Mathematical
Inequalities
& Applications
Volume 22, Number 2 (2019), 463–486 doi:10.7153/mia-2019-22-33

DILATION–COMMUTING OPERATORS ON POWER–WEIGHTED ORLICZ CLASSES

RON KERMAN, RAMA RAWAT AND RAJESH K. SINGH

(*Communicated by J. Pečarić*)

Abstract. Let Φ be a nondecreasing function from $\mathbb{R}_+ = (0,\infty)$ onto itself. Fix $\gamma \in \mathbb{R} = (-\infty,\infty)$ and let $L_{\Phi,t^\gamma}(\mathbb{R}_+)$ be the set of all Lebesgue-measurable functions f from $\mathbb{R}_+$ to $\mathbb{R}$ for which

$$\int_{\mathbb{R}_+} \Phi(k|f(t)|)\, t^\gamma dt < \infty$$

for some $k>0$. Define the gauge ρ_{Φ,t^γ} at $f \in L_{\Phi,t^\gamma}(\mathbb{R}_+)$ by

$$\rho_{\Phi,t^\gamma}(f) = \inf\left\{\lambda > 0 : \int_{\mathbb{R}_+} \Phi\left(\frac{|f(t)|}{\lambda}\right)\frac{t^\gamma}{\lambda} dt \leqslant 1\right\}.$$

Our principal goal in this paper is to find conditions on the nondecreasing functions Φ_1 and Φ_2, $\gamma \in \mathbb{R}$ and an operator T so that the assertions

$$\rho_{\Phi_1,t^\gamma}(Tf) \leqslant C\rho_{\Phi_2,t^\gamma}(f) \tag{G}$$

and

$$\int_{\mathbb{R}_+} \Phi_1(|(Tf)(t)|)\, t^\gamma dt \leqslant K \int_{\mathbb{R}_+} \Phi_2(K|f(s)|)\, s^\gamma ds, \tag{M}$$

concerning $f \in S(\mathbb{R}_+)$, the class of simple functions supported in $\mathbb{R}_+$, are equivalent and to then find necessary and sufficient conditions in order that (M) holds.

In addition, we investigate the connection between (G) and the assertion that

$$T : \mathring{L}_{\Phi_2,t^\gamma}(\mathbb{R}_+) \to L_{\Phi_1,t^\gamma}(\mathbb{R}_+),$$

where $\mathring{L}_{\Phi_2,t^\gamma}(\mathbb{R}_+)$ is the closure of $S(\mathbb{R}_+)$ in $L_{\Phi_2,t^\gamma}(\mathbb{R}_+)$.

1. Introduction

Let the operator T map the set, $S(\mathbb{R}_+)$, of simple, Lebesgue-measurable functions on $\mathbb{R}_+ = (0,\infty)$ into $M(\mathbb{R}_+)$, the class of Lebesgue-measurable functions on $\mathbb{R}_+$. Suppose that T is positively homogeneous in the sense that

$$|T(cf)| = |c||Tf|,\ f \in S(\mathbb{R}_+),\ c \in \mathbb{R},$$

with, moreover,

$$(Tf)(\lambda t) = T(f(\lambda\ \cdot))(t),\ \lambda, t \in \mathbb{R}_+.$$

Mathematics subject classification (2010): Primary 42B25, 26D15, Secondary 28A25.
Keywords and phrases: Dilation-commuting operators, Orlicz spaces, norm inequalities, modular inequalities, Hardy operator, maximal function, Hilbert transform.

We call such a T a *dilation-commuting operator*.

Our aim in this paper is to determine when certain dilation-commuting operators map functions in a so-called Orlicz class, $L_{\Phi_2,t^\gamma}(\mathbb{R}_+)$, into another such Orlicz class, $L_{\Phi_1,t^\gamma}(\mathbb{R}_+)$. Here, the $\Phi_i, i=1,2$, are nonnegative, nondecreasing functions on $\mathbb{R}_+$, $\gamma \in \mathbb{R}$ and, for any given nonnegative, nondecreasing function Φ from $\mathbb{R}_+$ onto itself,

$$L_{\Phi,t^\gamma}(\mathbb{R}_+) = \left\{ f \in M(\mathbb{R}_+) : \int_{\mathbb{R}_+} \Phi(k|f(t)|)t^\gamma dt < \infty, \text{ for some } k \in \mathbb{R}_+ \right\}.$$

One way to measure the size of an $f \in L_{\Phi,t^\gamma}(\mathbb{R}_+)$ is by its gauge

$$\rho_{\Phi,t^\gamma}(f) = \inf\left\{ \lambda > 0 : \int_{\mathbb{R}_+} \Phi\left(\frac{|f(t)|}{\lambda}\right) \frac{t^\gamma}{\lambda} dt \leqslant 1 \right\}.$$

The class $L_{\Phi,t^\gamma}(\mathbb{R}_+)$ can be shown to be a complete linear topological space under the metric

$$d_{\Phi,t^\gamma}(f,g) = \rho_{\Phi,t^\gamma}(f-g), \quad f,g \in L_{\Phi,t^\gamma}(\mathbb{R}_+).$$

The fundamental result in this paper, the one on which all others are based, is

THEOREM A. *Let T be a dilation-commuting operator from $S(\mathbb{R}_+)$ to $M(\mathbb{R}_+)$. Suppose Φ_1 and Φ_2 are nonnegative, nondecreasing functions from $\mathbb{R}_+$ onto itself and fix $\gamma \in \mathbb{R}, \gamma \neq -1$. Then, there exists $C > 0$, independent of $f \in S(\mathbb{R}_+)$, such that*

$$\rho_{\Phi_1,t^\gamma}(Tf) \leqslant C\rho_{\Phi_2,t^\gamma}(f) \tag{1.1}$$

if and only if

$$\int_{\mathbb{R}_+} \Phi_1\left(|(Tf)(t)|\right) t^\gamma dt \leqslant K \int_{\mathbb{R}_+} \Phi_2\left(K|f(s)|\right) s^\gamma ds, \tag{1.2}$$

in which $K > 0$ is independent of $f \in S(\mathbb{R}_+)$.

REMARKS 1.1. **1.** When T is linear, (1.1) implies

$$d_{\Phi_1,t^\gamma}(Tf,Tg) \leqslant C d_{\Phi_2,t^\gamma}(f,g), \quad f,g \in S(\mathbb{R}_+),$$

and hence

$$T : \mathring{L}_{\Phi_2,t^\gamma}(\mathbb{R}_+) \to L_{\Phi_1,t^\gamma}(\mathbb{R}_+) \tag{1.3}$$

continuously. Further, if Φ_1 and Φ_2 are convex, and hence $L_{\Phi_1,t^\gamma}(\mathbb{R}_+)$ and $L_{\Phi_2,t^\gamma}(\mathbb{R}_+)$ are Banach spaces, a well-known result from functional analysis [6, Chapter 1, Proposition 2.5] guarantees (1.1) equivalent to (1.3).

2. (1.2) is simpler than (1.1) and hence easier to work with.

3. A modular inequality, like (1.2), implies a gauge inequality, like (1.1), in a rather general context, as is seen in Proposition 3.1 below. Theorem A asserts the two inequalities are equivalent for dilation-commuting operators in the context of power weights, such weights being required for their homogeneity property.

4. One readily works out the variant of Theorem A in which $\mathbb{R}_+$ is replaced by $\mathbb{R}^n$, $n = 1,2,\ldots$, and t^γ by $|x|^\gamma = (x_1^2 + x_2^2 + \ldots + x_n^2)^{\gamma/2}$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. In this context $S(\mathbb{R}^n)$ denotes the class of simple functions supported in $\mathbb{R}^n \setminus \{(0,\ldots,0)\}$ and $\mathring{L}_{\Phi,|x|^\gamma}(\mathbb{R}^n)$ the closure of $S(\mathbb{R}^n)$ in $L_{\Phi,|x|^\gamma}(\mathbb{R}^n)$.

The specific dilation-commuting operators we focus on are the Hardy operators

$$(P_pf)(t) = t^{-\frac{1}{p}} \int_0^t f(s)s^{\frac{1}{p}-1}ds \text{ and } (Q_qf)(t) = t^{-\frac{1}{q}} \int_t^\infty f(s)s^{\frac{1}{q}-1}ds, \quad t \in \mathbb{R}_+,$$

where $p,q \in \mathbb{R}_+$ and $f \in S(\mathbb{R}_+)$; the Hardy-Littlewood maximal function

$$(Mf)(x) = \sup_{\substack{x\in I \\ I \text{ is an interval}}} \frac{1}{|I|}\int_I |f(y)|dy, \quad f \in S(\mathbb{R}),\ x \in \mathbb{R};$$

the Hilbert transform

$$(Hf)(x) = \frac{1}{\pi}\ (\mathrm{P})\int_{\mathbb{R}} \frac{f(y)}{x-y}dy = \lim_{\varepsilon\to 0^+} \frac{1}{\pi}\int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}dy,$$

with $f \in S(\mathbb{R})$, $x \in \mathbb{R}$.

REMARKS 1.2. **1.** The inequality (1.1) is characterized for $T = P_p$ and $T = Q_q$ in [4] when Φ_1 and Φ_2 are convex and $\gamma = 0$. Assuming, in addition, that $p = q = 1$, one can, using known results, characterize (1.1) for $T = M$ and $T = H$ as well.

2. Necessary and sufficient conditions to guarantee (1.2) are given in [2] for $T = M$ and (hence $T = H$), $\Phi_1 = \Phi_2 = \Phi$ is convex.

3. The results for M and H in $\mathbb{R}$ have analogues in $\mathbb{R}^n$, $n \geqslant 2$, involving the n-dimensional version of M and the Calderón-Zygmund operators discussed in [13].

The above operators are treated in Section 4, Section 5 and Section 6, respectively, following the proof of Theorem A in Section 3. Background on gauges like ρ_{Φ,t^γ} is given in Section 2; in particular, we explore when the continuity of a mapping such as (1.3) implies a corresponding gauge inequality like (1.1). Appendices I and II treat general modular inequalities for Hardy operators and Hardy-Littlewood maximal functions, in that order.

2. Orlicz classes

Let $(X, \mathscr{M}, \mu)$ be a totally σ-finite measure space and denote by $M(X)$ the set of μ-measurable functions from X to the real line $\mathbb{R}$. Given a nondecreasing function Φ from $\mathbb{R}_+$ onto itself its corresponding Orlicz class is

$$L_{\Phi,\mu}(X) = \left\{ f \in M(X) : \int_X \Phi(k|f(x)|)d\mu(x) < \infty, \text{ for some } k \in \mathbb{R}_+ \right\}.$$

The functional $\rho_{\Phi,\mu}$ defined at $f \in M(X)$ by

$$\rho_{\Phi,\mu}(f) = \inf\left\{ \lambda > 0 : \int_X \Phi\left(\frac{|f(x)|}{\lambda}\right)\frac{d\mu(x)}{\lambda} \leqslant 1 \right\}$$

is finite if and only if $f \in L_{\Phi,\mu}(X)$.

This functional has the following properties

1. $\rho_{\Phi,\mu}(f) = \rho_{\Phi,\mu}(|f|) \geqslant 0$, with $\rho_{\Phi,\mu}(f) = 0$ if and only if $f = 0$ μ-a.e.;
2. $\rho_{\Phi,\mu}(cf)$ is a nondecreasing function of c from $\mathbb{R}_+$ onto itself if $f \neq 0$ μ-a.e.;
3. $\rho_{\Phi,\mu}(f+g) \leqslant \rho_{\Phi,\mu}(f) + \rho_{\Phi,\mu}(g)$;
4. $0 \leqslant f_n \uparrow f$ implies $\rho_{\Phi,\mu}(f_n) \uparrow \rho_{\Phi,\mu}(f)$;
5. $\rho_{\Phi,\mu}(\chi_E) < \infty$ for all $E \subset X$ such that $\mu(E) < \infty$.

The functional $\rho_{\Phi,\mu}$ is a so-called F-norm on the linear space $L_{\Phi,\mu}(X)$ that makes it into a complete linear topological space under the metric

$$d_{\Phi,\mu}(f,g) = \rho_{\Phi,\mu}(f-g).$$

Our function Φ is said to be s-convex with fixed s, $0 < s \leqslant 1$, if

$$\Phi(\alpha x + \beta y) \leqslant \alpha^s \Phi(x) + \beta^s \Phi(y),$$

where $\alpha, \beta, x, y \in \mathbb{R}_+$ and $\alpha^s + \beta^s = 1$. For such a Φ, the functional

$$\rho^{(s)}_{\Phi,\mu}(f) = \inf\left\{ \lambda > 0 : \int_X \Phi\left(\frac{|f(x)|}{\lambda^{1/s}} \right) d\mu(x) \leqslant 1 \right\}$$

satisfies

$$\rho^{(s)}_{\Phi,\mu}(cf) = c^s \rho^{(s)}_{\Phi,\mu}(f), \quad c \geqslant 0,$$

as well as properties $1-5$ above, so, in particular, $\rho^{(1)}_{\Phi,\mu}(f)$ is a norm. One has $f \in M(X)$ belonging to $L_{\Phi,\mu}(X)$ if and only if $\rho^{(s)}_{\Phi,\mu}(f) < \infty$, with $L_{\Phi,\mu}(X)$ a complete linear topological space under the metric

$$d^{(s)}_{\Phi,\mu}(f,g) = \rho^{(s)}_{\Phi,\mu}(f-g), \quad f,g \in L_{\Phi,\mu}(X).$$

See [9, Theorem 1.2].

LEMMA 2.1. *Let $(X, \mathscr{M}, \mu)$ be a totally σ-finite measure space. Suppose Φ is a nondecreasing function from $\mathbb{R}_+$ onto itself which is s-convex for a fixed s, $0 < s \leqslant 1$. Then, the topologies induced on $L_{\Phi,\mu}(X)$ by the metrics $d_{\Phi,\mu}$ and $d^{(s)}_{\Phi,\mu}$ are homeomorphic.*

Proof. The equivalence of the topologies amounts to the assertion that, given $f, f_j \in L_\Phi(X,\mu)$, $j = 1, 2, \ldots$, one has

(i)

$$\lim_{j\to\infty} \rho_{\Phi,\mu}(f - f_j) = 0$$

if and only if

(ii)

$$\lim_{j\to\infty}\rho^{(s)}_{\Phi,\mu}(f-f_j)=0.$$

According to [9, Remarks 3, pp. 7–8], $\rho_{\Phi,\mu}(f)<1$ implies $\rho^{(s)}_{\Phi,\mu}(f)\leqslant\rho_{\Phi,\mu}(f)^s$ and $\rho^{(s)}_{\Phi,\mu}(f)<1$ implies $\rho_{\Phi,\mu}(f)\leqslant\rho^{(s)}_{\Phi,\mu}(f)^{\frac{1}{1+s}}$, $f\in M(X)$.

But, given (i), $\rho_{\Phi,\mu}(f-f_j)<1$ when j is sufficiently large. Restricting attention to those j, we get

$$\rho^{(s)}_{\Phi,\mu}(f-f_j)\leqslant\rho_{\Phi,\mu}(f-f_j)^s\to 0,\quad\text{as } j\to\infty.$$

Similarly, (ii) ensures, for j sufficiently large,

$$\rho_{\Phi,\mu}(f-f_j)\leqslant\rho^{(s)}_{\Phi,\mu}(f-f_j)^{\frac{1}{1+s}}\to 0,\quad\text{as } j\to\infty.\quad\square$$

Modulars, such as $\rho_{\Phi,\mu}$, were first studied in [10] and [11]. The s-convex modulars, like $\rho^{(s)}_{\Phi,\mu}$, appear in [12]. A systematic study of all this is given in [9].

PROPOSITION 2.1. *Let $(X,\mathscr{M},\mu)$ and $(Y,\mathscr{N},\nu)$ be totally σ-finite measure spaces. Suppose Φ_1 and Φ_2 are nondecreasing s-convex functions from $\mathbb{R}_+$ onto itself, where s is fixed in $(0,1]$. Then, any linear operator T mapping $L_{\Phi_2,\nu}(Y)$ into $L_{\Phi_1,\mu}(X)$ continuously with respect to the metrics $d_{\Phi_2,\nu}$ and $d_{\Phi_1,\mu}$ satisfies*

$$\rho^{(s)}_{\Phi_1,\mu}(Tf)\leqslant C\rho^{(s)}_{\Phi_2,\nu}(f),$$

in which $C=C(T)>0$ is independent of $f\in L_{\Phi_2,\nu}(Y)$.

Proof. Fix $f_0\in L_{\Phi_2,\nu}(Y)$. Since T is continuous at f_0, there is, in view of Lemma 2.1, a $\delta>0$ such that

$$\rho^{(s)}_{\Phi_1,\mu}(Tf-Tf_0)<1$$

for all $f\in L_{\Phi_2,\nu}(Y)$ satisfying $\rho^{(s)}_{\Phi_2,\nu}(f-f_0)<\delta$. Given $f\in L_{\Phi_2,\nu}(Y)$, set $g=\frac{\eta^{1/s}}{\rho^{(s)}_{\Phi_2,\nu}(f)^{1/s}}f$, for a fixed $\eta, 0<\eta<\delta$. Then,

$$\frac{\eta^{1/s}}{\rho^{(s)}_{\Phi_2,\nu}(f)^{1/s}}Tf=Tg=T(g+f_0)-Tf_0$$

and

$$\rho^{(s)}_{\Phi_1,\mu}\left(\frac{\eta^{1/s}}{\rho^{(s)}_{\Phi_2,\nu}(f)^{1/s}}Tf\right)=\rho^{(s)}_{\Phi_1,\mu}\left(T(g+f_0)-Tf_0\right)<1,$$

since

$$\rho^{(s)}_{\Phi_2,\nu}(g+f_0-f_0)=\rho^{(s)}_{\Phi_2,\nu}(g)\leqslant\eta<\delta.$$

Indeed,

$$\int_Y \Phi_2\left(\frac{g}{\eta^{1/s}}\right)d\nu=\int_Y \Phi_2\left(\frac{f}{\rho^{(s)}_{\Phi_2,\nu}(f)^{1/s}}\right)d\nu\leqslant 1.$$

Now,

$$\rho^{(s)}_{\Phi_1,\mu}\left(\frac{\eta^{1/s}}{\rho^{(s)}_{\Phi_2,\nu}(f)^{1/s}}Tf\right)<1,$$

implies,

$$\int_X \Phi_1\left(\frac{Tf}{\rho^{(s)}_{\Phi_2,\nu}(f)^{1/s}/\eta^{1/s}}\right)d\mu\leqslant 1,$$

which, in turn, means that

$$\rho^{(s)}_{\Phi_1,\mu}(Tf)\leqslant\eta^{-1}\rho^{(s)}_{\Phi_2,\nu}(f).\quad\square$$

Our particular concern in this paper is with the measure $\mu=t^{\gamma}dt$, $\gamma\in\mathbb{R}$, on the Lebesgue-measurable subsets of $\mathbb{R}_+$. For simplicity we write $\rho_{\Phi,t^{\gamma}}$ and $L_{\Phi,t^{\gamma}}$ rather than $\rho_{\Phi,t^{\gamma}dt}$ and $L_{\Phi,t^{\gamma}dt}$.

3. Proof of Theorem A

We will require the connection between a modular inequality, like (3.1), and certain gauge inequalities, (3.2). This connection is given, in some generality, in the following result.

PROPOSITION 3.1. *Let t,u,v and w be positive measurable functions, called weights, on $\mathbb{R}_+$. Suppose Φ_1 and Φ_2 are nonnegative, nondecreasing functions from $\mathbb{R}_+$ onto itself. Given $\varepsilon>0$, define the weighted gauge $\rho_{\Phi_2,u,\varepsilon v}$ by*

$$\rho_{\Phi_2,u,\varepsilon v}(f)=\inf\left\{\lambda>0:\int_{\mathbb{R}_+}\Phi_2\left(\frac{u(y)|f(y)|}{\lambda}\right)\frac{\varepsilon}{\lambda}v(y)dy\leqslant 1\right\},\quad f\in M(\mathbb{R}_+).$$

Define $\rho_{\Phi_1,t,\varepsilon w}$ similarly.

Then, a positively homogeneous operator T from $S(\mathbb{R}_+)$ to $M(\mathbb{R}_+)$ satisfies

$$\int_{\mathbb{R}_+}\Phi_1\left(t(x)|(Tf)(x)|\right)w(x)dx\leqslant K\int_{\mathbb{R}_+}\Phi_2\left(Ku(y)|f(y)|\right)v(y)dy,\tag{3.1}$$

if and only if it satisfies the uniform gauge inequalities

$$\rho_{\Phi_1,t,\varepsilon w}(Tf)\leqslant C\rho_{\Phi_2,u,\varepsilon v}(f),\tag{3.2}$$

in which $K>0$ is independent of $f\in S(\mathbb{R}_+)$ and $C>0$ is independent of both $f\in S(\mathbb{R}_+)$ and $\varepsilon>0$.

REMARK 3.1. Taking $\Phi_1 = \Phi_2 = \Phi$ convex and $t = u = 1$ yields a special case of Proposition 2.5 in [1]

A proof similar to the one for Proposition 3.1 yields the following result.

PROPOSITION 3.2. *Let t, u, v and w be weights on $\mathbb{R}_+$. Suppose Φ_1 and Φ_2 are nonnegative, nondecreasing functions from $\mathbb{R}_+$ onto itself, which are s-convex for some s, $0 < s \leqslant 1$. Given $\varepsilon > 0$, define the weighted s-gauge $\rho^{(s)}_{\Phi_2,u,\varepsilon v}$ by*

$$\rho^{(s)}_{\Phi_2,u,\varepsilon v}(f) = \inf\left\{\lambda > 0 : \int_{\mathbb{R}_+} \Phi_2\left(\frac{u(y)|f(y)|}{\lambda^{1/s}}\right)\varepsilon v(y)dy \leqslant 1\right\}, \quad f \in M(\mathbb{R}_+).$$

Define $\rho^{(s)}_{\Phi_1,t,\varepsilon w}$ similarly.

Then, a positively homogeneous operator T from $S(\mathbb{R}_+)$ to $M(\mathbb{R}_+)$ satisfies the modular inequality (3.1) if and only if it satisfies the uniform s-gauge inequalities

$$\rho^{(s)}_{\Phi_1,t,\varepsilon w}(Tf) \leqslant C^{(s)}\, \rho^{(s)}_{\Phi_2,u,\varepsilon v}(f),$$

in which $C^{(s)} > 0$ is independent of both $f \in S(\mathbb{R}_+)$ and $\varepsilon > 0$.

Proof of Proposition 3.1. Suppose (3.2) holds. Fix $f \in S(\mathbb{R}_+)$, $f \not\equiv 0$, and put

$$\varepsilon = \left(\int_{\mathbb{R}_+} \Phi_2\left(u(y)|f(y)|\right) v(y)dy\right)^{-1}.$$

Then,

$$\int_{\mathbb{R}_+} \Phi_2\left(u(y)|f(y)|\right)\varepsilon v(y)dy = 1,$$

so

$$\rho_{\Phi_2,u,\varepsilon v}(f) \leqslant 1,$$

whence (3.2) implies

$$\rho_{\Phi_1,t,\varepsilon w}(Tf) \leqslant C.$$

Thus,

$$\int_{\mathbb{R}_+} \Phi_1\left(\frac{t(x)|(Tf)(x)|}{C}\right)\frac{w(x)}{C}dx \leqslant \frac{1}{\varepsilon} = \int_{\mathbb{R}_+} \Phi_2\left(u(y)|f(y)|\right) v(y)dy.$$

Replacing f by Cf and using the fact that T is positively homogeneous yields (3.1), with $K = C$.

For the converse, fix $f \in S(\mathbb{R}_+)$ and $\varepsilon > 0$. Let $\alpha = \rho_{\Phi_2,u,\varepsilon v}(f)$, so that

$$\int_{\mathbb{R}_+} \Phi_2\left(\frac{u(y)|f(y)|}{\alpha}\right)\frac{\varepsilon}{\alpha}v(y)dy \leqslant 1.$$

By (3.1), then,

$$\begin{aligned}\int_{\mathbb{R}_+} \Phi_1\left(\frac{t(x)|(Tf)(x)|}{K\alpha}\right)\frac{\varepsilon}{K\alpha}w(x)dx &= \varepsilon\int_{\mathbb{R}_+}\Phi_1\left(\frac{t(x)|(Tf)(x)|}{K\alpha}\right)\frac{w(x)}{K\alpha}dx\\ &\leqslant \int_{\mathbb{R}_+}\Phi_2\left(\frac{u(y)|f(y)|}{\alpha}\right)\frac{\varepsilon}{\alpha}v(y)dy\\ &\leqslant 1,\end{aligned}$$

which amounts to

$$\rho_{\Phi_1,t,\varepsilon w}(Tf) \leqslant K\alpha = C\rho_{\Phi_2,u,\varepsilon v}(f),$$

with $C = K > 0$ independent of $f \in S(\mathbb{R}_+)$ and $\varepsilon > 0$. $\square$

Proof of Theorem A. According to Proposition 3.1, the modular inequality (1.2) is equivalent to the family of uniform gauge inequalities

$$\rho_{\Phi_1,\varepsilon t^\gamma}(Tf) \leqslant C\rho_{\Phi_2,\varepsilon t^\gamma}(f) \tag{3.3}$$

with $C > 0$ independent of both $f \in S(\mathbb{R}_+)$ and $\varepsilon > 0$.

In particular, (3.3) with $\varepsilon = 1$ is (1.1), so (1.2) implies (1.1).

Next, we prove (1.1) implies (3.3), which amounts to showing

$$\int_{\mathbb{R}_+}\Phi_1\left(\frac{|(Tf)(t)|}{C\rho_{\Phi_2,\varepsilon s^\gamma}(f)}\right)\frac{\varepsilon t^\gamma}{C\rho_{\Phi_2,\varepsilon s^\gamma}(f)}dt \leqslant 1.$$

Letting $z = \varepsilon^\delta t$, $\delta = \frac{1}{1+\gamma}$, the latter reads

$$\int_{\mathbb{R}_+}\Phi_1\left(\frac{|(Tf)(z/\varepsilon^\delta)|}{C\rho_{\Phi_2,\varepsilon s^\gamma}(f)}\right)\frac{z^\gamma}{C\rho_{\Phi_2,\varepsilon s^\gamma}(f)}dz \leqslant 1,$$

or, since T commutes with dilations,

$$\int_{\mathbb{R}_+}\Phi_1\left(\frac{|T(f(\frac{1}{\varepsilon^\delta}\cdot))(z)|}{C\rho_{\Phi_2,\varepsilon s^\gamma}(f)}\right)\frac{z^\gamma}{C\rho_{\Phi_2,\varepsilon s^\gamma}(f)}dz \leqslant 1.$$

But,

$$\begin{aligned}\rho_{\Phi_2,\varepsilon s^\gamma}(f) &= \inf\left\{\lambda > 0 : \int_{\mathbb{R}_+}\Phi_2\left(\frac{|f(s)|}{\lambda}\right)\frac{\varepsilon}{\lambda}s^\gamma ds \leqslant 1\right\}\\ &= \inf\left\{\lambda > 0 : \int_{\mathbb{R}_+}\Phi_2\left(\frac{\left|\left(f\left(\frac{1}{\varepsilon^\delta}y\right)\right)\right|}{\lambda}\right)\frac{y^\gamma}{\lambda}dy \leqslant 1\right\}\\ &= \rho_{\Phi_2,t^\gamma}\left(f\left(\frac{1}{\varepsilon^\delta}\cdot\right)\right),\end{aligned}$$

where in the first equality, we have made the change of variable $s = y/\varepsilon^\delta$. Altogether, then, (3.3) is the same as (1.1), with f replaced by $f\left(\frac{1}{\varepsilon^\delta}\cdot\right)$. $\square$

REMARK 3.2. Using Proposition 3.2, a proof similar to the one above yields the equivalence of (1.2) and the s-gauge inequality

$$\rho^{(s)}_{\Phi_1,t^\gamma}(Tf) \leqslant C^{(s)}\,\rho^{(s)}_{\Phi_2,t^\gamma}(f), \tag{3.4}$$

with $C^{(s)} > 0$ independent of $f \in S(\mathbb{R}_+)$.

Finally, in view of Lemma 2.1 and Proposition 2.1, (3.4) is equivalent to (1.3).

4. The operators P_p and Q_q

We will sometimes need to work with nonnegative, nondecreasing Φ on $\mathbb{R}_+$ that are Young functions, by which is meant

$$\Phi(t) = \int_0^t \phi(s)ds,\ t \in \mathbb{R}_+,$$

where ϕ is nondecreasing, left-continuous function on $\mathbb{R}_+$, with $\phi(0^+) = 0$ and $\lim_{s\to\infty}\phi(s) = \infty$. The Young function, Ψ, complementary to such a Φ is defined by

$$\Psi(t) = \int_0^t \phi^{-1}(s)ds,\ t \in \mathbb{R}_+.$$

where ϕ^{-1} denotes the left-continuous inverse of ϕ, defined by

$$\phi^{-1}(t) = \inf\{s \geqslant 0 : \phi(s) \geqslant t\},\quad t \in \mathbb{R}_+.$$

THEOREM B. *Fix $p,\gamma \in \mathbb{R}, \gamma \neq -1$. Let P_p be defined as in the introduction. Suppose that Φ_1 and Φ_2 are nonnegative, nondecreasing functions from $\mathbb{R}_+$ onto itself. Then, the following are equivalent:*

(4.1)

$$\rho_{\Phi_1,t^\gamma}(P_pf) \leqslant L\rho_{\Phi_2,t^\gamma}(f),$$

$L > 0$ being independent of $f \in S(\mathbb{R}_+)$;

(4.2)

$$\int_{\mathbb{R}_+} \Phi_1\left(|(P_pf)(t)|\right)t^\gamma dt \leqslant K\int_{\mathbb{R}_+} \Phi_2\left(K|f(s)|\right)s^\gamma ds,$$

in which $K > 0$ is independent of $f \in S(\mathbb{R}_+)$.

Moreover, when Φ_2 is a Young function with complementary function Ψ_2, (4.1) and (4.2) are each equivalent to

$$\int_0^t \Psi_2\left(\frac{\alpha(t)}{Cs^{1-\frac{1}{p}+\gamma}}\right)s^\gamma ds \leqslant \alpha(t) < \infty, \tag{4.3}$$

where

$$\alpha(t) = \int_t^\infty \Phi_1(s^{-\frac{1}{p}})s^\gamma ds,\quad t \in \mathbb{R}_+.$$

Finally, if Φ_1 *and* Φ_2 *are* s*-convex for some* s, $0 < s \leqslant 1$, *one has* (4.1) *and* (4.2) *each equivalent to*

$$P_p : \mathring{L}_{\Phi_2,t^\gamma}(\mathbb{R}_+) \to L_{\Phi_1,t^\gamma}(\mathbb{R}_+), \tag{4.4}$$

the mapping (4.4) *being continuous with respect to the metrics* d_{Φ_2,t^γ} *and* d_{Φ_1,t^γ}.

Proof of Theorem B. Since P_p commutes with dilations, (4.1) and (4.2) are equivalent, in view of Theorem A.

The inequality in (4.2) reads

$$\int_{\mathbb{R}_+} \Phi_1\left(t^{-\frac{1}{p}} \int_0^t f(s)s^{\frac{1}{p}-1}ds\right) t^\gamma dt \leqslant \int_{\mathbb{R}_+} \Phi_2\left(Kf(s)\right) s^\gamma ds,\ 0 \leqslant f \in S(\mathbb{R}_+).$$

Replacing $f(s)s^{\frac{1}{p}-1}$ by $g(s)$, we have

$$\int_{\mathbb{R}_+} \Phi_1\left(t^{-\frac{1}{p}} \int_0^t g(s)ds\right) t^\gamma dt \leqslant \int_{\mathbb{R}_+} \Phi_2\left(Kg(s)s^{1-\frac{1}{p}}\right) s^\gamma ds,\ 0 \leqslant f \in S(\mathbb{R}_+).$$

When Φ_2 is a Young function, then according to Proposition 7.2 (in Appendix I), this latter holds if and only if

$$\int_0^t \Psi_2\left(\frac{\alpha(\lambda,t)}{C\lambda y^{1-\frac{1}{p}+\gamma}}\right) y^\gamma dy \leqslant \alpha(\lambda,t) < \infty,$$

where

$$\alpha(\lambda,t) = \int_t^\infty \Phi_1\left(\lambda z^{-\frac{1}{p}}\right) z^\gamma dz,$$

the constant $C > 0$ being independent of $\lambda, t \in \mathbb{R}_+$. Letting $y = \lambda^p s$ and $z = \lambda^p s$ in the above integrals we obtain

$$\int_0^{\lambda^{-p}t} \Psi_2\left(\frac{\alpha(\lambda^{-p}t)}{Cs^{1-\frac{1}{p}+\gamma}}\right) s^\gamma ds \leqslant \alpha(\lambda^{-p}t) < \infty,$$

Replacing $\lambda^{-p}t$ by t yields (4.3).

In case Φ_1 and Φ_2 are s-convex, Lemma 2.1, Proposition 2.1 and Remark 3.2 ensure that (4.1), (4.2) and (4.4) are all equivalent. □

REMARK 4.1. The condition (4.3) is equivalent to the condition

$$\int_0^t \phi_2^{-1}\left(\frac{\alpha(t)}{Cs^{1-\frac{1}{p}+\gamma}}\right) s^{\frac{1}{p}-1} ds \leqslant C, \quad t \in \mathbb{R}_+, \tag{4.5}$$

since $\Psi_2(t) = \int_0^t \phi_2^{-1}(s)ds$ satisfies

$$\tfrac{1}{2}\phi_2^{-1}\left(\tfrac{t}{2}\right) \leqslant \tfrac{\Psi_2(t)}{t} \leqslant \phi_2^{-1}(t), \quad t \in \mathbb{R}_+.$$

Using (4.5) we are able to get more precise connections between the indices p and γ.

(1) $1-\frac{1}{p}+\gamma=0$. The condition (4.5) reads

$$p\phi_2^{-1}\left(\frac{\alpha(t)}{C}\right) \leqslant Ct^{-\frac{1}{p}}.$$

(2) $1-\frac{1}{p}+\gamma\neq 0$. We set $y=\frac{\alpha(t)}{s^{1-\frac{1}{p}+\gamma}}$ in the integral on the left side of the condition to get, with $\lambda(t)=\frac{\alpha(t)}{t^{1-\frac{1}{p}+\gamma}}$,

$$\int_{\lambda(t)}^{\infty}\phi_2^{-1}\left(\frac{y}{C}\right)\frac{dy}{y^{\frac{\gamma+1}{1-\frac{1}{p}+\gamma}}} \leqslant \left(1-\frac{1}{p}+\gamma\right)\alpha(t)^{\frac{1}{1-(1+\gamma)p}}, \tag{4.6}$$

when $1-\frac{1}{p}+\gamma>0$, and

$$\int_{0}^{\lambda(t)}\phi_2^{-1}\left(\frac{y}{C}\right)\frac{dy}{y^{\frac{\gamma+1}{1-\frac{1}{p}+\gamma}}} \leqslant -\left(1-\frac{1}{p}+\gamma\right)\alpha(t)^{\frac{1}{1-(1+\gamma)p}}, \tag{4.7}$$

when $1-\frac{1}{p}+\gamma<0$.

Observe that for the integral in (4.6) to make sense we require $\gamma+1>0$ or $\gamma>-1$.

Again, the change of variable $y=s^{-\frac{1}{p}}$ in the integral giving $\alpha(t)$ yields

$$\alpha(t)=p\int_0^{t^{-\frac{1}{p}}}\frac{\Phi_1(y)}{y}\frac{dy}{y^{(\gamma+1)p}}, \quad \text{when} \quad p>0, \tag{4.8}$$

and

$$\alpha(t)=-p\int_{t^{-\frac{1}{p}}}^{\infty}\frac{\Phi_1(y)}{y}\frac{dy}{y^{(\gamma+1)p}}, \quad \text{when} \quad p<0, \tag{4.9}$$

In (4.9) we need $\gamma+1<0$ or $\gamma<-1$.

Altogether, then, (4.3) amounts to (4.6) with $\alpha(t)$ given by (4.8), when $p>0$ and $\gamma>-1+\frac{1}{p}$ and to (4.7) with $\alpha(t)$ given by (4.9) when $p<0$ and $\gamma<-1+\frac{1}{p}$.

REMARK 4.2. Theorem B, with $\gamma=0$, helps to greatly simplify the proof of Proposition 6.2 in [7], in which proposition the condition (4.3), in the equivalent form (4.7), was used to construct the essentially largest Young function, Φ_1, that can appear with a fixed Young function, Φ_2, in an Orlicz-Sobolev inequality such as

$$\rho_{\Phi_1}(u) \leqslant \rho_{\Phi_2}(|\nabla u|);$$

here $C>0$ is independent of all infinitely differentiable u supported in a given bounded domain Ω of $\mathbb{R}^n$ with a Lipschitz boundary and $|\nabla u|^2=\left(\frac{\partial u}{\partial x_1}\right)^2+\ldots+\left(\frac{\partial u}{\partial x_n}\right)^2$.

COROLLARY 4.1. *Fix $q,\gamma\in\mathbb{R},\gamma\neq -1$. Let Q_q be defined as in the introduction. Suppose that Φ_1 and Φ_2 are nonnegative, nondecreasing functions from $\mathbb{R}_+$ onto itself. Then, the following are equivalent:*

(4.10)
$$\rho_{\Phi_1,t^\gamma}(Q_q f)\leqslant L\rho_{\Phi_2,t^\gamma}(f),$$

$L>0$ being independent of $f\in S(\mathbb{R}_+)$;

(4.11)
$$\int_{\mathbb{R}_+}\Phi_1\left(|(Q_q f)(t)|\right)t^\gamma dt\leqslant K\int_{\mathbb{R}_+}\Phi_2\left(K|f(s)|\right)s^\gamma ds,$$

in which $K>0$ is independent of $f\in S(\mathbb{R}_+)$.

Moreover, when Φ_1 and Φ_2 are Young functions with complementary functions Ψ_1 and Ψ_2, respectively, and $\gamma+\frac{1}{q}-1\neq 0$, (4.10) *and* (4.11) *are each equivalent to*

$$\int_0^t\Phi_1\left(\frac{\beta(t)}{Cs^{\frac{1}{q}}}\right)s^\gamma ds\leqslant\beta(t), \tag{4.12}$$

where
$$\beta(t)=\int_t^\infty\Psi_2(s^{\frac{1}{q}-1-\gamma})s^\gamma ds<\infty,\;\; t\in\mathbb{R}_+.$$

Finally, if Φ_1 and Φ_2 are s-convex for some s, $0<s\leqslant 1$, one has (4.10) *and* (4.11) *each equivalent to*
$$Q_q:\mathring{L}_{\Phi_2,t^\gamma}(\mathbb{R}_+)\to L_{\Phi_1,t^\gamma}(\mathbb{R}_+), \tag{4.13}$$

the mapping (4.13) *being continuous with respect to the metrics d_{Φ_2,t^γ} and d_{Φ_1,t^γ}.*

Proof. In view of Theorem A, (4.10) and (4.11) are equivalent, since Q_q commutes with dilations.

Given that Φ_1 and Φ_2 are s-convex, $0<s\leqslant 1$, Proposition 3.2 ensures (4.11), hence (4.10), is equivalent to

$$\rho^{(s)}_{\Phi_1,t^\gamma}(Q_q f)\leqslant L^{(s)}\,\rho^{(s)}_{\Phi_2,t^\gamma}(f),\quad f\in S(\mathbb{R}_+), \tag{4.14}$$

and hence, by Proposition 2.1, to

$$Q_q:\mathring{L}_{\Phi_2,t^\gamma}(\mathbb{R}_+)\to L_{\Phi_1,t^\gamma}(\mathbb{R}_+).$$

In particular, if $s=1$, namely, Φ_1 and Φ_2 are Young functions, having complementary functions Ψ_1 and Ψ_2, respectively, (4.14), with $s=1$, is equivalent to

$$\rho^{(1)}_{\Psi_2,t^\gamma}(P_r g)\leqslant K\rho^{(1)}_{\Psi_1,t^\gamma}(g),\quad g\in S(\mathbb{R}_+), \tag{4.15}$$

where $\frac{1}{r}=1-\frac{1}{q}+\gamma$. Theorem B ensures (4.15) holds if and only if (4.12) does. This completes the proof. □

5. The Hardy-Littlewood maximal operator M

THEOREM C. *Fix $\gamma > -1$. Let M be the Hardy-Littlewood maximal operator. Suppose Φ_1 and Φ_2 are nonnegative, nondecreasing functions from $\mathbb{R}_+$ onto itself. Then, the following are equivalent:*

(5.1)
$$\rho_{\Phi_1,|x|^\gamma}(Mf) \leqslant L\, \rho_{\Phi_2,|x|^\gamma}(f),$$
$L > 0$ being independent of $f \in L_{\Phi_2,|x|^\gamma}(\mathbb{R})$;

(5.2)
$$\int_{\mathbb{R}} \Phi_1\left((Mf)(x)\right)|x|^\gamma dx \leqslant K \int_{\mathbb{R}} \Phi_2(K|f(y)|)|y|^\gamma dy < \infty,$$
in which $K > 0$ is independent on $f \in M(\mathbb{R})$.

Moreover, when $\Phi_1 = \Phi_2 = \Phi$ is a Young function with complementary function Ψ, (5.1) *and* (5.2) *are each equivalent to*

(5.3) *(a)*
$$\Psi(2t) \leqslant C\Psi(t), \quad t \in \mathbb{R}_+$$
and

(b) $-1 < \gamma < 0$ or, if $\gamma \geqslant 0$,
$$\frac{1}{t}\int_0^t \phi^{-1}(s^{-\gamma})ds \leqslant C\phi^{-1}(Ct^{-\gamma}), \quad \phi = \frac{d\Phi}{dt},$$
for some $C \geqslant 1$ independent of $t \in \mathbb{R}_+$.

Proof of Theorem C. In view of Theorem A, (5.1) and (5.2) are equivalent. When $\Phi_1 = \Phi_2 = \Phi$ is a Young function, a special case of Theorem 1 in [2] ensures that (5.2) (hence (5.1)) holds if and only if
$$\Psi(2t) \leqslant C\Psi(t), \quad t \in \mathbb{R}_+,$$
and
$$\frac{1}{\mu_\gamma(I)}\int_I \Psi\left(\frac{1}{C}\frac{\Phi(\lambda)}{\lambda}\frac{\mu_\gamma(I)}{|I||x|^\gamma}\right)|x|^\gamma dx \leqslant \Phi(\lambda), \tag{5.4}$$
where $C \geqslant 1$ is independent of the bounded interval $I \subset \mathbb{R}$ and $\lambda \in \mathbb{R}_+$; here
$$\mu_\gamma(I) = \int_I |x|^\gamma dx.$$
Since
$$\tfrac{t}{2}\phi^{-1}(\tfrac{t}{2}) \leqslant \Psi(t) \leqslant t\phi^{-1}(t),\ t \in \mathbb{R}_+,$$
(5.4) is equivalent to
$$\frac{1}{|I|}\int_I \phi^{-1}\left(\frac{1}{C}\phi(\lambda)\frac{\mu_\gamma(I)}{|I|}\frac{1}{|x|^\gamma}\right)dx \leqslant C\lambda, \tag{5.5}$$
in which $C \geqslant 1$ does not depend on $I \subset \mathbb{R}$ or $\lambda \in \mathbb{R}_+$.

We observe that the assumption $\gamma > -1$ is necessary to guarantee $\mu_\gamma(I) < \infty$ for all intervals $I \subset \mathbb{R}$.

One readily shows that for $I = [a,b]$,

$$\frac{\mu_\gamma(I)}{|I|} \leqslant \max\left[1, \frac{1}{1+\gamma}\right] d^\gamma,$$

where $d = \max[|a|,|b|]$.

Assume, first that $ab \geqslant 0$, say $0 \leqslant a < b$. Then, (5.5) holds if

$$\frac{1}{b-a}\int_a^b \phi^{-1}\left(\phi(\lambda)\max\left[\tfrac{1}{C}, \tfrac{1}{C(1+\gamma)}\right]\left(\tfrac{b}{x}\right)^\gamma\right)dx \leqslant C\lambda,$$

which, when $-1 < \gamma < 0$, automatically holds with $C = \frac{1}{1+\gamma}$, since then $\frac{1}{C(\gamma+1)}\left(\frac{b}{x}\right)^\gamma \leqslant 1$. The same is true when $\gamma \geqslant 0$ and $a > \frac{b}{2}$ with $C = 2^\gamma$.

So, assume $\gamma \geqslant 0$ and $0 \leqslant a \leqslant \frac{b}{2}$. It suffices to show

$$\frac{1}{b}\int_0^b \phi^{-1}\left(\phi(\lambda)\frac{2}{C}\left(\frac{b}{x}\right)^\gamma\right)dx \leqslant \frac{C}{2}\lambda,$$

or, setting $x = by$,

$$\int_0^1 \phi^{-1}\left(\phi(\lambda)\frac{2}{C}y^{-\gamma}\right)dy \leqslant \frac{C}{2}\lambda.$$

Let $s = \phi(\lambda)^{-\frac{1}{\gamma}}\left(\frac{2}{C}\right)^{-\frac{1}{\gamma}}y$ to get

$$\int_0^{\phi(\lambda)^{-\frac{1}{\gamma}}\left(\frac{2}{C}\right)^{-\frac{1}{\gamma}}} \phi^{-1}\left(s^{-\gamma}\right)\phi(\lambda)^{\frac{1}{\gamma}}\left(\tfrac{2}{C}\right)^{\frac{1}{\gamma}}ds \leqslant \frac{C}{2}\lambda.$$

Taking $t = \phi(\lambda)^{-\frac{1}{\gamma}}\left(\frac{2}{C}\right)^{-\frac{1}{\gamma}}$, whence $\lambda = \phi^{-1}\left(\frac{C}{2}t^{-\gamma}\right)$, we then arrive at (5.3) (C), with C replaced by $\frac{C}{2}$.

Finally, suppose $ab < 0$, say $a < 0 < b$. In that case,

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b \phi^{-1}\left(\phi(\lambda)\max\left[\tfrac{1}{C}, \tfrac{1}{C(1+\gamma)}\right]\left(\tfrac{d}{x}\right)^\gamma\right)dx \\
&\qquad = \frac{1}{|a|+|b|}\left(\int_0^{|a|} + \int_0^{|b|}\right)\phi^{-1}\left(\phi(\lambda)\max\left[\tfrac{1}{C}, \tfrac{1}{C(1+\gamma)}\right]\left(\tfrac{d}{x}\right)^\gamma\right)dx \\
&\qquad \leqslant \frac{2}{d}\int_0^d \phi^{-1}\left(\phi(\lambda)\left[\tfrac{1}{C}, \tfrac{1}{C(1+\gamma)}\right]\left(\tfrac{d}{x}\right)^\gamma\right)dx,
\end{aligned}$$

whence (5.5) reduces to

$$\frac{1}{d}\int_0^d \phi^{-1}\left(\phi(\lambda)\max\left[\tfrac{2}{C}, \tfrac{2}{C(1+\gamma)}\right]\left(\tfrac{d}{x}\right)^\gamma\right)dx \leqslant \frac{C}{2}\lambda,$$

namely, to the previous case.

It only remains to show that (5.5) implies (5.3) (C). To this end, take $I = (0,t)$, $t > 0$ in the equivalent form of (5.5) to get

$$\frac{1}{t}\int_0^t \phi^{-1}\left(\tfrac{1}{C}\phi(\lambda)\tfrac{1}{\gamma+1}\left(\tfrac{t}{x}\right)^{\gamma}\right)dx \leqslant C\lambda.$$

Taking $\lambda = \phi^{-1}\left(C(\gamma+1)t^{-\gamma}\right)$ yields

$$\begin{aligned}\frac{1}{t}\int_0^t \phi^{-1}\left(x^{-\gamma}\right)dx &\leqslant C\phi^{-1}\left(C(\gamma+1)t^{-\gamma}\right)\\ &\leqslant C'\phi^{-1}(C't^{-\gamma}),\end{aligned}$$

with $C' = \max[1,\gamma+1]C$. □

6. The Hilbert transform H

THEOREM D. *Fix $\gamma \in \mathbb{R}$, $\gamma > -1$. Let H be the Hilbert transform. Suppose Φ_1 and Φ_2 are nonnegative, nondecreasing functions from $\mathbb{R}_+$ onto itself. Then, the following are equivalent:*

(6.1)
$$\rho_{\Phi_1,|x|^{\gamma}}(Hf) \leqslant L\,\rho_{\Phi_2,|x|^{\gamma}}(f),$$

$L > 0$ being independent of $f \in L_1\left(\frac{1}{1+|x|}\right) \cap L_{\Phi_2,|x|^{\gamma}}(\mathbb{R})$,

$$L_1\left(\tfrac{1}{1+|x|}\right) = \left\{f \in M(\mathbb{R}) : \int_{\mathbb{R}} \tfrac{|f(x)|}{1+|x|}dx < \infty\right\};$$

(6.2)
$$\int_{\mathbb{R}} \Phi_1\left(|(Hf)(x)|\right)|x|^{\gamma}dx \leqslant K\int_{\mathbb{R}} \Phi_2(K|f(y)|)|y|^{\gamma}dy < \infty,$$

in which $K > 0$ is independent of $f \in L_1\left(\frac{1}{1+|x|}\right)$.

Moreover, when $\Phi_1 = \Phi_2 = \Phi$ is a Young function with complementary function Ψ, (6.1) *and* (6.2) *are each equivalent to*

(6.3) *(a)*
$$\Phi(2t) \leqslant C\Phi(t),$$

(b)
$$\Psi(2t) \leqslant C\Psi(t), \quad t \in \mathbb{R}_+$$

and

(c) $-1 < \gamma < 0$ *or, if* $\gamma \geqslant 0$,

$$\frac{1}{t}\int_0^t \phi^{-1}(s^{-\gamma})ds \leqslant \phi^{-1}(Ct^{-\gamma}), \quad \phi = \frac{d\Phi}{dt},$$

for some $C \geqslant 1$ independent of $t \in \mathbb{R}_+$.

Finally, if Φ_1 *and* Φ_2 *are* s*-convex, for some* s, $0 < s \leqslant 1$, *one has* (6.1) *and* (6.2) *each equivalent to*

$$H : \mathring{L}_{\Phi_2,|x|^\gamma}(\mathbb{R}) \to L_{\Phi_1,|x|^\gamma}(\mathbb{R}), \tag{6.4}$$

the mapping (6.4) *being continuous with respect to the metrics* d_{Φ_2,t^γ} *and* d_{Φ_1,t^γ}.

The condition (6.3) (c) clearly holds if $\gamma \leqslant 0$. As for $\gamma > 0$, Lemma 6.1 to follow shows (6.3) (c) amounts to the condition A_ϕ for $|x|^\gamma$ in [8], provided one has (6.3) (a).

LEMMA 6.1. *Fix* $\gamma > 0$ *and let* $\Phi(t) = \int_0^t \phi(s)ds$ *be a Young function. Then, one has*

$$\frac{\varepsilon\,\mu_\gamma(I)}{|I|}\phi\left(\frac{1}{|I|}\int_I \phi^{-1}\left(\frac{1}{\varepsilon|x|^\gamma}\right)dx\right) \leqslant C, \tag{A_ϕ^γ}$$

for all bounded intervals $I \subset \mathbb{R}$ *and* $\varepsilon > 0$ *if and only if*

$$\frac{\int_0^{t_1}\phi^{-1}(s^{-\gamma})ds + \int_0^{t_2}\phi^{-1}(s^{-\gamma})ds}{t_1+t_2} \leqslant \phi^{-1}(Ct_2^{-\gamma}), \tag{6.5}$$

for some $C > 1$ *independent of* $0 \leqslant t_1 < t_2$.

If further, one has (6.3)*(*a*), then* (6.5) *can be replaced by* (6.3)*(*c*).*

Proof. Given $I = [a,b]$, the change of variable $y = \varepsilon^{\frac{1}{\gamma}}x$ in the integrals $\int_I \varepsilon|x|^\gamma dx$ and $\int_I \phi^{-1}\left(\frac{1}{\varepsilon|x|^\gamma}\right)dx$ in (A_ϕ^γ) yields $\varepsilon^{-\frac{1}{\gamma}}\int_{I_\varepsilon}|y|^\gamma dy$ and $\varepsilon^{-\frac{1}{\gamma}}\int_{I_\varepsilon}\phi^{-1}\left(\frac{1}{|y|^\gamma}\right)dy$, respectively, where $I_\varepsilon = [\varepsilon^{\frac{1}{\gamma}}a, \varepsilon^{\frac{1}{\gamma}}b]$. So ($A_\phi^\gamma$) becomes

$$\frac{\mu_\gamma(I_\varepsilon)}{|I_\varepsilon|}\phi\left(\frac{1}{|I_\varepsilon|}\int_{I_\varepsilon}\phi^{-1}\left(\frac{1}{|y|^\gamma}\right)dy\right) \leqslant C.$$

Since I_ε is arbitrary whenever I is, it suffices to verify (A_ϕ^γ) with $\varepsilon = 1$.

Now, if $ab \geqslant 0$, say, $0 \leqslant a < b$,

$$b^{-\gamma} \leqslant \frac{|I|}{\mu_\gamma(I)} \leqslant 2^{\gamma+1}b^{-\gamma}$$

while if $ab < 0$ with, say, $|a| < |b|$,

$$\frac{\gamma+1}{2}|b|^{-\gamma} \leqslant \frac{|I|}{\mu_\gamma(I)} \leqslant 2(\gamma+1)|b|^{-\gamma} \leqslant 2^{\gamma+1}|b|^{-\gamma}.$$

Thus, with $I = (-t_1, t_2)$, $0 \leqslant t_1 < t_2$, we have

$$\frac{1}{t_1+t_2}\left[\int_0^{t_1}\phi^{-1}(s^{-\gamma})ds + \int_0^{t_2}\phi^{-1}(s^{-\gamma})ds\right] = \frac{1}{|I|}\int_I \phi^{-1}(|y|^{-\gamma})dy$$

and

$$\phi^{-1}\left(C\frac{\gamma+1}{2}t_2^{-\gamma}\right) \leqslant \phi^{-1}\left(C\frac{|I|}{\mu_\gamma(I)}\right) \leqslant \phi^{-1}\left(C2^{\gamma+1}t_2^{-\gamma}\right),$$

that is, (A_ϕ^γ) is equivalent to (6.5) when $ab<0$. In particular, we have (A_ϕ^γ) implies (6.5).

It remains to show (6.5) implies (A_ϕ^γ) when $ab \geqslant 0$. In this case (A_ϕ^γ) holds if and only if

$$\frac{1}{b}\int_0^b \phi^{-1}(s^{-\gamma})ds \leqslant \phi^{-1}(Cb^{-\gamma}), \quad b>0, \tag{6.6}$$

since $\phi^{-1}(s^{-\gamma})$ decreases in s on $\mathbb{R}_+$.

Taking $t_1=0$ and $t_2=b$ in (6.5) yield (6.6).

Finally, (6.5) always implies (6.3) (c)-just take $t_1=0$ and $t_2=t$. Moreover,

$$\begin{aligned}\frac{\int_0^{t_1}\phi^{-1}(s^{-\gamma})ds+\int_0^{t_2}\phi^{-1}(s^{-\gamma})ds}{t_1+t_2} &\leqslant \frac{2}{t_2}\int_0^{t_2}\phi^{-1}(s^{-\gamma})ds \\ &\leqslant 2\,\phi^{-1}(Ct_2^{-\gamma}),\end{aligned} \tag{6.7}$$

ensures (6.3) (c) when (6.3) (a) holds, since (6.3) (a) is equivalent to $\phi(2t)\leqslant C\phi(t)$, which, on replacing t by $\phi^{-1}(t)$, yields

$$2\phi^{-1}(t)\leqslant\phi^{-1}(Ct), \quad t>0. \quad \square$$

Proof of Theorem D. The equivalence of (6.1), (6.2) and (6.4) follows from the variant of Theorem A for $|x|^\gamma$ on $\mathbb{R}$, since H is dilation-commuting.

The condition (6.3) (a) comes out of the inequality in (6.2) in the same way it comes out of the corresponding inequality for M in Theorem 7 of [2], but with

$$(Mf_m)(y)\geqslant C|E\cap B_m|\,|x-y|^{-1}, \quad y\notin B_m,$$

replaced by

$$(Hf_m)(y)\geqslant Cr_0\,|x-y|^{-1}, \quad y\notin B_m,$$

where $f_m=\chi_{B_m}$, $B_m=(x-2^{-m}r_0, x+2^{-m}r_0)$. Indeed, if, for instance, $y<x-2^{-m}r_0$,

$$\begin{aligned}-(Hf_m)(y)=\frac{1}{\pi}\int_{x-2^{-m}r_0}^{x+2^{-m}r_0}\frac{1}{y-z}dz &= \frac{1}{\pi}\log\left[\frac{x-y-2^{-m}r_0}{x-y+2^{-m}r_0}\right]\\ &=\frac{1}{\pi}\log\left[1-\frac{2^{-m}r_0}{x-y+2^{-m}r_0}\right]\\ &\geqslant\frac{1}{\pi}\,\frac{2^{-m}r_0}{x-y+2^{-m}r_0}\\ &\geqslant\frac{1}{\pi}\,\frac{2^{-m-1}r_0}{|x-y|}.\end{aligned}$$

Again, by Corollary 2.7 in [1], the modular inequality in (6.2) is equivalent to

$$\int_{\mathbb{R}}\Psi\left(|x|^{-\gamma}|(Hf)(x)|\right)|x|^\gamma dx\leqslant\int_{\mathbb{R}}\Psi(K|y|^{-\gamma}|f(y)|)|y|^\gamma dy<\infty,$$

which implies, by the argument above, the condition (6.3) (b).

Next, the argument in [8, p. 280], applied to (6.4) yields the (A_ϕ^γ) condition involving $|x|^\gamma$, provided one can replace $(Mf)(x)$ in

$$(Mf)(x) \geqslant \rho_{\Psi,\varepsilon|x|^\gamma}(\chi_I/\varepsilon|\cdot|^\gamma)\varepsilon\mu_\gamma(I)$$

by $|(Hf)(x)|$. In [8] f was a nonnegative, measurable function supported in I, with $\rho_{\Psi,|x|^\gamma}(f)=1$ and

$$\int_I f(x)dx = \rho_{\Psi,|x|^\gamma}(\chi_I/|\cdot|^\gamma).$$

But for this f and $x\in I+|I|$, one has

$$|(Hf)(x)| \geqslant \frac{1}{2\pi}\,\rho_{\Psi,|x|^\gamma}(\chi_I/|\cdot|^\gamma)\frac{\chi_J(x)}{|I|},$$

and so, as Φ satisfies the modular inequality in (6.2),

$$\int_J \Phi\left(\frac{\rho_{\Psi,|x|^\gamma}(\chi_I/|\cdot|^\gamma)}{|I|}\right)|y|^\gamma dy$$
$$\leqslant C\int_{\mathbb{R}} \Phi(|f(y)|)|y|^\gamma dy = C;$$

that is,

$$\Phi\left(\frac{\rho_{\Psi,|x|^\gamma}(\chi_I/|\cdot|^\gamma)}{|I|}\right)\mu_\gamma(J)\leqslant C.$$

Similarly, there holds

$$\Phi\left(\frac{\rho_{\Psi,|x|^\gamma}(\chi_J/|\cdot|^\gamma)}{|J|}\right)\mu_\gamma(I)\leqslant C,$$

whence

$$\Phi\left(\frac{\rho_{\Psi,|x|^\gamma}(\chi_J/|\cdot|^\gamma)}{|J|}\right)\mu_\gamma(J)\;\;\Phi\left(\frac{\rho_{\Psi,|x|^\gamma}(\chi_I/|\cdot|^\gamma)}{|I|}\right)\mu_\gamma(I)\leqslant C^2.$$

To get (A_ϕ^γ) (for $\varepsilon=1$, which is enough) it suffices to show

$$\Phi\left(\frac{\rho_{\Psi,|x|^\gamma}(\chi_J/|\cdot|^\gamma)}{|J|}\right)\mu_\gamma(J)\geqslant 1,$$

or, equivalently,

$$\frac{1}{\Phi^{-1}\left(\frac{1}{\mu_\gamma(I)}\right)}\,\rho_{\Psi,|x|^\gamma}(\chi_J/|\cdot|^\gamma)\geqslant |J|,$$

that is,

$$\rho_{\Phi,|x|^\gamma}(\chi_J)\;\;\rho_{\Psi,|x|^\gamma}(\chi_J/|\cdot|^\gamma)\geqslant |J|,$$

which inequality is essentially the generalized Hölder inequality

$$\begin{aligned}|J| &= \int_{\mathbb{R}} \chi_J(x)\frac{\chi_J(x)}{|x|^\gamma}|x|^\gamma dx \\ &\leqslant 2\rho_{\Phi,|x|^\gamma}(\chi_J)\ \ \rho_{\Psi,|x|^\gamma}(\chi_J/|\cdot|^\gamma).\end{aligned}$$

Finally, we prove conditions (6.3) (a), (6.3) (b) and (6.3) (c) imply (6.2). According to Theorem 7 in [8], $|x|^\gamma$ in (A^γ_ϕ), together with (6.3) (a) and (6.3) (b), implies $|x|^\gamma$ satisfies the A_∞ condition, namely, there exist constants $C,\delta > 0$ so that for any interval I and any measurable subset E of I,

$$\frac{\mu_\gamma(E)}{\mu_\gamma(I)} \leqslant C\left(\frac{|E|}{|I|}\right)^\delta.$$

The argument on p. 245 of [5] then ensures the maximal Hilbert transform, H^*, defined at $f \in L_1\left(\frac{1}{1+|y|}\right)$ by

$$(H^*f)(x) = \sup_{\varepsilon>0}\left|\frac{1}{\pi}\int_{|x-y|>\varepsilon}\frac{f(y)}{x-y}dy\right|, \quad x\in\mathbb{R},$$

satisfies, for any given $\alpha > 0$ and the $\delta > 0$ in the A_∞ condition,

$$\int_{\{H^*f>2\lambda,\ Mf\leqslant\alpha\lambda\}}|x|^\gamma dx \leqslant C\alpha^\delta\int_{\{Mf>\lambda\}}|x|^\gamma dx,$$

in which $C > 0$ does not depend on α, λ or $f \in L_1\left(\frac{1}{1+|y|}\right)$.

We thus have, since Φ satisfies (6.3) (a),

$$\begin{aligned}\int_{\mathbb{R}}\Phi((H^*f)(x))\,|x|^\gamma dx &= C\int_{\mathbb{R}_+}\phi(\lambda)\int_{\{H^*f>2\lambda\}}|x|^\gamma dx\,d\lambda \\ &\leqslant C\int_{\mathbb{R}_+}\phi(\lambda)\int_{\{Mf>\alpha\lambda\}}|x|^\gamma dx\,d\lambda + C\alpha^\delta\int_{\mathbb{R}_+}\phi(\lambda)\int_{\{H^*f>\lambda\}}|x|^\gamma dx\,d\lambda \\ &= \frac{C}{\alpha}\int_{\mathbb{R}_+}\phi(\lambda/\alpha)\int_{\{Mf>\lambda\}}|x|^\gamma dx\,d\lambda + C\alpha^\delta\int_{\mathbb{R}_+}\phi(\lambda)\int_{\{H^*f>\lambda\}}|x|^\gamma dx\,d\lambda \\ &\leqslant C'\int_{\mathbb{R}_+}\phi(\lambda)\int_{\{Mf>\lambda\}}|x|^\gamma dx\,d\lambda + C\alpha^\delta\int_{\mathbb{R}_+}\phi(\lambda)\int_{\{H^*f>\lambda\}}|x|^\gamma dx\,d\lambda\end{aligned}$$

Taking α such that $C\alpha^\delta < \frac{1}{2}$ we get

$$\begin{aligned}\int_{\mathbb{R}}\Phi(|(Hf)(x)|)\,|x|^\gamma dx &\leqslant \int_{\mathbb{R}}\Phi((H^*f)(x))\,|x|^\gamma dx \\ &\leqslant K\int_{\mathbb{R}}\Phi((Mf)(x))\,|x|^\gamma dx \\ &\leqslant \int_{\mathbb{R}}\Phi(K|f(x)|)\,|x|^\gamma dx,\end{aligned}$$

by Theorem C, since (6.3) (c) implies (5.3) (C). □

7. Appendix I

The two general results in this appendix are variants of Theorem 4.1 and 3.1 in [1].

PROPOSITION 7.1. *Let t,u,v and w be weights on $\mathbb{R}_+$. Suppose Φ_1 and Φ_2 are nonnegative nondecreasing functions from $\mathbb{R}_+$ onto itself. Then, the general weighted modular inequality for*

$$(If)(x) = \int_0^x f(y)dy, \quad 0 \leqslant f \in M(\mathbb{R}_+), x \in \mathbb{R}_+,$$

namely,

$$\int_{\mathbb{R}_+} \Phi_1(w(x)If(x))t(x)dx \leqslant \int_{\mathbb{R}_+} \Phi_2(Ku(y)f(y))v(y)dy \tag{7.1}$$

is equivalent to the weighted weak-type modular inequality

$$\int_{\{x\in\mathbb{R}_+:(If)(x)>\lambda\}} \Phi_1(\lambda w(x))t(x)dx \leqslant \int_{\mathbb{R}_+} \Phi_2(Ku(y)f(y))v(y)dy. \tag{7.2}$$

in both of which $K>0$, is independent of $0 \leqslant f \in M(\mathbb{R}_+)$ and in (7.2) is independent of λ as well.

Proof. Clearly, (7.1) implies (7.2). To prove the converse fix $f \geqslant 0$ and choose x_k so that $If(x_k) = 2^k, k = 0, \pm 1, \pm 2, \dots$ and set $I_k = [x_{k-1}, x_k)$ and $f_k = f\chi_{I_k}$. Then, arguing as in Proposition 4.1 in [1], we obtain, by (7.2),

$$\begin{aligned}\int_{\mathbb{R}_+} \Phi_1(w(x)(If)(x))t(x)dx &\leqslant \sum_{k\in\mathbb{Z}} \int_{\{x\in\mathbb{R}_+:I(8f_{k-1})(x)>2^k\}} \Phi_1(2^k w(x))t(x))dx \\ &\leqslant \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}_+} \Phi_2(8Kf_{k-1}(x)u(x))v(x)dx \\ &= \int_{\mathbb{R}_+} \Phi_2(8Kf(x)u(x))v(x)dx. \quad \square\end{aligned}$$

PROPOSITION 7.2. *Let t,u,v,w and Φ_1 and Φ_2 be as in the Proposition 7.1. Assume, moreover, that Φ_2 is a Young function. Then, (7.2) (and hence (7.1)) holds if and only if*

$$\int_0^x \Psi_2\left(\frac{\alpha(\lambda,x)}{C\lambda u(y)v(y)}\right) v(y)dy \leqslant \alpha(\lambda,x) < \infty, \tag{7.3}$$

where

$$\alpha(\lambda,x) = \int_x^\infty \Phi_1(\lambda w(y))t(y)dy,$$

and $C>0$ being independent of $\lambda, x \in \mathbb{R}_+$.

Proof. Suppose (7.2) holds and fix $x \in \mathbb{R}_+$. Since u and v are weights, they are positive a.e. and so

$$\Psi_2\left(\frac{1}{u(y)v(y)}\right)v(y) < \infty, \quad y\text{-a.e.}$$

Let the set $E_n \subseteq (0,x)$ be such that $E_n \uparrow (0,x)$

$$\int_{E_n}\Psi_2\left(\frac{1}{u(y)v(y)}\right)v(y) < \infty.$$

Fix $n \in \mathbb{Z}_+$. Then, as in the proof of Theorem 3.1 in [1], given $\lambda \in \mathbb{R}_+$, there exists an $\varepsilon > 0$ such that

$$\int_{E_n}\Psi_2\left(\frac{\varepsilon}{u(y)v(y)}\right)\frac{v(y)}{\varepsilon}dy = 2K\lambda.$$

Setting

$$f(y) = \frac{1}{K}\Psi_2\left(\frac{\varepsilon}{u(y)v(y)}\right)\frac{v(y)}{\varepsilon}\cdot\chi_{E_n}(y),$$

the subsequent part of the above-mentioned proof, with $(\Phi_1 \circ \Phi_2^{-1})(z)$ replaced by z, yields

$$\alpha(\lambda,x) \leqslant 2K\varepsilon$$

and then (7.3), with $C = 4K$.

The argument that (7.3) implies (7.2) is identical to the one that (3.2) implies (1.12) in [1]. □

8. Appendix II

Let $\Phi(t) = \int_0^t \phi(s)ds$, $t \in \mathbb{R}_+$ be a Young function, with complementary function $\Psi(x) = \int_0^x \phi^{-1}(y)dy$, and let w be a weight on $\mathbb{R}^n$. The conditions

$$\frac{1}{w(Q)}\int_Q \Psi\left(\frac{1}{C}\frac{\Phi(\lambda)}{\lambda}\frac{w(Q)}{|Q|}\frac{1}{w(x)}\right)w(x)dx \leqslant \Phi(\lambda) \tag{8.1}$$

and

$$\frac{\varepsilon w(Q)}{|Q|}\phi\left(\frac{1}{|Q|}\int_Q \phi^{-1}\left(\frac{1}{\varepsilon w(x)}\right)dx\right) \leqslant C, \tag{A_ϕ}$$

in which $C > 1$ is to be independent of λ, ε in $\mathbb{R}_+$ and Q is a cube in $\mathbb{R}^n$, with sides parallel to the coordinate axes, $w(Q) = \int_Q w(x)dx$, were introduced in [2] and [8], respectively. To put the two conditions on the same footing we will work with (8.1) in the equivalent form

$$\frac{1}{|Q|}\int_Q \phi^{-1}\left(\frac{1}{C}\phi(\lambda)\frac{w(Q)}{|Q|}\frac{1}{w(x)}\right)dx \leqslant C\lambda.$$

Our aim in this section is to compare (8.1) and (A_ϕ) in the context of power weights on $\mathbb{R}$, namely, the conditions (5.3) (C) and (6.5). We have already observed that (6.5) implies (5.3) (C). Indeed, (A_ϕ) implies (8.1) in general, as seen in

THEOREM 8.1. *Let $\Phi(t) = \int_0^t \phi(s)ds$, $t \in \mathbb{R}_+$, be a Young function and let w be a weight on $\mathbb{R}^n$. Then, (A_ϕ) implies* (8.1).

Proof. Writing (A_ϕ) in the form

$$\frac{1}{|Q|}\int_Q \phi^{-1}\left(\frac{1}{\varepsilon w(x)}\right)dx \leqslant \phi^{-1}\left(\frac{1}{\varepsilon}\frac{C|Q|}{w(Q)}\right),$$

then setting $\frac{1}{\varepsilon} = \phi(\lambda)\frac{w(Q)}{C|Q|}$, we obtain

$$\frac{1}{|Q|}\int_Q \phi^{-1}\left(\frac{1}{C}\phi(\lambda)\frac{w(Q)}{|Q|}\frac{1}{w(x)}\right)dx \leqslant \phi^{-1}(\phi(\lambda)) \leqslant \lambda,$$

which is, of course, (8.1). $\square$

We now show that to each power weight $w(x) = |x|^\gamma, \gamma > 0$, on $\mathbb{R}$ there corresponds a Young function, $\Phi_\gamma(t) = \int_0^t \phi_\gamma(s)ds$, $t \in \mathbb{R}_+$, for which (8.1) holds, but (A_ϕ) doesn't.

EXAMPLE 8.1. We define Φ_γ in terms of decreasing function χ as

$$\phi_\gamma^{-1}(t) = \chi(t^{-\frac{1}{\gamma}}), \quad t \in \mathbb{R}_+,$$

where

$$\chi(t) = \log(e/t), \quad 0 < t \leqslant 1,$$

and

$$\chi(t) = \begin{cases} \frac{1}{2^k}\left(1 - \frac{t-a_k}{2}\right), & a_k < t \leqslant a_k + 1, \\ \frac{1}{2^{k+1}}, & a_k + 1 < t \leqslant a_{k+1}, \end{cases}$$

with $a_0 = 1$ and $a_k = (k+3)!$, $k \geqslant 1$.

If (A_ϕ) held, one would have, on taking $t_1 = 0, t_2 = t$ in (6.5)

$$\frac{1}{t}\int_0^t \chi(s)ds \leqslant \chi\left(t/C^{\frac{1}{\gamma}}\right), \quad t \in \mathbb{R}_+,$$

for some $C > 1$. But, for $k \geqslant 1$,

$$\frac{1}{a_k}\int_0^{a_k} \chi(s)ds \geqslant \chi(a_k) = \frac{1}{2^k} = \chi\left(\tfrac{a_k}{k}\right).$$

It thus suffices to show

$$\frac{1}{t}\int_0^t \chi(s)ds \leqslant 4\chi\left(t/4^{\frac{1}{\gamma}}\right), \quad t \in \mathbb{R}_+.$$

This is readily done when $0 < t \leqslant 1$. For $t \in (a_k, a_{k+1}]$, $k \geqslant 0$, one has

$$\frac{1}{t}\int_0^t \chi(s)ds = \begin{cases} \frac{1}{t}\int_0^{a_k} \chi(s)ds + \frac{1}{t2^k}\left[(t-a_k) - \frac{(t-a_k)^2}{4}\right], & a_k < t \leqslant a_k + 1, \\ \frac{1}{t}\int_0^{a_k} \chi(s)ds + \frac{1}{2^{k+1}}\left[\frac{3}{2t} + 1 - \frac{(a_k+1)}{t}\right], & a_k + 1 < t \leqslant a_{k+1}. \end{cases}$$

If we can prove

$$\frac{1}{a_k}\int_0^{a_k}\chi(s)ds \leqslant 2\chi(a_k) \quad \text{for each } k, \tag{8.2}$$

then the above gives: for $a_k < t \leqslant a_k+1$,

$$\begin{aligned}\frac{1}{t}\int_0^t \chi(s)ds &\leqslant \frac{2a_k}{t}\chi(a_k)+\frac{1}{t2^k}\left[(t-a_k)-\frac{(t-a_k)^2}{4}\right]\\ &= \frac{1}{2^{k+1}}\left[\frac{4a_k}{t}+\frac{2}{t}\left((t-a_k)-\frac{(t-a_k)^2}{4}\right)\right]\\ &= \frac{1}{2^{k+1}}\left[\frac{2a_k}{t}+2-\frac{(t-a_k)^2}{2t}\right]\\ &\leqslant \frac{4}{2^k}\\ &= 4\chi(a_{k+1})\\ &\leqslant 4\chi\left(t/4^{\frac{1}{\gamma}}\right),\end{aligned}$$

and for $a_k+1 < t \leqslant a_{k+1}$

$$\begin{aligned}\frac{1}{t}\int_0^t \chi(s)ds &\leqslant \frac{2a_k}{t}\chi(a_k)+\frac{1}{2^{k+1}}\left[\frac{3}{2t}+1-\frac{(a_k+1)}{t}\right]\\ &= \frac{1}{2^{k+1}}\left[\left(3a_k+\frac{1}{2}\right)\frac{1}{t}+1\right]\\ &\leqslant \frac{1}{2^{k+1}}[3+1]\\ &= 4\chi(a_{k+1})\\ &\leqslant 4\chi\left(t/4^{\frac{1}{\gamma}}\right).\end{aligned}$$

We prove (8.2) by induction. It is readily shown for $k=0$. Assuming it holds for k, we prove it for $k+1$.

Indeed,

$$\begin{aligned}\frac{1}{a_{k+1}}\int_0^{a_{k+1}}\chi(s)ds &= \frac{a_k}{a_{k+1}}\frac{1}{a_k}\int_0^{a_k}\chi(s)ds+\frac{1}{a_{k+1}}\int_{a_k}^{1+a_k}\chi(s)ds+\frac{1}{a_{k+1}}\int_{1+a_k}^{a_{k+1}}\chi(s)ds\\ &\leqslant \frac{a_k}{a_{k+1}}2\chi(a_k)+\frac{1}{a_{k+1}}\frac{1}{2^k}\frac{3}{4}+\frac{1}{2^{k+1}}\left(1-\frac{1+a_k}{a_{k+1}}\right)\\ &= \frac{a_k}{a_{k+1}}\frac{2}{2^k}+\frac{1}{a_{k+1}}\frac{1}{2^k}\frac{3}{4}+\frac{1}{2^{k+1}}-\frac{1}{2^{k+1}}\frac{1}{a_{k+1}}-\frac{1}{2^{k+1}}\frac{a_k}{a_{k+1}}\\ &= \frac{1}{2^k}\left(2-\frac{1}{2}\right)\frac{1}{k+4}+\frac{1}{2^{k+1}}\frac{1}{2}\frac{1}{a_{k+1}}+\frac{1}{2^{k+1}}\\ &= \frac{1}{2^k}\left[\frac{3}{2(k+4)}+\frac{1}{2}+\frac{1}{4((k+4)!)}\right]\\ &< \frac{1}{2^k} = 2\chi(a_{k+1}).\end{aligned}$$

In view of [9, Theorem 1] and [2, Theorem 1], (A_ϕ) and (8.1) are equivalent if $\Psi(2t) \leqslant C\Psi(t)$, $t \in \mathbb{R}_+$, that is $\Psi \in \Delta_2$. Moreover, one can show this is also the case if $\Phi \in \Delta_2$. However, neither $\Psi \in \Delta_2$ nor $\Phi \in \Delta_2$ is necessary for the equivalence of (A_ϕ) and (8.1), since both conditions hold for *all* Young functions when $w(x) \equiv 1$.

Acknowledgement. The authors would like to thank the referee for helping them to clarify the presentation of this paper.

REFERENCES

[1] S. BLOOM AND R. KERMAN, *Weighted L_Φ integral inequalities for operators of Hardy type*, Studia Math. **110**, 1 (1994), 35–52.
[2] S. BLOOM AND R. KERMAN, *Weighted Orlicz space integral inequalities for the Hardy-Littlewood maximal operator*, Studia Math. **110**, 2 (1994), 149–167.
[3] C. BENNETT AND R. SHARPLEY, *Interpolation of operators*, Pure and Applied Mathematics, vol. **129**, Academic Press Inc., Boston, MA, 1988.
[4] A. CIANCHI, *Hardy inequalities in Orlicz spaces*, Trans. Amer. Math. Soc. **351**, 6 (1999), 2459–2478.
[5] R. COIFMAN AND C. FEFFERMAN, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. **51**, 3 (1974), 241–250.
[6] M. FABIAN, P. HABALA, P. HÁJEK, V. MONTESINOS AND V. ZIZLER, *Banach space theory*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, Springer, 2011.
[7] R. KERMAN AND L. PICK, *Explicit formulas for optimal rearrangement-invariant norms in Sobolev imbedding inequalities*, Studia Math. **206**, 2 (2011), 97–119.
[8] R. KERMAN AND A. TORCHINSKY, *Integral inequalities with weights for the Hardy maximal function*, Studia Math. **71**, 3 (1982), 277–284.
[9] L. MALIGRANDA, *Orlicz spaces and interpolation*, Sem. Math. 5, Dep. Mat., Univ. Estadual de Campinas, Campinas SP, Brazil (1989).
[10] W. MATUSZEWSKA AND W. ORLICZ, *On certain properties of φ-functions*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **8** (1960), 439–443.
[11] S. MAZUR AND W. ORLICZ, *On some classes of linear spaces*, Studia Math. **17** (1958), 97–119.
[12] W. ORLICZ, *A note on modular spaces. I*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **9** (1961), 157–162.
[13] A. TORCHINSKY, *Real-variable methods in harmonic analysis*, Pure and Applied Mathematics, vol. **123**, Academic Press, Inc., Orlando, FL, 1986.

(Received July 29, 2017)

Ron Kerman
Department of Mathematics
Brock University
St. Catharines, Ontario, L2S 3A1, Canada
e-mail: rkerman@brocku.ca

Rama Rawat
Department of Mathematics and Statistics
Indian Institute of Technology
Kanpur-208016, India
e-mail: rrawat@iitk.ac.in

Rajesh K. Singh
Department of Mathematics and Statistics
Indian Institute of Technology
Kanpur-208016, India
e-mail: agsinghraj@gmail.com