{\alpha,p}(\Omega)$ of all measurable functions f on Ω with the finite norm

$$\|f\|_{S_{\alpha,p}(\Omega)} = \left(\int_\Omega \left(\widetilde{A_\alpha|f|}(x)\right)^p dx\right)^{1/p} = \|A_\alpha|f|\|_{T_p(\Omega)};$$

for convenience set $f = 0$ outside Ω.

Remark that $S_{\alpha,p}(\Omega)$ is a WBFS (cf. [13, Lemma 7.14]).

COROLLARY 3.12. *Let* $1 < p < \infty$. *Suppose that* $p(\cdot)$ *satisfies* (P1′) *and* (P2). *Then*

$$L^{p(\cdot)}(\mathbf{B}) \hookrightarrow S_{\alpha,p_\alpha}(\mathbf{B}).$$

Proof. This corollary is proved by Theorem 3.10. □

THEOREM 3.13. *Suppose that* $p(\cdot)$ *satisfies* (P1′) *and* (P3). *Then* $A_\alpha : L^{p(\cdot)}(\mathbb{R}^n \setminus \mathbf{B}) \to T_{(p_\infty)_\alpha}(\mathbb{R}^n \setminus \mathbf{B})$.

Proof. To show $\|A_\alpha f\|_{T_{(p_\infty)_\alpha}(\mathbb{R}^n\setminus\mathbf{B})} \leqslant C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n\setminus\mathbf{B})}$, suppose that

$$\int_{\mathbb{R}^n\setminus\mathbf{B}} |f(x)|^{p(x)} dx \leqslant 1. \tag{17}$$

By our assumption,

$$|p(x) - p_\infty| \leqslant C/\log(e+|x|) \qquad \text{whenever } x \in \mathbb{R}^n \setminus \mathbf{B}.$$

Set $d = \inf_{x\in\mathbb{R}^n\setminus\mathbf{B}} p(x)$. Then by (P3)

$$q(x) := \max\left\{d, p_\infty - \frac{C}{\log(e+|x|)}\right\} \leqslant p(x) \leqslant p_\infty + \frac{C}{\log(e+|x|)} := \tilde{q}(x).$$

Hence $q(x) \leqslant p(x) \leqslant \tilde{q}(x)$ for $x \in \mathbb{R}^n \setminus \mathbf{B}$ and $q(\cdot)$ and $\tilde{q}(\cdot)$ satisfy (P1′) and (P3).

Since $|\nabla(1/\log(e+|x|))| \leqslant 1/e$, the functions q and $\widetilde{q}$ are Lipschitz and so, both satisfy (P4). Thus, by Lemma 3.9, $A_\alpha : L^{q(\cdot)}(\mathbb{R}^n) \to T_{q_\alpha(\cdot)}(\mathbb{R}^n)$ and $A_\alpha : L^{\widetilde{q}(\cdot)}(\mathbb{R}^n) \to T_{\widetilde{q}_\alpha(\cdot)}(\mathbb{R}^n)$. If we consider function with zero values on $\mathbf{B}$, then we obtain

$$A_\alpha : L^{q(\cdot)}(\mathbb{R}^n \setminus \mathbf{B}) \to T_{q_\alpha(\cdot)}(\mathbb{R}^n \setminus \mathbf{B}), \quad A_\alpha : L^{\tilde{q}(\cdot)}(\mathbb{R}^n \setminus \mathbf{B}) \to T_{\tilde{q}_\alpha(\cdot)}(\mathbb{R}^n \setminus \mathbf{B}). \tag{18}$$

Moreover, in view of Theorem 3.5 we have

$$T_{q_\alpha(\cdot)}(\mathbb{R}^n \setminus \mathbf{B}) \hookrightarrow T_{(p_\infty)_\alpha}(\mathbb{R}^n \setminus \mathbf{B}), \quad T_{\tilde{q}_\alpha(\cdot)}(\mathbb{R}^n \setminus \mathbf{B}) \hookrightarrow T_{(p_\infty)_\alpha}(\mathbb{R}^n \setminus \mathbf{B}). \tag{19}$$

Write

$$f = f\chi_{\{y:f(y)\geqslant 1\}} + f\chi_{\{y:0\leqslant f(y)<1\}} = f_1 + f_2. \tag{20}$$

By (17) and (20) we obtain

$$\begin{aligned} &\int_{\mathbb{R}^n\setminus\mathbf{B}} |f_1(x)|^{q(x)} dx + \int_{\mathbb{R}^n\setminus\mathbf{B}} |f_2(x)|^{\tilde{q}(x)} dx \\ &\leqslant \int_{\mathbb{R}^n\setminus\mathbf{B}} |f_1(x)|^{p(x)} dx + \int_{\mathbb{R}^n\setminus\mathbf{B}} |f_2(x)|^{p(x)} dx = \int_{\mathbb{R}^n\setminus\mathbf{B}} |f(x)|^{p(x)} dx \leqslant 1. \end{aligned}$$

By (18) we have

$$\int_{\mathbb{R}^n\setminus\mathbf{B}} \left|\widetilde{A_\alpha f_1}(x)\right|^{q_\alpha(x)} dx \leqslant C, \quad \int_{\mathbb{R}^n\setminus\mathbf{B}} \left|\widetilde{A_\alpha f_2}(x)\right|^{\tilde{q}_\alpha(x)} dx \leqslant C.$$

Finally, (19) yields

$$\begin{aligned} &\int_{\mathbb{R}^n\setminus\mathbf{B}} \left|\widetilde{A_\alpha f}(x)\right|^{(p_\infty)_\alpha} dx \\ &\leqslant C\int_{\mathbb{R}^n\setminus\mathbf{B}} \left|\widetilde{A_\alpha f_1}(x)\right|^{(p_\infty)_\alpha} dx + C\int_{\mathbb{R}^n\setminus\mathbf{B}} \left|\widetilde{A_\alpha f_2}(t)\right|^{(p_\infty)_\alpha} dx \leqslant C, \end{aligned}$$

which finishes the proof with Lemma 2.4. $\square$

COROLLARY 3.14. (cf. [14, Corollary 7.20]) *Let* $1 < p < \infty$. *Then*

$$\begin{array}{ll} A_\alpha : S_{\alpha,p_\alpha}(\mathbb{R}^n) \to T_{p_\alpha}(\mathbb{R}^n), & \text{(by Lemma 2.10)} \\ A : S_{\alpha,p_\alpha}(\mathbf{B}) \to S_{\alpha,p_\alpha}(\mathbf{B}) & \text{(by Lemma 2.10)}, \\ A_\alpha : S_{\alpha,p_\alpha}(\mathbb{R}^n \setminus \mathbf{B}) \to S_{\alpha,p_\alpha}(\mathbb{R}^n \setminus \mathbf{B}) & \text{(by Lemma 2.10)}, \\ A_\alpha : T_p(\mathbf{B}) \to T_{p_\alpha}(\mathbf{B}) & \text{(by Theorem 3.4 and Theorem 3.10)} \end{array}$$

and

$$A_\alpha : T_{p_\infty}(\mathbb{R}^n \setminus \mathbf{B}) \to T_{(p_\infty)_\alpha}(\mathbb{R}^n \setminus \mathbf{B}) \qquad \text{(by Theorem 3.5 and Theorem 3.13)}.$$

Moreover suppose that $r(\cdot), s(\cdot)$ *satisfy* (P1′) *and* (P2) *with a same* p. *Then*

$$A_\alpha : L^{r(\cdot)}(\mathbf{B}) \to L^{s(\cdot)}(\mathbf{B}) \qquad \text{(by Theorem 3.4 and Theorem 3.10)}.$$

Suppose that $r(\cdot), s(\cdot)$ *satisfy* (P1′) *and* (P3) *with the same* p. *Then*

$$A_\alpha : L^{r(\cdot)}(\mathbb{R}^n \setminus \mathbf{B}) \to L^{s(\cdot)}(\mathbb{R}^n \setminus \mathbf{B}) \qquad \text{(by Theorem 3.5 and Theorem 3.13)}.$$

4. Optimal pairs

DEFINITION 4.1. Let $\mathfrak{S} \subset \mathfrak{G}$. Assume $X, Y \in \mathfrak{S}$. Say that (X,Y) is an optimal pair for A_α with respect to $\mathfrak{S}$ if

$$A_\alpha : X \to Y, \tag{21}$$

$$\text{if } Z \in \mathfrak{S} \text{ with } A_\alpha : Z \to Y, \text{ then } Z \hookrightarrow X, \tag{22}$$

$$\text{if } Z \in \mathfrak{S} \text{ with } A_\alpha : X \to Z, \text{ then } Y \hookrightarrow Z. \tag{23}$$

LEMMA 4.2. ([14, Lemma 4.2]) *Let* $X, Y \in \mathfrak{G}$ *and* $A_\alpha : X \to T_Y$. *Suppose*

$$A_\alpha[|x|^{\alpha-n} h(x)] \in T_Y \qquad \text{when } h \in T_Y. \tag{24}$$

Then $(S_{\alpha,Y}, T_Y)$ *is an optimal pair for* A_α *with respect to* $\mathfrak{G}$.

REMARK 4.3. We note that (24) holds if and only if the inequality

$$\|A_\alpha[|x|^{\alpha-n} g]\|_Y \leqslant C\|g\|_Y$$

holds for every radial symmetric non-increasing function g. Such inequalities as (24) are investigated for many function spaces. See for example [6].

By Lemmas 2.8 and 4.2, we have the following lemma.

LEMMA 4.4. ([14, Lemma 4.3]) *Let* $X, Y \in \mathfrak{G}$ *and* $A_\alpha : X \to Y$, $M : Y \to Y$. *Suppose* (24) *holds. Then* $(S_{\alpha,Y}, T_Y)$ *is an optimal pair for* A_α *with respect to* $\mathfrak{G}$.

5. $L^{p(\cdot)}$ spaces and A_α

In this section we discuss optimal pairs A_α with respect to $\mathfrak{G}$ in Lemma 2.10. Recall that

$$1/p_\alpha(x) = 1/p(x) - (n-\alpha)/n.$$

LEMMA 5.1. *Suppose that* $q(\cdot)$ *satisfies* (P1′), (P3) *and* (P4). *Let* $n/q^- < \alpha \leqslant n$. *Assume* $h \in L^{q(\cdot)}(\mathbb{R}^n)$ *and set* $f(y) = |y|^{\alpha-n}|h(y)|$. *Then*

$$\|\widetilde{A_\alpha f}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leqslant C\|h\|_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

Proof. Set $f(y) = |y|^{\alpha-n}|h(y)|$ for $h \in L^{q(\cdot)}(\mathbb{R}^n)$. By [14, Lemma 3.3], Lemma 3.6 and Lemma 6.2 below, we have

$$\begin{aligned}
\|\widetilde{A_\alpha f}\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\leqslant C\|M(A_\alpha f)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\leqslant C\|A_\alpha f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\leqslant C\||x|^{n-\alpha}Mf\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\leqslant C\||y|^{n-\alpha}f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&= C\|h\|_{L^{q(\cdot)}(\mathbb{R}^n)},
\end{aligned}$$

as required. □

THEOREM 5.2. *Suppose that* $p(\cdot)$ *satisfies* (P1′), (P3) *and* (P4). *Let* $n(1 - 1/p^+) < \alpha \leqslant n$. *If* $X = L^{p(\cdot)}(\mathbb{R}^n)$ *and* $Y = L^{p_\alpha(\cdot)}(\mathbb{R}^n)$, *then* $(S_{\alpha,Y}, T_Y)$ *is an optimal pair for* A_α.

Proof. First we see from Lemmas 2.8 and 3.8 that $A_\alpha : X \to T_Y$. By Lemma 5.1 with $q(\cdot) = p_\alpha(\cdot)$, (24) in Lemma 4.2 holds. Hence it follows from Lemma 4.2 that $(S_{\alpha,Y}, T_Y)$ is an optimal pair for A_α. □

6. Weighted Lebesgue spaces and A_α

In this section, let $p(\cdot)$ and $q(\cdot)$ satisfy (P1'), (P3) and (P4). Let $\beta(\cdot)$ be a continuous function on $\mathbb{R}^n$ satisfying condition (P3), that is,

$$(\beta) \quad |\beta(x) - \beta| \leqslant C/\log(e+|x|) \qquad \text{for all } x \in \mathbb{R}^n;$$

here we write β for β_∞.

DEFINITION 6.1. Recall the definition of weighted Lebesgue spaces $L^{q(\cdot)}(\mathbb{R}^n, |x|^{\beta(\cdot)})$ as a set of all functions f with

$$\|f\|_{L^{q(\cdot)}(\mathbb{R}^n,|x|^{\beta(\cdot)})} = \inf\{\lambda > 0 : \int_{\mathbb{R}^n} (|f(x)/\lambda||x|^{\beta(x)})^{q(x)} dx \leqslant 1\} < \infty$$

(see [15]).

We know the following result (see [15, Theorem 1.1]).

LEMMA 6.2. *Let* $-n/q_\infty < \beta < n(1-1/q^-)$. *Then*

$$\|Mf\|_{L^{q(\cdot)}(\mathbb{R}^n,|x|^{\beta(\cdot)})} \leqslant C\|f\|_{L^{q(\cdot)}(\mathbb{R}^n,|x|^{\beta(\cdot)})}.$$

Now we prove the boundedness of A_α on weighted Lebesgue spaces.

LEMMA 6.3. *Let* $n(1-1/p^+) < \alpha \leqslant n$. *Let* $1/p^+ - 1/q^- > -1/q_\infty$ *and* $1/p^- - 1/q^+ < 1-1/q^-$. *Then*

$$\|A_\alpha f\|_{L^{q(\cdot)}(\mathbb{R}^n,|x|^{n(\frac{1}{p\alpha(\cdot)}-\frac{1}{q(\cdot)})})} \leqslant C\|f\|_{L^{q(\cdot)}(\mathbb{R}^n,|x|^{n(\frac{1}{p(\cdot)}-\frac{1}{q(\cdot)})})}.$$

Proof. Set $X = L^{q(\cdot)}(\mathbb{R}^n, |x|^{n(\frac{1}{p(\cdot)}-\frac{1}{q(\cdot)})})$ and $Y = L^{q(\cdot)}(\mathbb{R}^n, |x|^{n(\frac{1}{p\alpha(\cdot)}-\frac{1}{q(\cdot)})})$. Set $\beta(\cdot) = n(\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)})$. Since $1/p^+ - 1/q^- > -1/q_\infty$ and $1/p^- - 1/q^+ < 1 - 1/q^-$, $\beta(\cdot)$ satisfies (β) and $-n/q_\infty < \beta(\cdot) < n(1-1/q^-)$. We have

$$\begin{aligned}
&\int_{\mathbb{R}^n} |A_\alpha f(x)|^{q(x)} |x|^{n(\frac{q(x)}{p\alpha(x)}-1)} dx \\
&= \int_{\mathbb{R}^n} \left(\frac{1}{|B(0,|x|)|^{\alpha/n}} \int_{B(0,|x|)} |f(t)|dt\right)^{q(x)} |x|^{n(\frac{q(x)}{p\alpha(x)}-1)} dx \\
&\leqslant C\int_{\mathbb{R}^n} \left(\frac{1}{|x|^n} \int_{B(0,|x|)} |f(t)|dt\right)^{q(x)} |x|^{n(\frac{q(x)}{p\alpha(x)}-1)+q(x)(n-\alpha)} dx \\
&= C\int_{\mathbb{R}^n} \left(\frac{1}{|x|^n} \int_{B(0,|x|)} |f(t)|dt\right)^{q(x)} |x|^{\beta(x)q(x)} dx \\
&\leqslant C\int_{\mathbb{R}^n} (Mf(x)|x|^{\beta(x)})^{q(x)} dx.
\end{aligned}$$

By Lemma 6.2, we obtain

$$\|A_\alpha f\|_Y \leqslant C\|f\|_X,$$

as required. $\square$

Setting $\alpha = n$ in the previous lemma we obtain the next lemma.

REMARK 6.4. Let $1/p^+ - 1/q^- > -1/q_\infty$ and $1/p^- - 1/q^+ < 1 - 1/q^-$. Then

$$\|A_\alpha f\|_{L^{q(\cdot)}(\mathbb{R}^n,|x|^{n(\frac{1}{p(\cdot)}-\frac{1}{q(\cdot)})})} \leqslant C\|f\|_{L^{q(\cdot)}(\mathbb{R}^n,|x|^{n(\frac{1}{p(\cdot)}-\frac{1}{q(\cdot)})})}.$$

As an immediate consequence of Lemmas 6.2, 6.3 and 2.8, we obtain the following lemma.

LEMMA 6.5. *Let* $p^-, q^- > 1$ *and* $n(1-1/p^+) < \alpha \leqslant n$. *Let* $1/p^+ - 1/q^- > -1/q_\infty$ *and* $1/p^- - 1/q^+ < 1 - 1/q^-$. *Then*

$$\|A_\alpha f\|_{T_{L^{q(\cdot)}(\mathbb{R}^n, |x|^{n(\frac{1}{p\alpha(\cdot)} - \frac{1}{q(\cdot)})})}} \leqslant C\|f\|_{L^{q(\cdot)}(\mathbb{R}^n, |x|^{n(\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)})})}. \tag{25}$$

LEMMA 6.6. *Let* $n(1-1/p^+) < \alpha \leqslant n$. *Let* $1/p^+ - 1/q^- > -1/q_\infty$, $1/p^- - 1/q^+ < 1 - 1/q^-$ *and* $Y = L^{q(\cdot)}(\mathbb{R}^n, |x|^{n(\frac{1}{p\alpha(\cdot)} - \frac{1}{q(\cdot)})})$. *Assume* $h \in T_Y$ *and set* $f(x) = |x|^{\alpha-n}h(x)$. *Then*

$$\|A_\alpha f\|_{T_Y} \leqslant C\|h\|_{T_Y}.$$

Proof. Let $h \in T_Y$. Let $X = L^{q(\cdot)}(\mathbb{R}^n, |x|^{n(\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)})})$. By Lemma 6.5 with $f(x) = |x|^{\alpha-n}h(x)$, we have

$$\|A_\alpha f\|_{T_Y} \leqslant C\|f\|_X \leqslant C\|h\|_{T_Y},$$

since

$$\begin{aligned}
\int_{\mathbb{R}^n} |f(x)|^{q(x)}|x|^{n(\frac{q(x)}{p(x)}-1)}dx &= \int_{\mathbb{R}^n} (|x|^{\alpha-n}|h(x)|)^{q(x)}|x|^{n(\frac{q(x)}{p(x)}-1)}dx \\
&= \int_{\mathbb{R}^n} |h(x)|^{q(x)}|x|^{n(\frac{q(x)}{p\alpha(x)}-1)}dx \\
&\leqslant \int_{\mathbb{R}^n} \widetilde{h}(x)^{q(x)}|x|^{n(\frac{q(x)}{p\alpha(x)}-1)}dx. \qquad \square
\end{aligned}$$

We discuss optimal pairs A_α with respect to $\mathfrak{G}$ in Lemma 2.10. By Lemmas 6.5, 6.6 and 4.2, we obtain the following theorem.

THEOREM 6.7. *Let* $n(1-1/p^+) < \alpha \leqslant n$. *Let* $1/p^+ - 1/q^- > -1/q_\infty$ *and* $1/p^- - 1/q^+ < 1 - 1/q^-$. *If* $X = L^{q(\cdot)}(\mathbb{R}^n, |x|^{n(\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)})})$ *and* $Y = L^{q(\cdot)}(\mathbb{R}^n, |x|^{n(\frac{1}{p\alpha(\cdot)} - \frac{1}{q(\cdot)})})$, *then* $(S_{\alpha,Y}, T_Y)$ *is an optimal pair for* A_α.

Proof. Note from Lemma 6.5 that $A_\alpha : X \to T_Y$. Let $h \in T_Y$ and $f(x) = |x|^{\alpha-n}h(x)$. By Lemma 6.6, (24) in Lemma 4.2 holds. Hence, we see from Lemma 4.2 that $(S_{\alpha,Y}, T_Y)$ is an optimal pair for A_α. $\square$

7. Relation between T_Y and Y

We deal here with relations $T_Y = Y$ and $T_Y \subsetneqq Y$. Let us fix a weak Banach function space Y on $B(0,R)$, $0 < R \leqslant \infty$.

DEFINITION 7.1. Define for $\varepsilon > 0$ and $x \in B(0,R)$

$$c_\varepsilon(x) = \inf\{\|\chi_M\|_Y ; M \subset B(x,\varepsilon), |M| > 0\} \tag{26}$$

and

$$c(x) = \liminf_{\varepsilon \to 0_+} c_\varepsilon(x). \tag{27}$$

LEMMA 7.2. *The function $c(\cdot)$ is lower semi-continuous.*

Proof. Let $x_n \to x$ and choose $\lambda > 0$. There is a sequence $\varepsilon_n > 0$ tending to 0 with

$$c_{\varepsilon_n}(x) - \lambda \leqslant c(x) \leqslant c_{\varepsilon_n}(x) + \lambda$$

and we find n_0 such that $x_k \in B(x, \varepsilon_n)$ for $k > n_0$. From (27) there is a sequence $\eta_k > 0$ with $B(x_k, \eta_k) \subset B(x, \varepsilon_n)$

$$c_{\eta_k}(x_k) - \lambda \leqslant c(x_k) \leqslant c_{\eta_k}(x_k) + \lambda$$

and by (26) we can find a set $M_k \subset B(x_k, \eta_k)$ with

$$\|\chi_{M_k}\|_Y \leqslant c_{\eta_k}(x_k) + \lambda \leqslant c(x_k) + 2\lambda.$$

Since $M_k \subset B(x_k, \eta_k) \subset B(x, \varepsilon_n)$ we obtain

$$c(x) \leqslant c_{\varepsilon_n}(x) + \lambda \leqslant \|\chi_{M_k}\|_Y + \lambda \leqslant c(x_k) + 3\lambda$$

which means

$$c(x) \leqslant \liminf_{k \to \infty} c(x_k)$$

and finishes the proof. □

LEMMA 7.3. *Assume that there is $r > 0$ such that*

$$\inf\{c(x); |x| \geqslant r\} = 0. \tag{28}$$

Then $T_Y \subsetneqq Y$.

Proof. We know $T_Y \subseteq Y$. To see our lemma we must construct $f \in Y$ such that $f \notin T_Y$. From (28) there is a sequence x_n with $|x_n| \geqslant r$ and $c(x_n) \to 0$. By (27) there are $0 < \varepsilon_n < r/2$ such that $c_{\varepsilon_n}(x_n) \leqslant 1/(2n^3)$ and by (26) we can find sets $M_n \subset B(x_n, \varepsilon_n)$ with $|M_n| > 0$ such that $\|\chi_{M_n}\|_Y \leqslant 1/n^3$. Set

$$f(x) = \sum_{n=1}^{\infty} n \chi_{M_n}(x).$$

Then

$$\|f\|_Y \leqslant \sum_{n=1}^{\infty} n \|\chi_{M_n}\|_Y \leqslant \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

and so $f \in Y$. Moreover, fix n. Then for $|x| \leqslant r/2$ we have

$$\widetilde{f}(x) = \operatorname*{ess\,sup}_{|y| \geqslant |x|} |f(y)| \geqslant \operatorname*{ess\,sup}_{y \in M_n} |f(y)| \geqslant n.$$

Thus, $\widetilde{f}(x) = \infty$ on $B(0, r/2)$ which gives $\|\widetilde{f}\|_Y = \infty$ and so, $f \notin T_Y$. $\square$

Recall an easy fact $L_\infty \hookrightarrow Y$ provided $R < \infty$. Actually, taking f we have

$$\|f\chi_{B(0,R)}\|_Y \leqslant \|\chi_{B(0,R)}\|_Y \; \|f\|_\infty := D\|f\|_\infty. \tag{29}$$

LEMMA 7.4. *Let $0 \leqslant r_1 < R \leqslant \infty$ and denote $\Omega_1 = \{x; r_1 < |x| < R\}$. Assume that there is $\delta > 0$ with $c(x) \geqslant \delta$ for almost all $x \in \Omega_1$. Then*

$$\|f\chi_{\Omega_1}\|_\infty \leqslant \frac{1}{\delta}\|f\chi_{\Omega_1}\|_Y$$

holds for all f.

Proof. Let $A \subset \Omega_1$ be a set of full measure with $c(x) \geqslant \delta$ for all $x \in A$. Denote $d := \|f\chi_{\Omega_1}\|_\infty$. Choose an arbitrary $\lambda > 0$. Then there exists a set $M \subset \Omega_1$, $|M| > 0$ and $|f(z)| \geqslant d - \lambda$ for all $z \in M$. Let $x \in M$ be a Lebesgue point. By (27) there exists a sequence $\varepsilon_n \to 0$ such that $c(x) \leqslant c_{\varepsilon_n}(x) + \frac{1}{n}$. Set $P_n = M \cap B(x, \varepsilon_n)$. Then $|P_n| > 0$, $P_n \subset B(x, \varepsilon_n)$ and so, $c_{\varepsilon_n}(x) \leqslant \|\chi_{P_n}\|_Y$. Since $P_n \subset M$, we finally obtain

$$\delta \leqslant c(x) \leqslant c_{\varepsilon_n}(x) + \frac{1}{n} \leqslant \|\chi_{P_n}\|_Y + \frac{1}{n} \leqslant \frac{1}{d-\lambda}\|f\chi_{P_n}\|_Y + \frac{1}{n} \leqslant \frac{1}{d-\lambda}\|f\chi_{\Omega_1}\|_Y + \frac{1}{n}.$$

Letting $n \to \infty$, we obtain

$$(\|f\chi_{\Omega_1}\|_\infty - \lambda)\delta = (d-\lambda)\delta \leqslant \|f\chi_{\Omega_1}\|_Y.$$

Since λ was chosen arbitrarily, the lemma follows. $\square$

PROPOSITION 7.5. *Let $T_Y = Y$ and $0 < r_1 < r_2 < \infty$, $r_2 \leqslant R$. Denote $\Omega_2 = \{x; r_1 < |x| < r_2\}$. Then there exist two positive constants c_1, c_2 such that*

$$c_1\|f\chi_{\Omega_2}\|_\infty \leqslant \|f\chi_{\Omega_2}\|_Y \leqslant c_2\|f\chi_{\Omega_2}\|_\infty.$$

Proof. Since Ω_2 is bounded we have $\|f\chi_{\Omega_2}\|_Y \leqslant D\|f\chi_{\Omega_2}\|_\infty$ by (29). Let us prove the opposite inequality. By Lemma 7.3 there exists a $\delta > 0$ such that $c(x) \geqslant \delta$ in Ω_2. By Lemma 7.4 we obtain $\|f\chi_{\Omega_2}\|_\infty \leqslant 1/\delta\|f\chi_{\Omega_2}\|_Y$, which proves the lemma. $\square$

The previous proposition claims that the norm in Y on sets $\{x; r_1 < |x| < r_2\}$ behaves as L_∞ norm provided $T_Y = Y$.

In the next let us restrict ourselves to the case $R < \infty$. Define a function on $[0, R]$ by

$$\delta(r) = \operatorname{ess\,inf}\{c(x); r < |x| < R\}.$$

It is easy to see that $\delta(\cdot)$ is nondecreasing. We know in this case that $B(0,R)$ is bounded and the norm on sets $\{x; 0<r<|x|<R\}$ is in fact L_∞ norm provided $T_Y = Y$. Moreover, if $\delta := \lim_{r\to 0_+}\delta(r) > 0$ we have $c(x) \geqslant \delta$ and by Lemma 7.3 and Lemma 7.4 we know $Y = L_\infty$ with equivalent norms. In the next we will investigate a relation between properties $\lim_{r\to 0_+}\delta(r) = 0$ and $T_Y = Y$. We will find two examples of spaces Y. In both spaces the property $\lim_{r\to 0_+}\delta(r) = 0$ holds but the first one satisfies $T_Y = Y$ and the second one satisfies $T_Y \subsetneqq Y$.

EXAMPLE 7.6. There exists a space Y such that

(i) $\delta(r) > 0$ for all $r > 0$,

(ii) $\lim_{r\to 0_+}\delta(r) = 0$,

(iii) $T_Y = Y$.

Proof. Define

$$\|f\|_Y = \int_{\mathbf{B}} \operatorname*{ess\,sup}_{|t|\leqslant|y|<1} |f(y)|dt = \int_{\mathbf{B}} \tilde{f}(t)dt = \|\tilde{f}\|_Y = \|f\|_{T_Y};$$

set $f = 0$ outside $\mathbf{B}$ for convenience. Then $Y = T_Y$ by definition, which proves (iii).

Let us estimate $c(x)$ for $x \in \mathbf{B}$. By (27) and (26), we have

$$\begin{aligned} c(x) &= \liminf_{\varepsilon\to 0_+} c_\varepsilon(x) \leqslant \liminf_{\varepsilon\to 0_+} \int_{\mathbf{B}} \chi_{B(0,|x|+\varepsilon)}(t) \\ &= \liminf_{\varepsilon\to 0_+} C(|x|+\varepsilon)^n dt = C|x|^n. \end{aligned}$$

Thus

$$\lim_{r\to 0_+}\delta(r) = \lim_{r\to 0_+} \operatorname{ess\,inf}\{c(x); r\leqslant|x|\leqslant 1\} \leqslant \lim_{r\to 0_+} Cr^n = 0,$$

which proves (i) and (ii). $\square$

EXAMPLE 7.7. There exists a space Y such that

(i) $\delta(r) > 0$ for all $r > 0$,

(ii) $\lim_{r\to 0_+}\delta(r) = 0$,

(iii) $T_Y \subsetneqq Y$.

Proof. Consider the family of all measurable functions f such that

$$\|f\|_Y = \sum_{n=1}^{\infty} \frac{1}{n^2}\|f\|_{L^\infty(\frac{1}{n+1},\frac{1}{n})} < \infty.$$

Set

$$f=\sum_{n=1}^{\infty} n^2\chi_{(\frac{1}{n^2+1},\frac{1}{n^2})}.$$

Then

$$\begin{aligned}\tilde{f} = & 1^2\chi_{(\frac{1}{1^2+1},\frac{1}{1^2})}+1^2\chi_{(\frac{1}{1^2+2},\frac{1}{1^2+1})}+1^2\chi_{(\frac{1}{1^2+3},\frac{1}{1^2+2})}\\ & +2^2\chi_{(\frac{1}{2^2+1},\frac{1}{2^2})}+2^2\chi_{(\frac{1}{2^2+2},\frac{1}{2^2+1})}+2^2\chi_{(\frac{1}{2^2+3},\frac{1}{2^2+2})}++2^2\chi_{(\frac{1}{2^2+4},\frac{1}{2^2+3})}+2^2\chi_{(\frac{1}{2^2+5},\frac{1}{2^2+4})}\\ & +3^2\chi_{(\frac{1}{3^2+1},\frac{1}{3^2})}+3^2\chi_{(\frac{1}{3^2+2},\frac{1}{3^2+1})}+\cdots\\ & +\cdots.\end{aligned}$$

Hence

$$\begin{aligned}\|\tilde{f}\|_Y = & 1^2\cdot\frac{1}{(1^2)^2}+1^2\cdot\frac{1}{(1^2+1)^2}+1^2\cdot\frac{1}{(1^2+2)^2}\\ & +2^2\cdot\frac{1}{(2^2)^2}+2^2\cdot\frac{1}{(2^2+1)^2}+2^2\cdot\frac{1}{(2^2+2)^2}+2^2\cdot\frac{1}{(2^2+3)^2}+2^2\cdot\frac{1}{(2^2+4)^2}\\ & +3^2\cdot\frac{1}{(3^2)^2}+3^2\cdot\frac{1}{(3^2+1)^2}+\cdots+3^2\cdot\frac{1}{(4^2-1)^2}\\ & +\cdots\\ \geqslant & 1^2\cdot\frac{1}{(2^2-1)^2}\cdot(2^2-1^2)+2^2\cdot\frac{1}{(3^2-1)^2}\cdot(3^2-2^2)+\cdots\\ & +n^2\cdot\frac{1}{((n+1)^2-1)^2}\cdot((n+1)^2-n^2)+\cdots\\ = & \sum_{n=1}^{\infty}\frac{2n+1}{(n+2)^2}=\infty,\end{aligned}$$

which proves (iii).

Obviously, since $c(x)=1/n^2$ when $1/(n+1)<x<1/n$, we have the following:

- $\delta(r)>0$ for $0<r<1$;
- $\lim\limits_{r\to 0_+}\delta(r)=0$. □

REFERENCES

[1] C. BENNETT AND R. SHARPLEY, *Interpolations of operators*, Pure and Appl. Math., vol. 129, Academic Press, New York, 1988.

[2] C. CAPONE, D. CRUZ-URIBE AND A. FIORENZA, *The fractional maximal operator on variable L^p space*, Rev. Mat. Iberoamericana **23**, 3 (2007), 743–770.

[3] D. CRUZ-URIBE, A. FIORENZA AND C. J. NEUGEBAUER, *The maximal function on variable L^p space*, Ann. Acad. Sci. Fenn. Math. **28, 29**, (2003, 2004), 223–238, 247–249.

[4] D. E. EDMUNDS AND A. NEKVINDA, *Averaging operators on $l^{\{p_n\}}$ and $L^{p(x)}$*, Math. Inequal. Appl. **5** (2) (2002), 235–246.

[5] A. GOGATISHVILI, B. OPIC AND L. PICK, *Weighted inequalities for Hardy-type operators involving suprema*, Collect. Math. **57**, 3 (2006), 227–255.

[6] A. GOGATISHVILI AND V. D. STEPANOV, *Reduction theorems for weighted integral inequalities on the cone of monotone functions* (Russian), Uspekhi Mat. Nauk **68** (2013 no. 4 (412)), 3–68.
[7] K.-G. GROSSE-ERDMANN, *The blocking technique, weighted mean operators and Hardy's inequality*, Lect. Notes Math. 1679, Springer, Berlin etc., 1998.
[8] J. HEINONEN, T. KILPELÄINEN AND O. MARTIO, *Nonlinear potential theory of degenerate elliptic equations*, Clarendon Press, 1993.
[9] O. KOVÁČIK AND J. RÁKOSNÍK, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J., Praha **41** (1991), 592–618.
[10] A. KUFNER AND L. E. PERSSON, *Weighted inequalities of Hardy type*, World Scientific Publishing Co., Inc., River Edge, NJ, 2003.
[11] J. LANG AND A. NEKVINDA, *A difference between continuous and absolutely continuous norms in Banach function spaces*, Czechoslovak Math. J., Praha **47** (1997), 221–232.
[12] J. LANG, A. NEKVINDA AND J. RÁKOSNÍK, *Continuous norms and absolutely continuous norms in Banach function spaces are not the same*, Real Anal. Exch. **26**(1) (2000/2001), 345–364.
[13] Y. MIZUTA, A. NEKVINDA AND T. SHIMOMURA, *Hardy averaging operator on generalized Banach function spaces and duality*, Z. Anal. Anwend. **32** (2013), 233–255.
[14] Y. MIZUTA, A. NEKVINDA AND T. SHIMOMURA, *Optimal estimates for the fractional Hardy operator*, Studia Math. **227** (2015), 1–19.
[15] Y. MIZUTA AND T. SHIMOMURA, *Weighted Sobolev inequality in Musielak-Orlicz space*, J. Math. Anal. Appl. **338** (2012), 86–97.
[16] B. MUCKENHOUPT, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226.
[17] A. NEKVINDA AND L. PICK, *Optimal estimates for the Hardy averaging operator*, Math. Nachr. **283** (2010), 262–271.
[18] A. NEKVINDA AND L. PICK, *Duals of optimal spaces for the Hardy averaging operator*, Z. Anal. Anwend. **30** (2011), 435–456.
[19] E. M. STEIN, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, 1970.

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Yoshihiro Mizuta
4-13-11 Hachi-Hon-Matsu-Minami
Higashi-Hiroshima 739-0144, Japan
e-mail: `yomizuta@hiroshima-u.ac.jp`

Aleš Nekvinda
Faculty of Civil Engineering Czech technical University
Thákurova 7, 166 29 Praha 6, Czech Republic
e-mail: `nekvinda@fsv.cvut.cz`

Tetsu Shimomura
Department of Mathematics
Graduate School of Education, Hiroshima University
Higashi-Hiroshima 739-8524, Japan
e-mail: `tshimo@hiroshima-u.ac.jp`