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INTERVAL–TYPE THEOREMS CONCERNING QUASI–ARITHMETIC MEANS

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Abstract. Family of quasi-arithmetic means has a natural, partial order (point-wise order) $A^{[f]} \leqslant A^{[g]}$ if and only if $A^{[f]}(v) \leqslant A^{[g]}(v)$ for all admissible vectors v (f, g and, later, h are continuous, monotone and defined on a common interval).

Therefore one can introduce the notion of interval-type sets (sets $\mathscr{I}$ such that whenever $A^{[f]} \leqslant A^{[h]} \leqslant A^{[g]}$ for some $A^{[f]}, A^{[g]} \in \mathscr{I}$ then $A^{[h]} \in \mathscr{I}$ too).

Our aim is to give examples of interval-type sets involving vary smoothness assumptions of generating functions.

1. Introduction

In a recent paper [7] author introduced a new definition concerning means. A family $\mathscr{M}$ of means (functions) defined on a common domain is embedded in a natural partial order, that is for every $M, N \in \mathscr{M}$ we have

$$M \leqslant N \iff M(x) \leqslant N(x) \text{ for all } x.$$

In this setting we call $\mathscr{I} \subset \mathscr{M}$ to be an *interval-type set in* $\mathscr{M}$ (briefly: *interval-type set* or *interval*) if whenever $P \in \mathscr{M}$ and $M \leqslant P \leqslant N$ for some $M, N \in \mathscr{I}$ then also $P \in \mathscr{I}$.

Many families of means are linearly ordered by this process. For example one of the most classical result in a theory of means states that power means are linearly ordered, that is if we denote by $\mathscr{P}_p$ the p-th power mean, then $(\{\mathscr{P}_p\}_{p\in\mathbb{R}}, \leqslant)$ is isomorphic to $(\mathbb{R}, \leqslant)$ under the natural isomorphism $\mathscr{P}_p \mapsto p$. In particular all intervals in this family could be trivially described.

Situation becomes much more interesting if there appear means which are not comparable among each other. Perhaps the most famous family of this type are quasi-arithmetic means. They were introduced in series of nearly simultaneous papers in a beginning of 1930s [1, 3, 5] as a generalization of already mentioned family of power

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means. For a continuous and strictly monotone function $f\colon I \to \mathbb{R}$ (I is an interval) and a vector $a=(a_1,a_2,\dots,a_n)\in I^n$, $n\in\mathbb{N}$ we define

$$A^{[f]} := f^{-1}\left(\frac{f(a_1)+f(a_2)+\cdots+f(a_n)}{n}\right).$$

It is easy to verify that for $I=\mathbb{R}_+$ and $f=\pi_p$, where $\pi_p(x):=x^p$ if $p\neq 0$ and $\pi_0(x):=\ln x$, the mean $A^{[f]}$ coincides with $\mathscr{P}_p$ (this fact had been already noticed by Knopp [2] before quasi-arithmetic means were formally introduced).

Then, whenever f and g are defined on a common interval I, we get

$$A^{[f]} \leqslant A^{[g]} \text{ if and only if } A^{[f]}(a) \leqslant A^{[g]}(a) \text{ for all } a \in \bigcup_{n=1}^{\infty} I^n.$$

As we do not define comparability of means defined on two different intervals, throughout all quasi-arithmetic means are considered on an arbitrary, but common, interval (from now on denoted by I). We will be dealing with interval-type sets in a family of quasi-arithmetic means defined on I (we will call them briefly interval-type sets or intervals).

Let us recall some simple, however important, results from our previous paper [7]. It could be proved that interval-type sets inherit many properties of regular intervals in $\mathbb{R}$. For example intersection of any number of intervals are again an interval, increasing sum of intervals are again an interval and so on – proofs of this facts are elementary and omitted here; for detailed discussion we refer the reader to [7]. Moreover, if $D\subset \bigcup_{n=1}^{\infty} I^n$ and $L,\,U\colon D\to\mathbb{R}$ are arbitrary functions then both

$$[L,+\infty) := \{A^{[f]}\colon L(v)\leqslant A^{[f]}(v) \text{ for all } v\in D\},$$
$$(L,+\infty) := \{A^{[f]}\colon L(v)< A^{[f]}(v) \text{ for all } v\in D\}$$

are intervals. Similarly we can define all possible intervals of this type involving $-\infty$. Having this we define bounded intervals of this type as an intersection; for example $[L,U) := [L,+\infty)\cap(-\infty,U)$ etc.

Furthermore, as we have only a partial order, it is reasonable to define, for every family $\mathscr{F}$ of quasi-arithmetic means, the smallest interval-type set containing $\mathscr{F}$. We will denote such a set by

$$[[\mathscr{F}]] := \bigcap\big\{\mathscr{I}\colon \mathscr{I} \text{ is an interval and } \mathscr{F}\subset\mathscr{I}\big\}.$$

In a special case when each pair of elements in $\mathscr{F}$ has both the lower and the upper bound in $\mathscr{F}$ we obtain

$$[[\mathscr{F}]] = \bigcup_{\substack{X,Y\in\mathscr{F}\\ X\leqslant Y}} [X,Y]. \tag{1}$$

Proof of this equality is elementary and we omit it.

2. Comparability among quasi-arithmetic means

It could happen that two intervals has a non-empty intersection although its sum is not an interval. Indeed, the family

$$[A^{[f]}]^* := (-\infty, A^{[f]}] \cup [A^{[f]}, +\infty)$$

is a family of all quasi-arithmetic means which are comparable with $A^{[f]}$. Investigating properties of this set is somehow outside the scope of the present paper, as it is not an interval. Let us just notice that for arithmetic mean $[A^{[\pi_1]}]^*$ is a family of quasi-arithmetic means generated by either convex functions or concave functions, which is the classical application of Jensen inequality.

In fact Jensen inequality is closely related with comparability of quasi-arithmetic means. In what follows we will present a number of equivalent conditions in a series of propositions. They will be uniquely numerated, as we will refer to each of them just by mentioning its identifier.

PROPOSITION 1. *Let $f, g \colon I \to \mathbb{R}$ be a continuous and monotone functions. Then $A^{[f]} \leqslant A^{[g]}$ if and only if*

i. g is increasing and $g \circ f^{-1}$ is convex or g is decreasing and $g \circ f^{-1}$ is concave,

ii. f is increasing and $f \circ g^{-1}$ is concave or f is decreasing and $f \circ g^{-1}$ is convex.

In fact this proposition possess a lot of symmetries as we have the well-known equality condition (cf. [5])

$$A^{[f]} = A^{[g]} \iff \begin{pmatrix} \text{there exists } \alpha, \beta \in \mathbb{R} \text{ with } \alpha \neq 0 \\ \text{such that } f = \alpha \cdot g + \beta \end{pmatrix}. \tag{2}$$

It is easy to observe that $g \circ f^{-1}$ is continuous, so its convexity, t-convexity for given $t \in [0,1]$, Jensen convexity ($1/2$-convexity) are all equivalent. Therefore we obtain a number of conditions which provide comparability of quasi-arithmetic means. This is a folk result in a theory of means

PROPOSITION 2. *Let $f, g \colon I \to \mathbb{R}$ be a continuous and monotone functions. Then the following conditions are equivalent to $A^{[f]} \leqslant A^{[g]}$*

iii. $A^{[f]}(a) \leqslant A^{[g]}(a)$ for all $a \in \bigcup_{n=1}^{\infty} I^n$;

iv. $A^{[f]}(a) \leqslant A^{[g]}(a)$ for some $k \in \mathbb{N}$ and all $a \in I^k$;

v. $A_{\xi}^{[f]}(a) \leqslant A_{\xi}^{[g]}(a)$ for some $\xi \in (0,1)$ and all $a \in I^2$, where $A_{\xi}^{[f]}(a) := f^{-1}(\xi f(a_1) + (1-\xi) f(a_2))$;

vi. $A_{\xi}^{[f]}(a) \leqslant A_{\xi}^{[g]}(a)$ for all $\xi \in (0,1)$ and all $a \in I^2$.

Additionally we have a condition, in the spirit of Páles [8] (see also [6]) *vii.* $\frac{f(y)-f(x)}{f(z)-f(x)} \geqslant \frac{g(y)-g(x)}{g(z)-g(x)}$ for all $x,y,z \in I$, $x<y<z$;

It is worth mentioning that the substitution $(x,y,z) := (a_2, A_\xi^{[f]}(a), a_1)$ proves the equivalence (vii) $\iff$ (vi). Furthermore, we know that a differentiable function is convex/concave if and only if its derivative is non-decreasing/non-increasing. Applying this we get next comparability conditions.

PROPOSITION 3. *Let $f,g\colon I \to \mathbb{R}$ be a monotone and differentiable functions with $f' \cdot g' \neq 0$. Then $A^{[f]} \leqslant A^{[g]}$ if and only if one of the following conditions is satisfied*

viii. f and g are of the same monotonicity (both increasing or both decreasing) and f'/g' is non-increasing (equivalently g'/f' is non-decreasing)

viii'. f and g are of the converse monotonicity (one increasing, second decreasing) and f'/g' is non-decreasing (equivalently g'/f' is non-increasing).

Now we turn into the result of Mikusiński [4]. He, and independently Łojasiewicz (compare [4, footnote 2]), expressed handy tool to compare quasi-arithmetic means in terms of operator $f \mapsto f''/f'$ (the negative of this operator is used to be called an Arrow-Pratt index). More precisely their result reads

PROPOSITION 4. *Let I be an interval, $f,g \in \mathscr{C}^2(I)$, $f' \cdot g' \neq 0$ on I. Then $A^{[f]} \leqslant A^{[g]}$ if and only if*

ix. $\frac{f''(x)}{f'(x)} \leqslant \frac{g''(x)}{g'(x)}$ *for all $x \in I$.*

Using this result we immediately obtain some "Mikusiński-type intervals"

$$\tilde{\mathscr{M}}(x_0,U) := \left\{ A^{[f]} : \begin{array}{c} f \text{ is twice continuously differentiable in some} \\ \text{neighborhood of } x_0,\ f'(x_0) \neq 0,\ \frac{f''(x_0)}{f'(x_0)} \in U \end{array} \right\},$$

where $x_0 \in I$ and $U \subset \mathbb{R}$ is an interval. Nevertheless $\tilde{\mathscr{M}}(x_0,U)$ is usually not an interval-type set, therefore we extend this set to an interval in the way that was described in the introduction

$$\mathscr{M}(x_0,U) := \big[\big[\tilde{\mathscr{M}}(x_0,U)\big]\big].$$

This lead us to the following problem. A family $\tilde{\mathscr{M}}(x_0,U)$ contains only $\mathscr{C}^2$ functions around x_0 with $f'(x_0) \neq 0$. But what about $\mathscr{M}(x_0,U)$?

By (1) we know that for all $A^{[h]} \in \mathscr{M}(x_0,U)$ we have

$$A^{[f]} \leqslant A^{[h]} \leqslant A^{[g]} \tag{3}$$

for some $f,g \in \mathscr{C}^2(V)$, $f' \cdot g' \neq 0$ and open interval $V \ni x_0$.

In fact it does not imply that the second derivative of h at x_0 exists, which can be illustrated in a simple example

EXAMPLE 1. Let $I=(0,2)$, $f(x)=x$, $g(x)=x^2$, $x_0=1$,

$$h(x)=\begin{cases} x & x\in(0,1]\,,\\ \frac{x^2+1}{2} & x\in(1,2)\,.\end{cases}$$

Then, by (viii) , assertion (3) holds but h is $\mathscr{C}^1$ only.

Despiting this drawback, it can be proved that if $f,g\in\mathscr{C}^2(I)$ with nonvanishing derivative and (3) holds, then h is continuously differentiable for all $x\in U$ and also h' is nowhere vanishing.

Nevertheless to obtain an interval-type set assumption on f, g, and h have to be the same. Thus we want to prove that if f and g are continuously differentiable with nonvanishing derivative, then so is h. Equivalently, family of quasi-arithmetic means generated by $\mathscr{C}^1$ functions with nonvanishing derivative is an interval (it will be done in Theorem 9).

3. Interval-type sets in a family quasi-arithmetic means

In the following section we will prove a number of examples of interval-type sets involving vary smoothness assumptions of generating functions. Let us first prove some abstract theorem.

THEOREM 5. *Let I be a compact interval, $x_0\in I$, $f_0\colon I\to\mathbb{R}$ with $f_0(x_0)=0$. Let $\mathscr{F}\subset\mathscr{C}(I)$ be an interval such that $\mathscr{F}\subseteq o(f_0)$ in a right/left neighborhood of x_0. Then the family*

$$A^{[f_0+\mathscr{F}]}:=\{A^{[f_0+f]}\colon f\in\mathscr{F} \text{ and } f_0+f \text{ is strictly monotone on } I\}.$$

is an interval.

Proof. We want to bind the cases where x_0 is in the interior of the interval and is the endpoint. In the proof we will concern right neighborhood of the point, therefore we have $x_0\neq\sup I$. Second case is completely analogous. Similarly assume that f_0 is increasing.

Take any $x_1\in I$ such that $x_1>x_0$. Let $r_1,r_2\in\mathscr{F}$ and $f:=f_0+r_1$, $g:=f_0+r_2$. By the definition there holds $f(x_0)=g(x_0)=0$. Denote

$$\hat{f}(x):=\frac{f(x)-f(x_0)}{f(x_1)-f(x_0)}=\frac{f(x)}{f(x_1)},\quad \hat{g}(x):=\frac{g(x)-g(x_0)}{g(x_1)-g(x_0)}=\frac{g(x)}{g(x_1)}.$$

Let us consider an arbitrary function $\tilde{h}\colon I\to\mathbb{R}$ satisfying

$$A^{[f]}\leqslant A^{[\tilde{h}]}\leqslant A^{[g]}.$$

By (2), there exists a unique function h such that $h(x_0)=0$, $h(x_1)=1$, and $A^{[\tilde{h}]}=A^{[h]}$.

Then, by (vii) , we get $\hat{g}(x) \leqslant h(x) \leqslant \hat{f}(x)$ for all $x \in (x_0, x_1)$. Thus

$$\frac{g(x)}{g(x_1)} \leqslant h(x) \leqslant \frac{f(x)}{f(x_1)}, \quad x \in (x_0, x_1).$$

Therefore

$$\frac{f_0(x) + r_2(x)}{g(x_1)} \leqslant h(x) \leqslant \frac{f_0(x) + r_1(x)}{f(x_1)}, \quad x \in (x_0, x_1).$$

Then

$$\frac{f_0(x)}{g(x_1)} + \frac{1}{g(x_1)} \cdot r_2(x) \leqslant h(x) \leqslant \frac{f_0(x)}{f(x_1)} + \frac{1}{f(x_1)} \cdot r_1(x) \tag{4}$$

It implies

$$\left(\frac{1}{g(x_1)} - \frac{1}{f(x_1)}\right) f_0(x) \leqslant \frac{1}{f(x_1)} \cdot r_1(x) - \frac{1}{g(x_1)} \cdot r_2(x)$$

But $\frac{1}{f(x_1)} r_1(x) - \frac{1}{g(x_1)} r_2(x) \in o(f_0)$ in a right neighborhood of x_0. Thus

$$\frac{1}{g(x_1)} - \frac{1}{f(x_1)} \leqslant 0.$$

Consequently $g(x_1) \geqslant f(x_1)$. As x_1 was an arbitrary number greater than x_0 we obtain $g(x) \geqslant f(x)$ for $x > x_0$. Thus

$$r_2(x) \geqslant r_1(x) \text{ for } x > x_0. \tag{5}$$

Now observe that

$$\frac{f(x)}{f(x_1)} \leqslant h(x) \leqslant \frac{g(x)}{g(x_1)} \text{ for } x > x_1.$$

$$\frac{f_0(x) + r_1(x)}{f_0(x_1) + r_1(x_1)} \leqslant h(x) \leqslant \frac{f_0(x) + r_2(x)}{f_0(x_1) + r_2(x_1)} \text{ for } x > x_1.$$

Denote

$$r_\beta = (\beta - 1) r_2 + (2 - \beta) r_1 \in \mathscr{F}, \quad \beta \in [1, 2].$$

Then there exists $\beta(x, x_1) \in [1, 2]$ such that

$$h(x) = \frac{f_0(x) + r_{\beta(x,x_1)}(x)}{f_0(x_1) + r_{\beta(x,x_1)}(x_1)} \text{ for } x > x_1.$$

For $x_1 \geqslant x_0$ and $x \geqslant x_0$ we have

$$\mathscr{F}(x) \ni r_1(x) \leqslant r_{\beta(x,x_1)}(x) \leqslant r_2(x) \in \mathscr{F}(x).$$

Thus $r_{\beta(x,x_1)}(x) \in \mathscr{F}(x)$ (it means that $\mathscr{F}$ is considered as variable of x). Similarly $r_{\beta(x,x_1)}(x_1) \in \mathscr{F}(x_1)$. Furthermore, by Taylor's theorem, $\frac{1}{p + \mathscr{F}(x)} = \frac{1}{p} + o(x - x_0)$. Thus, for $x > x_1$,

$$h(x) = \frac{f_0(x) + r_{\beta(x,x_1)}(x)}{f_0(x_1) + \mathscr{F}(x_1)} = (f_0(x) + r_{\beta(x,x_1)}(x)) \cdot \left(\frac{1}{f_0(x_1)} + o(x_1 - x_0)\right)$$

Recall that x_1 was fixed but arbitrary, so we can substitute $x_1 \leftarrow s$, where $s > x_0$. Furthermore we can consider $h_s(x) := f_0(s) \cdot h(x)$, as their generate the same quasi-arithmetic mean. Then, for $x > s > x_0$,

$$\begin{aligned} h_s(x) &= f_0(s) \cdot (f_0(x) + r_{\beta(x,s)}(x)) \Big(\frac{1}{f_0(s)} + o(s - x_0) \Big) \\ &= (f_0(x) + r_{\beta(x,s)}(x)) \cdot (1 + f_0(s) \cdot o(s - x_0)) \\ &= (f_0(x) + r_{\beta(x,s)}(x)) \cdot (1 + o(f_0(s) \cdot (s - x_0))). \end{aligned}$$

Thus we get a family of functions $\mathscr{H} = \{h_s(x)\}_{s>x_0}$

$$h_s(x) = (f_0(x) + r_{\beta(x,s)}(x)) \cdot (1 + o(f_0(s) \cdot (s - x_0))), x > s > x_0.$$

By the definition of β we have

$$\begin{aligned} h_s(x) &\geqslant (f_0(x) + r_1(x)) \cdot (1 + o(f_0(s) \cdot (s - x_0))), x > s > x_0; \\ h_s(x) &\leqslant (f_0(x) + r_2(x)) \cdot (1 + o(f_0(s) \cdot (s - x_0))), x > s > x_0. \end{aligned}$$

We can now pass $s \to x_0$ and obtain

$$h_{x_0}(x) := \lim_{s \to x_0} h_s(x) \in [f_0(x) + r_1(x), f_0(x) + r_2(x)], x > x_0.$$

Therefore $h_{x_0} \in f_0 + \mathscr{F}$. Furthermore, as $A^{[h]} = A^{[h_s]}$ for all $s > x_0$ then, applying (vii) , $A^{[h_{x_0}]} = A^{[h]}$. Finally $A^{[\tilde{h}]} = A^{[h]} = A^{[h_{x_0}]} \in A^{[f_0 + \mathscr{F}]}$. $\square$

This theorem has a very useful corollary

COROLLARY 6. *Let I be an interval, $x_0 \in I$. The family of quasi-arithmetic means generated by right-(left-)sided differentiable function at x_0 with $f'_+(x_0) = 0$ ($f'_-(x_0) = 0$) is an interval-type set.*

Proof. Let $A^{[f]} \leqslant A^{[h]} \leqslant A^{[g]}$. Suppose $f(x_0) = g(x_0) = 1$. We know that

$$f(x) = 1 + o(x - x_0) \quad \text{and} \quad g(x) = 1 + o(x - x_0) \quad \text{for } x > x_0.$$

Take $f_0 \equiv 1$ and $\mathscr{F} = o(x - x_0)$. Then the pair f_0, $\mathscr{F}$ satisfies all conditions of Theorem 5. Furthermore $f, g \in f_0 + \mathscr{F}$, therefore we get

$$h \in f_0 + \mathscr{F} = 1 + o(x - x_0) \text{ for } x > x_0.$$

It implies that $h'_+(x_0)$ exists and $h'_+(x_0) = 0$. $\square$

3.1. Interval-type sets involving smoothness assumptions

In the following section we are going to present some interval-type sets in a family of quasi-arithmetic means involving smoothness assumptions of their generating functions. Recall that all means are considered on a common interval I.

First result will concern existence of one-sided derivative at certain point

THEOREM 7. *Let $x_0 \in \operatorname{int} I$ and $f\colon I \to \mathbb{R}$ be a continuous and monotone functions which has a right-(left-)sided differentiable function at x_0 with $f'_+(x_0) \neq 0$ $[f'_-(x_0) \neq 0]$. If $A^{[g]} \in [A^{[f]}]^*$ for some $g\colon I \to \mathbb{R}$ then g is right-(left-)sided differentiable at x_0 too and $g'_+(x_0) \neq 0$ $[g'_-(x_0) \neq 0]$.*

Proof. By (2) we may assume $f(x_0) = g(x_0) = 0$. Then we have

$$\begin{aligned}\lim_{x \to x_0^+} \frac{g(x) - g(x_0)}{x - x_0} &= \lim_{y \to 0^+} \frac{g \circ f^{-1}(y)}{f^{-1}(y) - x_0} \\ &= \lim_{y \to 0^+} \frac{g \circ f^{-1}(y)}{y} \cdot \lim_{y \to 0^+} \frac{y}{f^{-1}(y) - x_0} \\ &= \lim_{y \to 0^+} \frac{g \circ f^{-1}(y)}{y} \cdot \lim_{x \to x_0^+} \frac{f(x)}{x - x_0}\end{aligned}$$

But $g \circ f^{-1}(0) = g(x_0) = 0$. Moreover, as $A^{[g]}$ is comparable with $A^{[f]}$, we know that $g \circ f^{-1}$ is either convex or concave (see i and ii). In particular there exists a one-side derivative $(g \circ f^{-1})_+(0)$. Moreover, as $g \circ f^{-1}$ is strictly monotone and convex or concave in some neighborhood of 0, we get $(g \circ f^{-1})_+(0) \neq 0$. Furthermore $f'_+(x_0)$ exists and is nonzero.

Finally we obtain that there exists $g'_+(x_0)$ and

$$g'_+(x_0) = (g \circ f^{-1})'_+(0) \cdot f'_+(x_0) \neq 0. \quad \square$$

Having this already proved we have an immediate corollary

COROLLARY 8. *Quasi-arithmetic means generated by functions which are right-(left-)sided differentiable functions at certain point $x_0 \in I$ with $f'_-(x_0) \neq 0$ $[f'_+(x_0) \neq 0]$ is an interval.*

This result can be somehow improved. Namely if both derivatives $f'(x_0)$ and $g'(x_0)$ exists, are nonzero and (3) holds, then it is also the case in h. In can be formally expressed in term of the following

THEOREM 9. *Quasi-arithmetic means generated by a functions differentiable at certain point $x_0 \in I$ with $f'(x_0) \neq 0$ is an interval.*

Proof. Let f, g, and h be strictly increasing, and $A^{[f]} \leqslant A^{[h]} \leqslant A^{[g]}$. If f, g are differentiable at x_0 and $f'(x_0)g'(x_0) \neq 0$ then, by Corollary 8, we know that both $h'_+(x_0)$ and $h'_-(x_0)$ exists and are nonzero. Now, by ii, h is convex with respect to f. Thus $h'_-(x_0) \leqslant h'_+(x_0)$.

Similarly, by i, h is concave with respect to g and, consequently, $h'_-(x_0) \geqslant h'_+(x_0)$. $\square$

Furthermore, this result could be rearrange in the case of continuous derivative

PROPOSITION 10. *Quasi-arithmetic means generated by a functions belonging to $\mathscr{C}^1(I)$ with nowhere vanishing derivative is an interval.*

Proof. Suppose that f, g, h are increasing, $f, g \in \mathscr{C}^1(I)$, $f' \cdot g' \neq 0$ and $A^{[f]} \leqslant A^{[h]} \leqslant A^{[g]}$. Then, by Theorem 9, h is differentiable and $h' \neq 0$. Moreover, by (viii) , we know that h'/f' is non-decreasing and h'/g' is non-increasing.

Let $x_0 \in I$. We can take affine transformations such that $f(x_0) = g(x_0) = h(x_0) = 0$ and $f'(x_0) = g'(x_0) = h'(x_0) = 1$. Thus (by (viii)) $f'(x) \leqslant h'(x) \leqslant g'(x)$ for all $x > x_0$. Similarly $g'(x) \leqslant h'(x) \leqslant f'(x)$ for $x < x_0$. Therefore

$$l := \min(f', g') \leqslant h' \leqslant \max(f', g') =: u\,.$$

But $l(x_0) = u(x_0) = 1$ and both l and u are continuous so h' is continuous at the point x_0. But x_0 was arbitrary so $h \in \mathscr{C}^1(I)$. □

We are now heading toward one-sided differentiability in certain point (without vanishing or nonvanishing assumptions). First we will prove a very useful lemma.

LEMMA 11. *If $g, h \colon I \to \mathbb{R}$ are right-(left-)sided differentiable at $x_0 \in \operatorname{int} I$ with $g'_+(x_0) = 0$ and $h'_+(x_0) \neq 0$ ($g'_-(x_0) = 0$ and $h'_-(x_0) \neq 0$) then $A^{[g]}$ is not comparable with $A^{[h]}$.*

Proof. Take $\varepsilon > 0$ such that $x_0 + \varepsilon \in I$. There exist affine transformations $\hat{g}$ and $\hat{h}$ of g and h, respectively, such that

- $\hat{g}(x_0) = \hat{h}(x_0) = 0$,
- $\hat{h}(x_0 + \varepsilon) = 1$,
- $\hat{g}(x_0 + \varepsilon) = 2$.

We obviously have $\hat{h}'(x_0) > 0$ and $\hat{g}'(x_0) = 0$. It implies that $\hat{g}(x) < \hat{h}(x)$ is some right neighborhood of x_0. Let $\xi > x_0$ be the smallest number such that $\hat{g}(\xi) = \hat{h}(\xi)$. Now, as $\hat{g}(x) < \hat{h}(x)$ for all $x \in (x_0, \xi)$ we obtain

$$\hat{g}^{-1}(y) > \hat{h}^{-1}(y) \text{ for all } y \in \big(0, \hat{g}(\xi)\big).$$

in particular, for $y := \hat{g}(\xi)/2$ we get

$$\begin{aligned} A^{[g]}_{1/2}(x_0, \xi) &= A^{[\hat{g}]}_{1/2}(x_0, \xi) = \hat{g}^{-1}(\hat{g}(\xi)/2) = \hat{g}^{-1}(y) \\ &> \hat{h}^{-1}(y) = \hat{h}^{-1}\Big(\frac{\hat{h}(x_0) + \hat{h}(\xi)}{2}\Big) = A^{[\hat{h}]}_{1/2}(x_0, \xi) = A^{[h]}_{1/2}(x_0, \xi) \end{aligned}$$

Therefore $A^{[g]} \nleqslant A^{[h]}$. To obtain the converse we can adapt this proof assuming $\varepsilon < 0$ (regarding we will consider maximal ξ and some sings will be changed). □

Having this already proved we can skip the assumption about nonvanishing one-sided derivative and obtain

THEOREM 12. *The family of quasi-arithmetic means generated by functions differentiable at some point $x_0 \in \operatorname{int} I$ is an interval-type set.*

Proof. Suppose that f and g are differentiable at x_0 and $A^{[f]} \leqslant A^{[h]} \leqslant A^{[g]}$. As $A^{[f]}$ and $A^{[g]}$ are comparable, by Lemma 11 we get that either $f'(x_0) = g'(x_0) = 0$ or $f'(x_0) \cdot g'(x_0) \neq 0$. In the first case we can use Corollary 6 to obtain differentiability of h at x_0, while in the second case we use Theorem 9. □

In the case when I is open we can take an intersection of these interval-type sets over all $x_0 \in I$ to obtain

COROLLARY 13. *The family of quasi-arithmetic means defined on an open interval I generated by differentiable functions is an interval-type set.*

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