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WEIGHTED NORM INEQUALITIES FOR PARAMETRIC LITTLEWOOD–PALEY OPERATORS

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Abstract. In this paper, we establish the boundedness of parametric Littlewood-Paley operators from Musielak-Orlicz Hardy space to Musielak-Orlicz space. The endpoint weak type estimates are also obtained. Part of these results are new even for classical Hardy space of Fefferman and Stein.

1. Introduction

The impact of the theory of Hardy space $H^p(\mathbb{R}^n)$ with $p \in (0, 1]$ in the last forty years has been significant. Hardy space first appeared in the work of Hardy [13] in 1914. Its study was based on complex methods and its theory was one-dimensional. The higher dimensional Euclidean theory of Hardy space was developed by Fefferman and Stein [9] who proved a variety of characterizations for them. Later, the advent of its atomic or molecular characterizations enabled the extension of $H^p(\mathbb{R}^n)$ to far more general settings such as space of homogeneous type in the sense of Coifman and Weiss [3]. It is well-known that, when $p \in (0, 1]$, Hardy space $H^p(\mathbb{R}^n)$ is a good substitute of the Lebesgue space $L^p(\mathbb{R}^n)$ in the study for the boundedness of operators. For example, when $p \in (0, 1]$, the Riesz transforms are not bounded on $L^p(\mathbb{R}^n)$, however, they are bounded on $H^p(\mathbb{R}^n)$.

Recently, Ky [19] introduced a new Musielak-Orlicz Hardy space $H^{\varphi}(\mathbb{R}^n)$, which unifies the classical Hardy space [10], the weighted Hardy space [32], the Orlicz Hardy space [14, 15, 16, 17], and the weighted Orlicz Hardy space, in which the spatial and the time variables may not be separable. Apart from interesting theoretical considerations, the motivation to study $H^{\varphi}(\mathbb{R}^n)$ comes from applications to elasticity, fluid dynamics, image processing, nonlinear PDEs and the calculus of variation (see, for example, [4, 5]). More Musielak-Orlicz-type spaces are referred to [2, 8, 21, 24, 25, 35, 36, 37].

On the other hand, various fields of analysis and differential equations require the theory of various function space, for examples, Lebesgue space, Hardy space, various forms of Lipschitz space, BMO space and Sobolev space. From the original definitions of these spaces, it may not appear that they are very closely related. There exist,

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however, various unified approaches to their study. The Littlewood-Paley theory, which arises naturally from the consideration of the Dirichlet problem, provides one of the most successful unifying perspectives on these function spaces (see [11] for more details). And, it remains closely related to the theory of Fourier multipliers (see [12, Chapter 5]).

Suppose that S^{n-1} is the unit sphere in the n-dimensional Euclidean space $\mathbb{R}^n$ ($n \geqslant 2$). Throughout this paper let Ω be a homogeneous function of degree zero on $\mathbb{R}^n$ which is locally integrable and satisfies the cancellation condition

$$\int_{S^{n-1}} \Omega(x')\, d\sigma(x') = 0,$$

where $d\sigma$ is the Lebesgue measure and $x' := x/|x|$ for any $x \neq \mathbf{0}$. For a function f on $\mathbb{R}^n$, *parametric Littlewood-Paley operators* $\mu^{\rho}_{\Omega,S}$ and $\mu^{\rho,*}_{\Omega,\lambda}$ are, respectively, defined by setting, for any $x \in \mathbb{R}^n$,

$$\mu^{\rho}_{\Omega,S}(f)(x) = \left(\int\int_{|y-x|<t} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} f(z)\, dz \right|^2 \frac{dydt}{t^{n+2\rho+1}} \right)^{1/2}$$

and

$$\mu^{\rho,*}_{\Omega,\lambda}(f)(x) = \left[\int\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} f(z)\, dz \right|^2 \frac{dydt}{t^{n+2\rho+1}} \right]^{1/2},$$

where $\rho \in (0,\infty)$ and $\lambda \in (1,\infty)$. The operators $\mu^{\rho}_{\Omega,S}$ and $\mu^{\rho,*}_{\Omega,\lambda}$ were first studied by Sakamoto and Yabuta [30] in 1999. They showed that if $\Omega \in \mathrm{Lip}_{\alpha}(S^{n-1})$ with $\alpha \in (0,1]$, then $\mu^{\rho}_{\Omega,S}$ and $\mu^{\rho,*}_{\Omega,\lambda}$ are bounded on $L^p(\mathbb{R}^n)$ with $p \in (1,\infty)$. In 2009, Xue and Ding [33] obtained a celebrated result that $\mu^{\rho}_{\Omega,S}$ and $\mu^{\rho,*}_{\Omega,\lambda}$ are bounded on $L^p_{\omega}(\mathbb{R}^n)$ with $p \in (1,\infty)$ under weaker smoothness condition of Ω, where $\omega \in A_p$ and A_p denotes the Muckenhoupt weight class. As for their Hardy space boundedness, Ding et al. [6, 7] showed that, if Ω satisfies some weaker smoothness condition, then $\mu^{\rho}_{\Omega,S}$ and $\mu^{\rho,*}_{\Omega,\lambda}$ are bounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. More conclusions of Littlewood-Paley operators are referred to [1, 26, 27, 28, 20, 22, 31, 34].

Motivated by all of the above mentioned facts, a natural and interesting problem arises, that is to say, whether $\mu^{\rho}_{\Omega,S}$ and $\mu^{\rho,*}_{\Omega,\lambda}$ are bounded from Musielak-Orlicz Hardy space $H^{\varphi}(\mathbb{R}^n)$ to Musielak-Orlicz space $L^{\varphi}(\mathbb{R}^n)$. In this paper we shall answer this problem affirmatively. Not only that, we also discuss boundedness of $\mu^{\rho}_{\Omega,S}$ and $\mu^{\rho,*}_{\Omega,\lambda}$ from Musielak-Orlicz Hardy space $H^{\varphi}(\mathbb{R}^n)$ to weak Musielak-Orlicz space $WL^{\varphi}(\mathbb{R}^n)$ at the critical index.

The present paper is built up as follows. In Section 2, we recall some notions concerning Muckenhoupt weights, growth functions and Musielak-Orlicz Hardy space $H^{\varphi}(\mathbb{R}^n)$. Then we state the boundedness of $\mu^{\rho}_{\Omega,S}$ and $\mu^{\rho,*}_{\Omega,\lambda}$ from $H^{\varphi}(\mathbb{R}^n)$ to $L^{\varphi}(\mathbb{R}^n)$ or to $WL^{\varphi}(\mathbb{R}^n)$ (see Theorems 1-4 below), the proofs of which are given in Sections 3 and 4. In the process of the proofs of main results, some boundedness criterions of operators on $H^{\varphi}(\mathbb{R}^n)$ (see [22, Lemma 3.12] and [29, Theorem 3.14]) play an indispensable role.

Finally, we make some conventions on notation. Let $\mathbb{Z}_+ := \{1, 2, \ldots\}$ and $\mathbb{N} := \{0\} \cup \mathbb{Z}_+$. For any $\beta := (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, let $|\beta| := \beta_1 + \cdots + \beta_n$. Throughout this paper the letter C will denote a *positive constant* that may vary from line to line but will remain independent of the main variables. The *symbol* $P \lesssim Q$ stands for the inequality $P \leqslant CQ$. If $P \lesssim Q \lesssim P$, we then write $P \sim Q$. For any set $E \subset \mathbb{R}^n$, we use $E^{\complement}$ to denote the set $\mathbb{R}^n \setminus E$, $|E|$ its *n-dimensional Lebesgue measure* and χ_E its *characteristic function*. For any $s \in \mathbb{R}$, $\lfloor s \rfloor$ denotes the unique integer such that $s - 1 < \lfloor s \rfloor \leqslant s$. If there are no special instructions, any space $\mathscr{X}(\mathbb{R}^n)$ is denoted simply by $\mathscr{X}$. For instance, $L^2(\mathbb{R}^n)$ is simply denoted by L^2. For any index $q \in [1, \infty]$, q' denotes the *conjugate index* of q, namely, $1/q + 1/q' = 1$. For any set E of $\mathbb{R}^n$, $t \in [0, \infty)$ and measurable function φ, let $\varphi(E, t) := \int_E \varphi(x, t)\, dx$ and $\{|f| > t\} := \{x \in \mathbb{R}^n : \ |f(x)| > t\}$. As usual, for any $x \in \mathbb{R}^n$, $r \in (0, \infty)$ and $\alpha \in (0, \infty)$, let $B(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}$ and $\alpha B(x, r) := B(x, \alpha r)$.

2. Notions and main results

In this section, we first recall the notion concerning the Musielak-Orlicz Hardy space H^{φ} via the non-tangential grand maximal function, and then present the boundedness of parametric Littlewood-Paley operators from Musielak-Orlicz Hardy space to Musielak-Orlicz space, or to weak Musielak-Orlicz space at the critical index.

Recall that a nonnegative function φ on $\mathbb{R}^n \times [0, \infty)$ is called a *Musielak-Orlicz function* if, for any $x \in \mathbb{R}^n$, $\varphi(x, \cdot)$ is an Orlicz function on $[0, \infty)$ and, for any $t \in [0, \infty)$, $\varphi(\cdot, t)$ is measurable on $\mathbb{R}^n$. Here a function $\phi : [0, \infty) \to [0, \infty)$ is called an *Orlicz function*, if it is nondecreasing, $\phi(0) = 0$, $\phi(t) > 0$ for any $t \in (0, \infty)$, and $\lim_{t \to \infty} \phi(t) = \infty$.

Given a Musielak-Orlicz function φ on $\mathbb{R}^n \times [0, \infty)$, φ is said to be of *uniformly lower* (resp. *upper*) *type* p with $p \in \mathbb{R}$, if there exists a positive constant $C := C_{\varphi}$ such that, for any $x \in \mathbb{R}^n$, $t \in [0, \infty)$ and $s \in (0, 1]$ (resp. $s \in [1, \infty)$),

$$\varphi(x, st) \leqslant C s^p \varphi(x, t).$$

The *critical uniformly lower type index* and the *critical uniformly upper type index* of φ are, respectively, defined by

$$i(\varphi) := \sup\{p \in \mathbb{R} : \ \varphi \text{ is of uniformly lower type } p\}, \tag{1}$$

and

$$I(\varphi) := \inf\{p \in \mathbb{R} : \ \varphi \text{ is of uniformly upper type } p\}. \tag{2}$$

Observe that $i(\varphi)$ or $I(\varphi)$ may not be attainable, namely, φ may not be of uniformly lower type $i(\varphi)$ or of uniformly upper type $I(\varphi)$ (see [23, p. 415] for more details).

DEFINITION 1. Let $q \in [1, \infty)$. A locally integrable function $\varphi(\cdot, t) : \mathbb{R}^n \to [0, \infty)$ is said to satisfy the *uniformly Muckenhoupt condition* $\mathbb{A}_q$, denoted by $\varphi \in \mathbb{A}_q$, if there exists a positive constant C such that, for any ball $B\subset\mathbb{R}^n$ and $t\in(0,\infty)$, when $q=1$,

$$\frac{1}{|B|}\int_B \varphi(x,t)\,dx\left\{\operatorname*{ess\,sup}_{y\in B}[\varphi(y,t)]^{-1}\right\}\leqslant C$$

and, when $q\in(1,\infty)$,

$$\frac{1}{|B|}\int_B \varphi(x,t)\,dx\left\{\frac{1}{|B|}\int_B[\varphi(y,t)]^{-\frac{1}{q-1}}\,dy\right\}^{q-1}\leqslant C.$$

For $\varphi\in\mathbb{A}_q$ with $q\in[1,\infty)$, we have the following properties as the classical Muckenhoupt weight.

LEMMA 1. ([19, Lemma 4.5]) *Let $\varphi\in\mathbb{A}_q$ with $q\in[1,\infty)$. Then the following statements hold true:*

(i) *there exists a positive constant C such that, for any ball $B\subset\mathbb{R}^n$, $\lambda\in(1,\infty)$ and $t\in(0,\infty)$,*

$$\varphi(\lambda B,t)\leqslant C\lambda^{nq}\varphi(B,t).$$

(ii) *if $q\neq 1$, there exists a positive constant C such that, for any ball $B(x_0,r)\subset\mathbb{R}^n$ and $t\in(0,\infty)$,*

$$\int_{B^{\complement}}\frac{\varphi(x,t)}{|x-x_0|^{nq}}\,dx\leqslant C\frac{\varphi(B(x_0,r),t)}{r^{nq}}.$$

Define $\mathbb{A}_\infty:=\bigcup_{q\in[1,\infty)}\mathbb{A}_q$. It is well-known that if $\varphi\in\mathbb{A}_q$ with $q\in[1,\infty]$, then $\varphi^{\varepsilon}\in\mathbb{A}_q$ for any $\varepsilon\in(0,1]$ and $\varphi^{\eta}\in\mathbb{A}_q$ for some $\eta\in(1,\infty)$. Also, if $\varphi\in\mathbb{A}_q$ with $q\in(1,\infty)$, then $\varphi\in\mathbb{A}_r$ for any $r\in(q,\infty)$ and $\varphi\in\mathbb{A}_d$ for some $d\in(1,q)$. Thus, the *critical weight index* of $\varphi\in\mathbb{A}_\infty$ is defined as follows:

$$q(\varphi):=\inf\{q\in[1,\infty):\ \varphi\in\mathbb{A}_q\}. \tag{3}$$

Observe that, if $q(\varphi)\in(1,\infty)$, then $\varphi\notin\mathbb{A}_{q(\varphi)}$, and there exists $\varphi\notin\mathbb{A}_1$ such that $q(\varphi)=1$ (see [18] for more details).

DEFINITION 2. ([19, Definition 2.1]) A function $\varphi:\mathbb{R}^n\times[0,\infty)\to[0,\infty)$ is called a *growth function* if the following conditions are satisfied:

(i) φ is a Musielak-Orlicz function;

(ii) $\varphi\in\mathbb{A}_\infty$;

(iii) φ is of uniformly lower type p for some $p\in(0,1]$ and of uniformly upper type 1.

Throughout the paper, we always assume that φ is a growth function.

Recall that the *Musielak-Orlicz space* L^φ is defined to be the space of all measurable functions f such that, for some $\eta \in (0,\infty)$,

$$\int_{\mathbb{R}^n} \varphi\left(x, \frac{|f(x)|}{\eta}\right) dx < \infty$$

equipped with the (quasi-)norm

$$\|f\|_{L^\varphi} := \inf\left\{\eta \in (0,\infty) : \int_{\mathbb{R}^n} \varphi\left(x, \frac{|f(x)|}{\eta}\right) dx \leqslant 1\right\}.$$

Similarly, the *weak Musielak-Orlicz space* WL^φ is defined to be the space of all measurable functions f such that, for some $\eta \in (0,\infty)$,

$$\sup_{t\in(0,\infty)} \varphi\left(\{|f|>t\}, \frac{t}{\eta}\right) < \infty$$

equipped with the quasi-norm

$$\|f\|_{WL^\varphi} := \inf\left\{\eta \in (0,\infty) : \sup_{t\in(0,\infty)} \varphi\left(\{|f|>t\}, \frac{t}{\eta}\right) \leqslant 1\right\}.$$

In what follows, we denote by $\mathscr{S}$ the *space of all Schwartz functions* and by $\mathscr{S}'$ its *dual space* (namely, the *space of all tempered distributions*). For any $m \in \mathbb{N}$, let $\mathscr{S}_m$ be the *space* of all $\psi \in \mathscr{S}$ such that $\|\psi\|_{\mathscr{S}_m} \leqslant 1$, where

$$\|\psi\|_{\mathscr{S}_m} := \sup_{\alpha\in\mathbb{N}^n,\,|\alpha|\leqslant m+1} \sup_{x\in\mathbb{R}^n} (1+|x|)^{(m+2)(n+1)} |\partial^\alpha \psi(x)|.$$

Then, for any $m \in \mathbb{N}$ and $f \in \mathscr{S}'$, the *non-tangential grand maximal function* f_m^* of f is defined by setting, for all $x \in \mathbb{R}^n$,

$$f_m^*(x) := \sup_{\psi\in\mathscr{S}_m} \sup_{|y-x|<t,\,t\in(0,\infty)} |f * \psi_t(y)|,$$

where, for any $t \in (0,\infty)$, $\psi_t(\cdot) := t^{-n}\psi(\frac{\cdot}{t})$. When

$$m = m(\varphi) := \left\lfloor n\left(\frac{q(\varphi)}{i(\varphi)} - 1\right)\right\rfloor, \tag{4}$$

we denote f_m^* simply by f^*, where $q(\varphi)$ and $i(\varphi)$ are as in (3) and (1), respectively.

DEFINITION 3. ([19, Definition 2.2]) Let φ be a growth function as in Definition 2. The *Musielak-Orlicz Hardy space* H^φ is defined as the space of all $f \in \mathscr{S}'$ such that $f^* \in L^\varphi$ endowed with the (quasi-)norm

$$\|f\|_{H^\varphi} := \|f^*\|_{L^\varphi}.$$

Below we recall some notions about the kernel Ω. Let $\alpha \in (0,1]$. A function Ω is said to satisfy the *Lipschitz condition of order* α if there exists a positive constant C such that

$$|\Omega(y') - \Omega(z')| \leqslant C|y' - z'|^{\alpha} \text{ for any } y', z' \in S^{n-1}.$$

A function $\Omega \in L^2(S^{n-1})$ is said to satisfy the $L^{2,\alpha}$*-Dini condition* (when $\alpha = 0$, it is called the L^2*-Dini condition*) if

$$\int_0^1 \frac{\omega_2(\delta)}{\delta^{1+\alpha}}\, d\delta < \infty,$$

where

$$\omega_2(\delta) := \sup_{\|\gamma\|<\delta} \left(\int_{S^{n-1}} |\Omega(\gamma x') - \Omega(x')|^2 \, d\sigma(x') \right)^{1/2}$$

and γ denotes a rotation on S^{n-1} with $\|\gamma\| := \sup_{y' \in S^{n-1}} |\gamma y' - y'|$.

The relationship between the Lipschitz condition and the Dini-type condition is not clear up to now.

The main results of this paper are as follows, the proofs of which are given in Sections 3 and 4.

THEOREM 1. *Let* $\alpha \in (0,1]$, $\rho \in (n/2, \infty)$ *and* $\beta \in (0, \min\{1/2, \alpha, \rho - n/2\})$. *Suppose* φ *is a growth function as in Definition 2 with* $p \in (n/(n+\beta), 1]$ *and* $\varphi \in \mathbb{A}_{p(1+\beta/n)}$. *If*

(i) Ω *satisfies the Lipschitz condition of order* α, *or*

(ii) Ω *satisfies the* $L^{2,\alpha}$*-Dini condition,*

then there exists a positive constant C *independent of* f *such that*

$$\left\| \mu^{\rho}_{\Omega,S}(f) \right\|_{L^{\varphi}} \leqslant C \|f\|_{H^{\varphi}}.$$

THEOREM 2. *Let* $\alpha \in (0,1]$, $\rho \in (n/2, \infty)$ *and* $\beta \in (0, \min\{1/2, \alpha, \rho - n/2\})$. *Suppose* φ *is a growth function as in Definition 2 with* $p := n/(n+\beta)$, $\varphi \in \mathbb{A}_1$ *and* $I(\varphi) \in (0,1)$, *where* $I(\varphi)$ *is as in* (2). *If*

(i) Ω *satisfies the Lipschitz condition of order* α, *or*

(ii) Ω *satisfies the* $L^{2,\alpha}$*-Dini condition,*

then there exists a positive constant C *independent of* f *such that*

$$\left\| \mu^{\rho}_{\Omega,S}(f) \right\|_{WL^{\varphi}} \leqslant C \|f\|_{H^{\varphi}}.$$

THEOREM 3. *Let* $\alpha \in (0,1]$, $\rho \in (n/2, \infty)$, $\lambda \in (2, \infty)$ *and* $\beta \in (0, \min\{1/2, \alpha, \rho - n/2, (\lambda - 2)n/3\})$. *Suppose* φ *is a growth function as in Definition 2 with* $p \in (n/(n+\beta), 1]$ *and* $\varphi \in \mathbb{A}_{p(1+\beta/n)}$. *If*

(i) Ω *satisfies the Lipschitz condition of order* α, *or*

(ii) Ω *satisfies the* $L^{2,\alpha}$*-Dini condition,*

then there exists a positive constant C *independent of* f *such that*

$$\left\|\mu_{\Omega,\lambda}^{\rho,*}(f)\right\|_{L^\varphi} \leqslant C\|f\|_{H^\varphi}.$$

THEOREM 4. *Let* $\alpha \in (0,1]$, $\rho \in (n/2,\infty)$, $\lambda \in (2,\infty)$ *and* $\beta \in (0, \min\{1/2, \alpha, \rho - n/2, (\lambda-2)n/3\})$. *Suppose* φ *is a growth function as in Definition 2 with* $p := n/(n+\beta)$, $\varphi \in \mathbb{A}_1$ *and* $I(\varphi) \in (0,1)$, *where* $I(\varphi)$ *is as in* (2). *If*

(i) Ω *satisfies the Lipschitz condition of order* α, *or*

(ii) Ω *satisfies the* $L^{2,\alpha}$*-Dini condition,*

then there exists a positive constant C *independent of* f *such that*

$$\left\|\mu_{\Omega,\lambda}^{\rho,*}(f)\right\|_{WL^\varphi} \leqslant C\|f\|_{H^\varphi}.$$

REMARK 1.

(i) Let ω be a classic Muckenhoupt weight and ϕ an Orlicz function.

(a) When $\varphi(x,t) := \omega(x)\phi(t)$ for all $(x,t) \in \mathbb{R}^n \times [0,\infty)$, we have $H^\varphi = H^\phi_\omega$. In this case, Theorems 1-4 hold true for weighted Orlicz Hardy space. Even when $\varphi(x,t) := \phi(t)$, the above results are also new.

(b) When $\varphi(x,t) := \omega(x)t^p$ for all $(x,t) \in \mathbb{R}^n \times [0,\infty)$, H^φ is reduced to weighted Hardy space H^p_ω. In this case, Theorems 1-4 are new and, even for Hardy space H^p (namely, $\omega \equiv 1$), Theorems 2 and 4 are also new.

(ii) We only prove Theorems 1 and 4, since the proofs of Theorems 2 and 3 are analogous.

3. Proof of Theorem 1

To show Theorem 1, we need some notions and auxiliary lemmas.

DEFINITION 4. ([19, Definition 2.4]) Let φ be a growth function as in Definition 2 and $s \in [m(\varphi),\infty) \cap \mathbb{N}$, where $m(\varphi)$ is as in (4). A measurable function a is called a (φ,∞,s)*-atom* if there exists some ball $B \subset \mathbb{R}^n$ such that the following conditions are satisfied:

(i) a is supported in B;

(ii) $\|a\|_{L^\infty} \leqslant \|\chi_B\|^{-1}_{L^\varphi}$;

(iii) $\int_{\mathbb{R}^n} a(x)x^\alpha dx = 0$ for any $\alpha \in \mathbb{N}^n$ with $|\alpha| \leqslant s$.

LEMMA 2. ([6, Lemma 2.1] or [29, Lemma 4.11]) *Let $\rho\in(0,\infty)$. Suppose that $\Omega\in L^2(S^{n-1})$ satisfies the L^2-Dini condition. Then there exists a positive constant C such that, for any $y\in B(\mathbf{0},R/2)$ with $R\in(0,\infty)$,*

$$\left(\int_{R\leqslant|x|<2R}\left|\frac{\Omega(x-y)}{|x-y|^{n-\rho}}-\frac{\Omega(x)}{|x|^{n-\rho}}\right|^2dx\right)^{1/2}\leqslant CR^{\rho-n/2}\left(\frac{|y|}{R}+\int_{|y|/2R}^{|y|/R}\frac{\omega_2(\delta)}{\delta}d\delta\right).$$

LEMMA 3. *Let $\alpha\in(0,1]$, $\rho\in(n/2,\infty)$ and $\beta\in(0,\min\{1/2,\alpha,\rho-n/2\})$. Suppose b is a multiple of a (φ,∞,s)-atom associated with some ball $B:=B(x_0,r)$. If*

(i) *Ω satisfies the Lipschitz condition of order α, or*

(ii) *Ω satisfies the $L^{2,\alpha}$-Dini condition,*

then there exists a positive constant C independent of b such that, for any $x\in(64B)^\complement$,

$$\mu^\rho_{\Omega,S}(b)(x)\leqslant C\|b\|_{L^\infty}\frac{r^{n+\beta}}{|x-x_0|^{n+\beta}}.$$

Proof. We show this lemma by borrowing some ideas from the proof of [6, Theorem 1]. The trick of the proof is to find a subtle segmentation. For any $x\in(64B)^\complement$, write

$$\begin{aligned}\mu^\rho_{\Omega,S}(b)(x)&=\left(\int\int_{|y-x|<t}\left|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}b(z)\,dz\right|^2\frac{dydt}{t^{n+2\rho+1}}\right)^{1/2}\\&\leqslant\left(\int\int_{\substack{|y-x|<t\\ y\in16B}}\left|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}b(z)\,dz\right|^2\frac{dydt}{t^{n+2\rho+1}}\right)^{1/2}\\&\quad+\left(\int\int_{\substack{|y-x|<t\\ y\in(16B)^\complement\\ t\leqslant|y-x_0|+8r}}\cdots\right)^{1/2}+\left(\int\int_{\substack{|y-x|<t\\ y\in(16B)^\complement\\ t>|y-x_0|+8r}}\cdots\right)^{\frac12}=:\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3.\end{aligned}$$

The estimates of I_1 and I_2 can be showed by the usual argument (see [6, pp. 1541-1542]) with some slight modifications. For the sake of completeness we provide the proofs.

For I_1, by $x\in(64B)^\complement$, $y\in16B$ and $z\in B$, we know that

$$t>|y-x|\geqslant|x-x_0|-|y-x_0|>|x-x_0|-|x-x_0|/4>|x-x_0|/2 \text{ and } |y-z|<32r.$$

From this, $\Omega\in L^2(S^{n-1})$ and $\beta<\rho-n/2$, it follows that, for any $x\in(64B)^\complement$,

$$\mathrm{I}_1=\left(\int\int_{\substack{|y-x|<t\\ y\in16B}}\left|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}b(z)\,dz\right|^2\frac{dydt}{t^{n+2\rho+1}}\right)^{1/2}$$

$$\leqslant \left[\int\int_{\substack{y\in 16B \\ t>|x-x_0|/2}} \left(\int_{|y-z|<32r} \frac{|\Omega(y-z)|}{|y-z|^{n-\rho}} |b(z)|\, dz \right)^2 \frac{dydt}{t^{n+2\rho+1}} \right]^{1/2}$$
$$\leqslant \|b\|_{L^\infty} \left[\int\int_{\substack{y\in 16B \\ t>|x-x_0|/2}} \left(\int_{|z|<32r} \frac{|\Omega(z)|}{|z|^{n-\rho}}\, dz \right)^2 \frac{dydt}{t^{n+2\rho+1}} \right]^{1/2}$$
$$\lesssim \|b\|_{L^\infty} \left(\int_{16B} 1\, dy \int_{\frac{|x-x_0|}{2}}^{\infty} \frac{dt}{t^{n+2\rho+1}} \right)^{1/2} \left(\int_{S^{n-1}} \int_0^{32r} \frac{|\Omega(z')|}{u^{n-\rho}} u^{n-1}\, du d\sigma(z') \right)$$
$$\sim \|b\|_{L^\infty} \frac{r^{\rho+n/2}}{|x-x_0|^{\rho+n/2}} \lesssim \|b\|_{L^\infty} \frac{r^{n+\beta}}{|x-x_0|^{n+\beta}},$$

which is desired.

For I_2, by $x \in (64B)^{\complement}$, $y \in (16B)^{\complement}$, $z \in B$ and the mean value theorem, we know that

$$|y-z| \sim |y-x_0|; \tag{5}$$

$$|y-x_0| - 2r \leqslant |y-x_0| - |x_0-z| \leqslant |y-z| < t \leqslant |y-x_0| + 8r; \tag{6}$$

$$|x-x_0| \leqslant |x-y| + |y-x_0| \leqslant t + |y-x_0| \leqslant 2|y-x_0| + 8r \leqslant 3|y-x_0|; \tag{7}$$

$$\left| \frac{1}{(|y-x_0|-2r)^{n+2\rho}} - \frac{1}{(|y-x_0|+8r)^{n+2\rho}} \right| \lesssim \frac{r}{|y-x_0|^{n+2\rho+1}}. \tag{8}$$

From Minkowski's inequality for integrals, (5)–(8), $\Omega \in L^2(S^{n-1})$ and $\beta < 1/2$, we deduce that, for any $x \in (64B)^{\complement}$,

$$\mathrm{I}_2 = \left(\int\int_{\substack{|y-x|<t \\ y\in(16B)^{\complement} \\ t\leqslant |y-x_0|+8r}} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b(z)\, dz \right|^2 \frac{dydt}{t^{n+2\rho+1}} \right)^{1/2}$$
$$\leqslant \int_B |b(z)| \left(\int\int_{\substack{|y-x|<t \\ y\in(16B)^{\complement} \\ t\leqslant|y-x_0|+8r \\ |y-z|<t}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dydt}{t^{n+2\rho+1}} \right)^{1/2} dz$$
$$\lesssim \int_B |b(z)| \left[\int_{\substack{y\in(16B)^{\complement} \\ |x-x_0|\leqslant 3|y-x_0|}} \frac{|\Omega(y-z)|^2}{|y-x_0|^{2n-2\rho}} \left(\int_{|y-x_0|-2r}^{|y-x_0|+8r} \frac{dt}{t^{n+2\rho+1}} \right) dy \right]^{1/2} dz$$
$$\lesssim \int_B |b(z)| \left(\int_{\substack{y\in(16B)^{\complement} \\ |x-x_0|\leqslant 3|y-x_0|}} \frac{|\Omega(y-z)|^2}{|y-x_0|^{2n-2\rho}} \frac{r}{|y-x_0|^{n+2\rho+1}}\, dy \right)^{1/2} dz$$

$$\lesssim \int_B |b(z)| \left(\int_{\substack{|y-z|\geqslant r \\ |x-x_0|\leqslant 3|y-x_0|}} \frac{|\Omega(y-z)|^2}{|y-x_0|^{n-2\beta+1}} \frac{r}{|x-x_0|^{2n+2\beta}}\, dy \right)^{1/2} dz$$

$$\lesssim \|b\|_{L^\infty} \frac{r^{n+1/2}}{|x-x_0|^{n+\beta}} \left(\int_{S^{n-1}} \int_r^\infty \frac{|\Omega(y')|^2}{u^{n-2\beta+1}} u^{n-1}\, du d\sigma(y') \right)^{1/2}$$

$$\sim \|b\|_{L^\infty} \frac{r^{n+1/2}}{|x-x_0|^{n+\beta}} r^{\beta-1/2} \sim \|b\|_{L^\infty} \frac{r^{n+\beta}}{|x-x_0|^{n+\beta}},$$

which is also desired.

It remains to estimate I_3. It is apparent from $t > |y-x_0|+8r$ that $B \subset \{z \in \mathbb{R}^n : |y-z| < t\}$. By this, the vanishing moments of b, and Minkowski's inequality for integrals, we obtain that, for any $x \in (64B)^\complement$,

$$\mathrm{I}_3 = \left[\int\int_{\substack{|y-x|<t \\ y\in(16B)^\complement \\ t>|y-x_0|+8r}} \left| \int_{|y-z|<t} \left(\frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}} \right) b(z)\, dz \right|^2 \frac{dydt}{t^{n+2\rho+1}} \right]^{1/2}$$

$$\leqslant \int_B |b(z)| \left(\int\int_{\substack{y\in(16B)^\complement \\ t>\max\{|y-x|,\,|y-x_0|+8r,\,|y-z|\}}} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}} \right|^2 \frac{dydt}{t^{n+2\rho+1}} \right)^{1/2} dz$$

$$\leqslant \int_B |b(z)| \left(\int\int_{\substack{y\in(16B)^\complement \\ t>\max\{|y-x|,\,|y-x_0|+8r,\,|y-z|\} \\ |x-x_0|\leqslant 3|y-x_0|}} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}} \right|^2 \frac{dydt}{t^{n+2\rho+1}} \right)^{1/2} dz$$

$$+ \int_B |b(z)| \left(\int\int_{\substack{y\in(16B)^\complement \\ t>\max\{|y-x|,\,|y-x_0|+8r,\,|y-z|\} \\ |x-x_0|>3|y-x_0|}} \cdots \right)^{1/2} dz =: \mathrm{I}_3' + \mathrm{I}_3''.$$

For I_3', we consider two cases.

Case (i). Lipschitz condition of order α. We first claim that, for any $y \in (16B)^\complement$ and $z \in B$,

$$\left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}} \right| \lesssim \frac{|z-x_0|^\alpha}{|y-x_0|^{n-\rho+\alpha}}.$$

Indeed, by the mean value theorem and the assumption that Ω satisfies the Lipschitz condition of order α, we obtain that, for any $y \in (16B)^{\complement}$ and $z \in B$,

$$\begin{aligned}
\left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}}\right| &\leqslant \left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-z)}{|y-x_0|^{n-\rho}}\right| + \left|\frac{\Omega(y-z)}{|y-x_0|^{n-\rho}} - \frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}}\right| \\
&\lesssim \left|\frac{1}{|y-z|^{n-\rho}} - \frac{1}{|y-x_0|^{n-\rho}}\right| \\
&\quad + \frac{1}{|y-x_0|^{n-\rho}}\left|\Omega\left(\frac{y-z}{|y-z|}\right) - \Omega\left(\frac{y-x_0}{|y-x_0|}\right)\right| \\
&\lesssim \frac{|z-x_0|}{|y-x_0|^{n-\rho+1}} + \frac{1}{|y-x_0|^{n-\rho}}\left|\frac{y-z}{|y-z|} - \frac{y-x_0}{|y-x_0|}\right|^{\alpha} \\
&\lesssim \frac{1}{|y-x_0|^{n-\rho}}\frac{|z-x_0|}{|y-x_0|} + \frac{1}{|y-x_0|^{n-\rho}}\left(\frac{|z-x_0|}{|y-x_0|}\right)^{\alpha} \\
&\lesssim \frac{|z-x_0|^{\alpha}}{|y-x_0|^{n-\rho+\alpha}}.
\end{aligned}$$

This inequality and the fact that $\beta < \alpha$ yield that, for any $x \in (64B)^{\complement}$,

$$\begin{aligned}
\mathrm{I}_3' &\leqslant \int_B |b(z)| \left(\iint_{\substack{y\in(16B)^{\complement} \\ t>|y-x_0| \\ |x-x_0|\leqslant 3|y-x_0|}} \left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}}\right|^2 \frac{dydt}{t^{n+2\rho+1}}\right)^{1/2} dz \\
&= \int_B |b(z)| \left[\int_{\substack{y\in(16B)^{\complement} \\ |x-x_0|\leqslant 3|y-x_0|}} \left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}}\right|^2 \left(\int_{|y-x_0|}^{\infty} \frac{dt}{t^{n+2\rho+1}}\right) dy\right]^{1/2} dz \\
&\sim \int_B |b(z)| \left[\int_{\substack{y\in(16B)^{\complement} \\ |x-x_0|\leqslant 3|y-x_0|}} \left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}}\right|^2 \frac{1}{|y-x_0|^{n+2\rho}} dy\right]^{1/2} dz \\
&\lesssim \frac{\|b\|_{L^\infty}}{|x-x_0|^{n+\beta}} \int_B \left(\int_{(16B)^{\complement}} \left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}}\right|^2 \frac{1}{|y-x_0|^{2\rho-n-2\beta}} dy\right)^{1/2} dz \\
&\lesssim \frac{\|b\|_{L^\infty}}{|x-x_0|^{n+\beta}} \int_B \left(\int_{(16B)^{\complement}} \frac{|z-x_0|^{2\alpha}}{|y-x_0|^{2n-2\rho+2\alpha}} \frac{1}{|y-x_0|^{2\rho-n-2\beta}} dy\right)^{1/2} dz \\
&\lesssim \|b\|_{L^\infty} \frac{r^{n+\alpha}}{|x-x_0|^{n+\beta}} \left(\int_{B^{\complement}} \frac{1}{|y-x_0|^{n-2\beta+2\alpha}} dy\right)^{1/2} \\
&\sim \|b\|_{L^\infty} \frac{r^{n+\alpha}}{|x-x_0|^{n+\beta}} \left(\int_{S^{n-1}} \int_r^{\infty} \frac{1}{u^{n-2\beta+2\alpha}} u^{n-1}\, du d\sigma(y')\right)^{1/2} \\
&\sim \|b\|_{L^\infty} \frac{r^{n+\alpha}}{|x-x_0|^{n+\beta}} r^{\beta-\alpha} \sim \|b\|_{L^\infty} \frac{r^{n+\beta}}{|x-x_0|^{n+\beta}}.
\end{aligned}$$

Case (ii). $L^{2,\alpha}$-Dini condition. One may use Lemma 2 and the assumption that Ω satisfies the $L^{2,\alpha}$-Dini condition to deduce that, for any $x \in (64B)^{\complement}$,

$$\begin{aligned}
\mathrm{I}_3' &\lesssim \frac{\|b\|_{L^\infty}}{|x-x_0|^{n+\beta}} \int_B \left(\int_{(16B)^{\complement}} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}} \right|^2 \frac{1}{|y-x_0|^{2\rho-n-2\beta}} dy \right)^{1/2} dz \\
&\lesssim \|b\|_{L^\infty} \frac{1}{|x-x_0|^{n+\beta}} \\
&\quad \times \int_B \sum_{j=4}^{\infty} \left(\int_{2^j r \leqslant |y-x_0| < 2^{j+1} r} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}} \right|^2 \frac{1}{|y-x_0|^{2\rho-n-2\beta}} dy \right)^{1/2} dz \\
&\sim \|b\|_{L^\infty} \frac{1}{|x-x_0|^{n+\beta}} \\
&\quad \times \int_B \sum_{j=4}^{\infty} \frac{1}{(2^j r)^{\rho-n/2-\beta}} \left(\int_{2^j r \leqslant |y-x_0| < 2^{j+1} r} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}} \right|^2 dy \right)^{1/2} dz \\
&\lesssim \|b\|_{L^\infty} \frac{1}{|x-x_0|^{n+\beta}} \int_{|z|<r} \sum_{j=4}^{\infty} \frac{(2^j r)^{\rho-n/2}}{(2^j r)^{\rho-n/2-\beta}} \left(\frac{|z|}{2^j r} + \int_{\frac{|z|}{2^{j+1} r}}^{\frac{|z|}{2^j r}} \frac{\omega_2(\delta)}{\delta} d\delta \right) dz \\
&\lesssim \|b\|_{L^\infty} \frac{1}{|x-x_0|^{n+\beta}} \int_{|z|<r} \sum_{j=4}^{\infty} \frac{(2^j r)^{\rho-n/2}}{(2^j r)^{\rho-n/2-\beta}} \left[\frac{|z|}{2^j r} + \left(\frac{|z|}{2^j r} \right)^{\alpha} \right] dz \\
&\lesssim \|b\|_{L^\infty} \frac{1}{|x-x_0|^{n+\beta}} \int_{|z|<r} \sum_{j=4}^{\infty} (2^j r)^{\beta} 2^{-j\alpha} dz \sim \|b\|_{L^\infty} \frac{r^{n+\beta}}{|x-x_0|^{n+\beta}},
\end{aligned}$$

where the last "$\sim$" is due to $\beta < \alpha$.

Now we are interested in I_3''. Noticing that $t > \max\{|y-x|, |y-x_0|+8r, |y-z|\}$ and $|x-x_0| > 3|y-x_0|$, we see that

$$t > |y-x| \geqslant |x-x_0| - |y-x_0| > |x-x_0|/2.$$

From this, $\beta < \rho - n/2$ and the argument same as in I_3', it follows that, for any $x \in (64B)^{\complement}$,

$$\begin{aligned}
\mathrm{I}_3'' &\leqslant \int_B |b(z)| \left[\int_{(16B)^{\complement}} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}} \right|^2 \left(\int_{\substack{t>|y-x_0| \\ t>|x-x_0|/2}} \frac{dt}{t^{n+2\rho+1}} \right) dy \right]^{1/2} dz \\
&\lesssim \frac{\|b\|_{L^\infty}}{|x-x_0|^{n+\beta}} \int_B \left[\int_{(16B)^{\complement}} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}} \right|^2 \left(\int_{|y-x_0|}^{\infty} \frac{dt}{t^{2\rho-n-2\beta+1}} \right) dy \right]^{1/2} dz \\
&\sim \frac{\|b\|_{L^\infty}}{|x-x_0|^{n+\beta}} \int_B \left(\int_{(16B)^{\complement}} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}} \right|^2 \frac{1}{|y-x_0|^{2\rho-n-2\beta}} dy \right)^{1/2} dz \\
&\lesssim \|b\|_{L^\infty} \frac{r^{n+\beta}}{|x-x_0|^{n+\beta}}.
\end{aligned}$$

Combining the estimates of I_1, I_2, I_3' and I_3'', we obtain the desired inequality. This finishes the proof of Lemma 3. □

Proof of Theorem 1. Obviously, $\mu_{\Omega,S}^{\rho}$ is a positive sublinear operator and bounded on L^2. Thus, by the boundedness criterions of operators from H^{φ} to L^{φ} (see [22, Lemma 3.12] or [29, Theorem 3.11]), Theorem 1 will be proved by showing that $\mu_{\Omega,S}^{\rho}$ maps all multiple of a (φ,∞,s)-atoms into the bounded elements of L^{φ} uniformly, namely, there exists a positive constant C such that, for any $\eta\in(0,\infty)$ and multiple of a (φ,∞,s)-atom b associated with some ball $B:=B(x_0,r)\subset\mathbb{R}^n$,

$$\int_{\mathbb{R}^n}\varphi\left(x,\frac{\mu_{\Omega,S}^{\rho}(b)(x)}{\eta}\right)dx\leqslant C\varphi\left(B,\frac{\|b\|_{L^\infty}}{\eta}\right).$$

For any $\eta\in(0,\infty)$, write

$$\int_{\mathbb{R}^n}\varphi\left(x,\frac{\mu_{\Omega,S}^{\rho}(b)(x)}{\eta}\right)dx=\int_{64B}\varphi\left(x,\frac{\mu_{\Omega,S}^{\rho}(b)(x)}{\eta}\right)dx+\int_{(64B)^\complement}\cdots=:\mathrm{P}_1+\mathrm{P}_2.$$

For P_1, noticing that $p>n/(n+\beta)$, we see that $\varphi\in\mathbb{A}_2$. From the uniformly upper type 1 property of φ, the weighted L^2-boundedness of $\mu_{\Omega,S}^{\rho}$ with $\varphi\in\mathbb{A}_2$ (see [33, Theorem 1]), and Lemma 1(i) with $\varphi\in\mathbb{A}_2$, it follows that, for any $\eta\in(0,\infty)$,

$$\begin{aligned}\mathrm{P}_1&\lesssim\int_{64B}\left(1+\frac{|\mu_{\Omega,S}^{\rho}(b)(x)|}{\|b\|_{L^\infty}}\right)^2\varphi\left(x,\frac{\|b\|_{L^\infty}}{\eta}\right)dx\\&\lesssim\int_{64B}\left(1+\frac{|\mu_{\Omega,S}^{\rho}(b)(x)|^2}{\|b\|_{L^\infty}^2}\right)\varphi\left(x,\frac{\|b\|_{L^\infty}}{\eta}\right)dx\\&\lesssim\varphi\left(64B,\frac{\|b\|_{L^\infty}}{\eta}\right)+\frac{1}{\|b\|_{L^\infty}^2}\int_{\mathbb{R}^n}|\mu_{\Omega,S}^{\rho}(b)(x)|^2\varphi\left(x,\frac{\|b\|_{L^\infty}}{\eta}\right)dx\\&\lesssim\varphi\left(64B,\frac{\|b\|_{L^\infty}}{\eta}\right)+\frac{1}{\|b\|_{L^\infty}^2}\int_{B}|b(x)|^2\varphi\left(x,\frac{\|b\|_{L^\infty}}{\eta}\right)dx\\&\lesssim\varphi\left(B,\frac{\|b\|_{L^\infty}}{\eta}\right).\end{aligned}$$

For P_2, by Lemma 3, the uniformly lower type p property of φ, and Lemma 1(ii) with $\varphi\in\mathbb{A}_{p(1+\beta/n)}$, we know that, for any $\eta\in(0,\infty)$,

$$\begin{aligned}\mathrm{P}_2&\lesssim\int_{(64B)^\complement}\varphi\left(x,\frac{\|b\|_{L^\infty}}{\eta}\frac{r^{n+\beta}}{|x-x_0|^{n+\beta}}\right)dx\\&\lesssim r^{(n+\beta)p}\int_{B^\complement}\frac{1}{|x-x_0|^{(n+\beta)p}}\varphi\left(x,\frac{\|b\|_{L^\infty}}{\eta}\right)dx\lesssim\varphi\left(B,\frac{\|b\|_{L^\infty}}{\eta}\right).\end{aligned}$$

Combining the estimates of P_1 and P_2, we obtain the desired inequality. This finishes the proof of Theorem 1. □

4. Proof of Theorem 4

To show Theorem 4, we need to establish the following lemma.

LEMMA 4. *Let* $\alpha \in (0,1]$, $\rho \in (n/2,\infty)$, $\lambda \in (2,\infty)$ *and* $\beta \in (0, \min\{1/2, \alpha, \rho - n/2, (\lambda-2)n/3\})$. *Suppose* b *is a multiple of a* (φ,∞,s)*-atom associated with some ball* $B := B(x_0, r)$. *If*

(i) Ω *satisfies the Lipschitz condition of order* α, *or*

(ii) Ω *satisfies the* $L^{2,\alpha}$*-Dini condition,*

then there exists a positive constant C *independent of* b *such that, for any* $x \in (64B)^{\complement}$,

$$\mu_{\Omega,\lambda}^{\rho,*}(b)(x) \leqslant C\|b\|_{L^\infty} \frac{r^{n+\beta}}{|x-x_0|^{n+\beta}}.$$

Proof. We show this lemma by borrowing some ideas from the proof of [7, Theorem 1.1]. By Lemma 3, we know that, for any $x \in (64B)^{\complement}$,

$$\begin{aligned}
\mu_{\Omega,\lambda}^{\rho,*}(b)(x) &= \left[\int\int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left|\int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b(z)\,dz\right|^2 \frac{dydt}{t^{n+2\rho+1}}\right]^{1/2} \\
&\leqslant \left[\int\int_{|y-x|<t} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left|\int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b(z)\,dz\right|^2 \frac{dydt}{t^{n+2\rho+1}}\right]^{1/2} \\
&\quad + \left[\int\int_{|y-x|\geqslant t} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left|\int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b(z)\,dz\right|^2 \frac{dydt}{t^{n+2\rho+1}}\right]^{1/2} \\
&\leqslant \mu_{\Omega,S}^{\rho}(b)(x) \\
&\quad + \left[\int\int_{|y-x|\geqslant t} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left|\int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b(z)\,dz\right|^2 \frac{dydt}{t^{n+2\rho+1}}\right]^{1/2} \\
&\leqslant C\|b\|_{L^\infty} \frac{r^{n+\beta}}{|x-x_0|^{n+\beta}} \\
&\quad + \left[\int\int_{|y-x|\geqslant t} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left|\int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b(z)\,dz\right|^2 \frac{dydt}{t^{n+2\rho+1}}\right]^{1/2} \\
&=: C\|b\|_{L^\infty} \frac{r^{n+\beta}}{|x-x_0|^{n+\beta}} + \mathrm{J}.
\end{aligned}$$

Thus, to show Lemma 4, it suffices to prove that, for any $x \in (64B)^{\complement}$,

$$\mathrm{J} \lesssim \|b\|_{L^\infty} \frac{r^{n+\beta}}{|x-x_0|^{n+\beta}}.$$

For any $x \in (64B)^{\complement}$, write

$$\mathrm{J} \leqslant \left[\int\int_{\substack{|y-x|\geqslant t\\ y\in 16B}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left|\int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b(z)\,dz\right|^2 \frac{dydt}{t^{n+2\rho+1}}\right]^{1/2}$$
$$+\left[\int\int_{\substack{|y-x|\geqslant t\\ y\in(16B)^{\complement}\\ t\leqslant|y-x_0|+8r}} \cdots\right]^{1/2} + \left[\int\int_{\substack{|y-x|\geqslant t\\ y\in(16B)^{\complement}\\ t>|y-x_0|+8r}} \cdots\right]^{1/2} =: \mathrm{J}_1+\mathrm{J}_2+\mathrm{J}_3.$$

For J_1, from Minkowski's inequality for integrals, $\beta < \min\{(\lambda-2)n/3, \rho - n/2\}$, $|y-x| \sim |x-x_0|$ with $x \in (64B)^{\complement}$ and $y \in 16B$, and $\Omega \in L^2(S^{n-1})$, it follows that, for any $x \in (64B)^{\complement}$,

$$\mathrm{J}_1 \leqslant \int_B |b(z)| \left[\int\int_{\substack{|y-x|\geqslant t\\ y\in 16B\\ |y-z|<32r,\,|y-z|<t}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dydt}{t^{n+2\rho+1}}\right]^{1/2} dz$$
$$\lesssim \int_B |b(z)| \left[\int\int_{\substack{|y-x|\geqslant t\\ |y-z|<32r}} \left(\frac{t}{|y-z|}\right)^{2\rho-n-2\beta}\right.$$
$$\left.\times\left(\frac{t}{|x-x_0|}\right)^{2n+3\beta} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dydt}{t^{n+2\rho+1}}\right]^{1/2} dz$$
$$\sim \int_B |b(z)| \left[\int_{|y-z|<32r} \frac{|\Omega(y-z)|^2}{|x-x_0|^{2n+3\beta}|y-z|^{n-2\beta}} \left(\int_0^{|y-x|} t^{\beta-1}dt\right) dy\right]^{1/2} dz$$
$$\sim \int_B |b(z)| \left(\int_{|y-z|<32r} \frac{|y-x|^{\beta}|\Omega(y-z)|^2}{|x-x_0|^{2n+3\beta}|y-z|^{n-2\beta}} dy\right)^{1/2} dz$$
$$\sim \frac{1}{|x-x_0|^{n+\beta}} \int_B |b(z)| \left(\int_{|y|<32r} \frac{|\Omega(y)|^2}{|y|^{n-2\beta}} dy\right)^{1/2} dz$$
$$\sim \|b\|_{L^\infty} \frac{r^n}{|x-x_0|^{n+\beta}} \left(\int_{S^{n-1}} \int_0^{32r} \frac{|\Omega(y')|^2}{u^{n-2\beta}} u^{n-1}\,dud\sigma(y')\right)^{1/2}$$
$$\sim \|b\|_{L^\infty} \frac{r^{n+\beta}}{|x-x_0|^{n+\beta}},$$

which is desired.

For J_2, write

$$\mathrm{J}_2 \leqslant \left[\int\int_{\substack{|y-x|\geqslant t\\ y\in(16B)^{\complement}\\ t\leqslant|y-x_0|+8r\\ |x-x_0|>3|y-x_0|}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left|\int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b(z)\,dz\right|^2 \frac{dydt}{t^{n+2\rho+1}}\right]^{1/2}$$
$$+\left[\int\int_{\substack{y\in(16B)^{\complement}\\ |x-x_0|\leqslant 3|y-x_0|\\ |y-x_0|-2r<t\leqslant|y-x_0|+8r}} \left|\int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b(z)\,dz\right|^2 \frac{dydt}{t^{n+2\rho+1}}\right]^{1/2} =: \mathrm{J}_2'+\mathrm{J}_2''.$$

The estimate of J_2'' is quite similar to that given earlier for the estimate of I_2 in Lemma 3 and so is omitted. We are now turning to the estimate of J_2'.

For J_2', by $x \in (64B)^{\complement}$, $y \in (16B)^{\complement}$, $z \in B$, $|x-x_0| > 3|y-x_0|$ and the mean value theorem, we know that

$$|y-z| \sim |y-x_0|; \tag{9}$$

$$|y-x_0| - 2r \leqslant |y-x_0| - |x_0-z| \leqslant |y-z| < t \leqslant |y-x_0| + 8r; \tag{10}$$

$$|x-y| \geqslant |x-x_0| - |y-x_0| > |x-x_0|/2; \tag{11}$$

$$\left| \frac{1}{(|y-x_0|-2r)^{2\rho-n-2\beta}} - \frac{1}{(|y-x_0|+8r)^{2\rho-n-2\beta}} \right| \lesssim \frac{r}{|y-x_0|^{2\rho-n-2\beta+1}}. \tag{12}$$

By Minkowski's inequality for integrals, $\beta < \min\{(\lambda-2)n/3, 1/2\} < (\lambda-2)n/2$ and (9)–(12) and $\Omega \in L^2(S^{n-1})$, we know that, for any $x \in (64B)^{\complement}$,

$$\begin{aligned}
\mathrm{J}_2' &= \left[\iint_{\substack{|y-x|\geqslant t\\ y\in(16B)^{\complement}\\ t\leqslant|y-x_0|+8r\\ |x-x_0|>3|y-x_0|}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b(z)\,dz \right|^2 \frac{dydt}{t^{n+2\rho+1}} \right]^{1/2} \\
&\leqslant \int_B |b(z)| \left[\iint_{\substack{|y-x|\geqslant t\\ y\in(16B)^{\complement}\\ t\leqslant|y-x_0|+8r\\ |x-x_0|>3|y-x_0|\\ |y-z|<t}} \left(\frac{t}{t+|x-y|}\right)^{2n+2\beta} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dydt}{t^{n+2\rho+1}} \right]^{1/2} dz \\
&\lesssim \int_B |b(z)| \left[\iint_{\substack{y\in(16B)^{\complement}\\ |x-y|>|x-x_0|/2\\ |y-x_0|-2r\leqslant t\leqslant|y-x_0|+8r}} \left(\frac{t}{|x-y|}\right)^{2n+2\beta} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dydt}{t^{n+2\rho+1}} \right]^{1/2} dz \\
&\lesssim \frac{1}{|x-x_0|^{n+\beta}} \int_B |b(z)| \left[\int_{(16B)^{\complement}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \left(\int_{|y-x_0|-2r}^{|y-x_0|+8r} \frac{dt}{t^{2\rho-n-2\beta+1}} \right) dy \right]^{1/2} dz \\
&\lesssim \frac{1}{|x-x_0|^{n+\beta}} \int_B |b(z)| \left(\int_{(16B)^{\complement}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{r}{|y-x_0|^{2\rho-n-2\beta+1}} dy \right)^{1/2} dz \\
&\lesssim \|b\|_{L^\infty} \frac{r^{n+1/2}}{|x-x_0|^{n+\beta}} \left(\int_{S^{n-1}} \int_r^\infty \frac{|\Omega(y')|^2}{u^{n-2\beta+1}} u^{n-1}\,du\,d\sigma(y') \right)^{1/2} \\
&\sim \|b\|_{L^\infty} \frac{r^{n+1/2}}{|x-x_0|^{n+\beta}} r^{\beta-1/2} \sim \|b\|_{L^\infty} \frac{r^{n+\beta}}{|x-x_0|^{n+\beta}},
\end{aligned}$$

which is also desired.

For J_3, noticing that $t > |y-x_0| + 8r$, we see that, for any $y \in (16B)^{\complement}$,

$$B \subset \{z \in \mathbb{R}^n : \ |z-y| < t\} \tag{13}$$

and

$$t+|x-y| \geqslant t+|x-x_0|-|y-x_0| \geqslant |x-x_0|+8r > |x-x_0|. \tag{14}$$

From (13), the vanishing moments of b, Minkowski's inequality for integrals, (14), $\beta < \min\{\alpha, \rho - n/2, (\lambda-2)n/3\} < (\lambda-2)n/2$ and the argument same as in I_3' of Lemma 3, it follows that, for any $x \in (64B)^{\complement}$,

$$\begin{aligned}
\mathrm{J}_3 &= \left[\iint_{\substack{|y-x|\geqslant t\\ y\in(16B)^{\complement}\\ t>|y-x_0|+8r}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left|\int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b(z)\,dz\right|^2 \frac{dydt}{t^{n+2\rho+1}}\right]^{1/2} \\
&\leqslant \int_B |b(z)| \\
&\quad \times \left[\iint_{\substack{|y-x|\geqslant t\\ y\in(16B)^{\complement}\\ t>|y-x_0|+8r\\ |y-z|<t\\ t+|x-y|>|x-x_0|}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}}\right|^2 \frac{dydt}{t^{n+2\rho+1}}\right]^{1/2} dz \\
&\lesssim \int_B |b(z)| \left[\iint_{\substack{|y-x|\geqslant t\\ y\in(16B)^{\complement}\\ t>|y-x_0|+8r\\ |y-z|<t\\ t+|x-y|>|x-x_0|}} \left(\frac{t+|x-y|}{|x-x_0|}\right)^{2n+2\beta} \left(\frac{t}{t+|x-y|}\right)^{\lambda n}\right. \\
&\qquad\qquad \left. \times \left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}}\right|^2 \frac{dydt}{t^{n+2\rho+1}}\right]^{1/2} dz \\
&\lesssim \int_B |b(z)| \left[\int\int_{\substack{y\in(16B)^{\complement}\\ t>|y-x_0|}} \frac{1}{|x-x_0|^{2n+2\beta}} \frac{t^{\lambda n}}{(t+|x-y|)^{\lambda n-2n-2\beta}}\right. \\
&\qquad\qquad \left. \times \left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}}\right|^2 \frac{dydt}{t^{n+2\rho+1}}\right]^{1/2} dz \\
&\lesssim \frac{1}{|x-x_0|^{n+\beta}} \int_B |b(z)| \left[\int_{(16B)^{\complement}} \left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}}\right|^2\right. \\
&\qquad\qquad \left. \times \left(\int_{|y-x_0|}^{\infty} \frac{dt}{t^{2\rho-n-2\beta+1}}\right) dy\right]^{1/2} dz \\
&\lesssim \frac{\|b\|_{L^\infty}}{|x-x_0|^{n+\beta}} \int_B \left(\int_{(16B)^{\complement}} \left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}}\right|^2 \frac{1}{|y-x_0|^{2\rho-n-2\beta}} dy\right)^{1/2} dz \\
&\lesssim \|b\|_{L^\infty} \frac{r^{n+\beta}}{|x-x_0|^{n+\beta}}.
\end{aligned}$$

Combining the estimates of J_1, J_2 and J_3, we obtain the desired inequality. This finishes the proof of Lemma 4. □

Proof of Theorem 4. Obviously, $\mu_{\Omega,\lambda}^{\rho,*}$ is a positive sublinear operator and bounded on L^2. Thus, by the boundedness criterions of operators from H^φ to WL^φ (see [29, Theorem 3.14]), Theorem 4 will be proved by showing that $\mu_{\Omega,\lambda}^{\rho,*}$ maps all multiple of (φ,∞,s)-atoms into the bounded elements of WL^φ uniformly, namely, there exists a positive constant C such that, for any $\eta\in(0,\infty)$ and multiple of a (φ,∞,s)-atom b associated with some ball $B:=B(x_0,r)\subset\mathbb{R}^n$,

$$\sup_{t\in(0,\infty)}\varphi\left(\left\{\mu_{\Omega,\lambda}^{\rho,*}(b)>t\right\},\frac{t}{\eta}\right)\leqslant C\varphi\left(B,\frac{\|b\|_{L^\infty}}{\eta}\right).$$

For any $\eta\in(0,\infty)$, write

$$\begin{aligned}&\sup_{t\in(0,\infty)}\varphi\left(\left\{\mu_{\Omega,\lambda}^{\rho,*}(b)>t\right\},\frac{t}{\eta}\right)\\&\leqslant\sup_{t\in(0,\infty)}\varphi\left(\left\{x\in 64B:\ \mu_{\Omega,\lambda}^{\rho,*}(b)(x)>t\right\},\frac{t}{\eta}\right)\\&\quad+\sup_{t\in(0,\infty)}\varphi\left(\left\{x\in(64B)^{\complement}:\ \mu_{\Omega,\lambda}^{\rho,*}(b)(x)>t\right\},\frac{t}{\eta}\right)=:\mathrm{Q}_1+\mathrm{Q}_2.\end{aligned}$$

For Q_1, by the uniformly upper type 1 property of φ, the weighted L^2-boundedness of $\mu_{\Omega,\lambda}^{\rho,*}$ with $\varphi\in\mathbb{A}_2$ (see [33, Theorem 1]), and Lemma 1(i) with $\varphi\in\mathbb{A}_2$, we obtain that, for any $\eta\in(0,\infty)$,

$$\begin{aligned}\mathrm{Q}_1&=\sup_{t\in(0,\infty)}\int_{\left\{x\in 64B:\ \mu_{\Omega,\lambda}^{\rho,*}(b)(x)>t\right\}}\varphi\left(x,\frac{t}{\eta}\right)dx\\&\leqslant\int_{64B}\varphi\left(x,\frac{\mu_{\Omega,\lambda}^{\rho,*}(b)(x)}{\eta}\right)dx\\&\lesssim\int_{64B}\left(1+\frac{|\mu_{\Omega,\lambda}^{\rho,*}(b)(x)|}{\|b\|_{L^\infty}}\right)^2\varphi\left(x,\frac{\|b\|_{L^\infty}}{\eta}\right)dx\\&\lesssim\int_{64B}\left(1+\frac{|\mu_{\Omega,\lambda}^{\rho,*}(b)(x)|^2}{\|b\|_{L^\infty}^2}\right)\varphi\left(x,\frac{\|b\|_{L^\infty}}{\eta}\right)dx\\&\lesssim\varphi\left(64B,\frac{\|b\|_{L^\infty}}{\eta}\right)+\frac{1}{\|b\|_{L^\infty}^2}\int_{\mathbb{R}^n}|\mu_{\Omega,\lambda}^{\rho,*}(b)(x)|^2\varphi\left(x,\frac{\|b\|_{L^\infty}}{\eta}\right)dx\\&\lesssim\varphi\left(64B,\frac{\|b\|_{L^\infty}}{\eta}\right)+\frac{1}{\|b\|_{L^\infty}^2}\int_B|b(x)|^2\varphi\left(x,\frac{\|b\|_{L^\infty}}{\eta}\right)dx\\&\lesssim\varphi\left(B,\frac{\|b\|_{L^\infty}}{\eta}\right).\end{aligned}$$

For Q_2, from Lemma 4, Lemma 1(i) with $\varphi \in \mathbb{A}_1$, and the uniformly lower type $p = n/(n+\beta)$ property of φ, we deduce that, for any $\eta \in (0,\infty)$,

$$\begin{aligned}
\mathrm{Q}_2 &\lesssim \sup_{t\in(0,\infty)} \varphi\left(\left\{x \in (64B)^{\complement}:\ \|b\|_{L^\infty}\frac{r^{n+\beta}}{|x-x_0|^{n+\beta}} > t\right\}, \frac{t}{\eta}\right)\\
&\lesssim \sup_{t\in(0,\infty)} \varphi\left(\left\{x \in B^{\complement}:\ |x-x_0|^{n+\beta} < \frac{\|b\|_{L^\infty}}{t} r^{n+\beta}\right\}, \frac{t}{\eta}\right)\\
&\sim \sup_{t\in(0,\infty)} \varphi\left(\left\{x \in \mathbb{R}^n:\ r \leqslant |x-x_0| < \left(\frac{\|b\|_{L^\infty}}{t}\right)^{\frac{1}{n+\beta}} r\right\}, \frac{t}{\eta}\right)\\
&\lesssim \sup_{t\in(0,\|b\|_{L^\infty})} \varphi\left(\left\{x \in \mathbb{R}^n:\ |x-x_0| < \left(\frac{\|b\|_{L^\infty}}{t}\right)^{\frac{1}{n+\beta}} r\right\}, \frac{t}{\eta}\right)\\
&\sim \sup_{t\in(0,\|b\|_{L^\infty})} \varphi\left(\left[\frac{\|b\|_{L^\infty}}{t}\right]^{\frac{1}{n+\beta}} B, \frac{t}{\eta}\right)\\
&\lesssim \sup_{t\in(0,\|b\|_{L^\infty})} \left(\frac{\|b\|_{L^\infty}}{t}\right)^{p} \varphi\left(B, \frac{t}{\eta}\right)\\
&\lesssim \sup_{t\in(0,\|b\|_{L^\infty})} \left(\frac{\|b\|_{L^\infty}}{t}\right)^{p} \left(\frac{t}{\|b\|_{L^\infty}}\right)^{p} \varphi\left(B, \frac{\|b\|_{L^\infty}}{\eta}\right)\\
&\sim \varphi\left(B, \frac{\|b\|_{L^\infty}}{\eta}\right).
\end{aligned}$$

Combining the estimates of Q_1 and Q_2, we obtain the desired inequality. This finishes the proof of Theorem 4. $\square$

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