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LOCAL ONE–SIDED MAXIMAL FUNCTION ON FRACTIONAL SOBOLEV SPACES

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Abstract. In this article we study the boundedness of local one sided maximal operators on weighted fractional Sobolev Spaces. As a consequence we obtain a Lebesgue differentiation theorem for functions in fractional Sobolev spaces.

1. Introduction and preliminaries

Let $\Omega \subset \mathbb{R}^n$ be an open set. For $f \in L^{1,loc}(\Omega)$ and $x \in \Omega$, the centered local Hardy-Littlewood maximal operator M_Ω is defined as

$$M_\Omega f(x) := \sup_{0<r<dist(x,\partial\Omega)} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy, \quad x \in \Omega \tag{1}$$

where $B(x,r)$ denotes the ball centered at x of radius r and $|B(x,r)|$ denotes the volume of the ball. When $\Omega = \mathbb{R}^n$, then the supremum is over all $r > 0$ and in that case we denote M_Ω simply by M.

Maximal operators play a very crucial part in differentiation theory and are often used in establishing the different kind of convergences for certain integral averages. Over the last two decades, there has also been considerable development in understanding boundedness and smoothness properties of the maximal operators defined on Sobolev spaces. In this direction, the first remarkable achievement was by J. Kinnunen in [3], where he proved that M is a bounded operator on the classical Sobolev spaces $W^{1,p}(\mathbb{R}^n)$ for $p > 1$. Then in [2], the authors observed this phenomenon for any translation invariant operator. More precisely, they proved

THEOREM 1.1. ([2]) *Assume $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, $1 < p < \infty$, is bounded and sub-linear. If $T(\tau_y f) = \tau_y(Tf)$ for all $f \in L^p(\mathbb{R}^n)$ and for every $y \in \mathbb{R}^n$, where $\tau_y f(.) = f(. - y)$. Then T is bounded on $W^{1,p}(\mathbb{R}^n)$.*

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Naturally, for $p=1$, M cannot be bounded on $W^{1,1}(\mathbb{R}^n)$. However, in [14], H. Tanaka proved that the operator $f \to |\nabla(\tilde{M}f)|$ is bounded from $W^{1,1}(\mathbb{R})$ to $L^1(\mathbb{R})$, where $\tilde{M}$ is the uncentered Hardy-Littlewood maximal operator defined by

$$\tilde{M}f(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|dy,$$

where the supremum is over all balls B containing x. Recently, in [11], Tanaka's result was extended for the centered maximal function M on $\mathbb{R}$, which completely answers a question raised by Hajłasz and Onninen in [2].

When it comes to the study of M_Ω, the supremum in (1) is taken only over those balls which are contained in Ω, so one of the main difficulties in handling this operator is that it does not commute with translation. In [5], J. Kinnunen and P. Lindqvist established the boundedness of M_Ω on $W^{1,p}(\Omega)$. Then the continuity of M_Ω on $W^{1,p}(\Omega)$ was elegantly achieved in [6]. In recent times the study of local maximal function is a growing area of interest. In their two consecutive articles ([8, 9]), H.Luiro and A.V.Vähäkangas explored the local maximal operator on fractional Sobolev spaces. First in [8], they proved the boundedness result on fractional Sobolev spaces $W^{s,p}(\Omega)$. To describe these results in full glory we need to first recall the following preliminaries.

DEFINITION 1.2. A weight w on $\mathbb{R}^n$ is a locally integrable function such that $w(x) > 0$ a.e on $\mathbb{R}^n$.

In his celebrated article [12], B. Muckenhoupt characterized the weights w for which the classical Hardy-Littlewood maximal operator is bounded on $L^p(w)$ and and these class of weights are commonly referred as A_p weights. The result is stated as the following theorem.

THEOREM 1.3. ([12])

1. *Assume* $1 < p < \infty$. *There exist a constant* $C_{p,w} > 0$ *such that the inequality*

$$\int_{\mathbb{R}^n} Mf(x)^p w(x)dx \leqslant C_{p,w} \int_{\mathbb{R}^n} |f(x)|^p w(x)dx$$

holds for all $f \in L^p(w)$ *if and only if* w *satisfies the following* A_p *condition*

$$[w]_{A_p} := \sup_Q \left(\frac{1}{|Q|} \int_Q w \right) \left(\frac{1}{|Q|} \int_Q w^{-\frac{1}{p-1}} \right)^{p-1} < \infty,$$

where the supremum is taken over all cubes with sides parallel to co-ordinate axes.

2. *Let* $p = 1$. *There exists a constant* $C_w > 0$ *such that*

$$||Mf||_{L^{1,\infty}(w)} \leqslant C_w ||f||_{L^1(w)}$$

holds for all $f \in L^1(w)$ *if and only if* w *satisfies the following* A_1 *condition*

$$\frac{1}{|Q|}\int_Q w \lesssim ess \inf_{y\in Q} w(y)$$

for all cubes Q *with sides parallel to coordinate axes.*

It is very natural to ask whether the local Hardy-Littlewood maximal operator is bounded on fractional Sobolev spaces for A_p weights. In [9], the authors answered the above question affirmatively by defining a proper weighted analogue of fractional Sobolev spaces. These spaces appear naturally in the study of fractional weighted Hardy-type inequalities and potential theory.

DEFINITION 1.4. [8, 9] Let $s > 0$, $1 \leqslant p < \infty$, and w be a weight on $\mathbb{R}^n$. Fix an open set $\Omega \subsetneq \mathbb{R}^n$. Then we define the fractional weighted Sobolev space $Z^{s,p,w}(\Omega)$ as

$$Z^{s,p,w}(\Omega) = \{f \in L^p(\Omega) : ||f||^p_{Z^{s,p,w}(\Omega)} = ||f||^p_{L^p(\Omega)} + |f|^p_{Z^{s,p,w}(\Omega)} < \infty\},$$

where,

$$|f|_{Z^{s,p,w}(\Omega)} = \left(\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{sp}} w(x-y)dydx\right)^{\frac{1}{p}}. \tag{2}$$

Among other facts, it is also shown in [9] that $||f||_{Z^{s,p,w}(\Omega)}$ is a semi-norm on $Z^{s,p,w}(\Omega)$. One can observe that if we take $w(x) = |x|^{\varepsilon - n}$ then we recover classical fractional Sobolev spaces. For more in this direction we encourage the reader to consult [9, 15] and the references therein. Now we are in a position to state the main result in [9].

THEOREM 1.5. [9] *Assume* $\emptyset \neq \Omega \subset \mathbb{R}^n$ *is an open set,* $0 \leqslant s \leqslant 2$ *and* $1 < p < \infty$. *Fix a measurable function* $R : \Omega \to [0,\infty)$ *satisfying* $0 \leqslant R(x) \leqslant dist(x, \partial\Omega)$ *for every* $x \in \Omega$. *Let* M_R *denotes the following maximal function*

$$M_R f(x) := \sup_{0\leqslant r\leqslant R(x)} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|dy.$$

Then for every A_p *weight* w *in* $\mathbb{R}^n$, *there exists a constant* $C = C(n,p,[w]_{A_p}) > 0$ *such that*

$$\int_\Omega\int_\Omega \frac{|M_R f(x) - M_R f(y)|^p}{(|x-y| + |R(x)-R(y)|)^{sp}} w(x-y)dydx \leqslant C\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{sp}} w(x-y)dydx$$

for all $f \in Z^{s,p,w}(\Omega)$. *Here we follow the convention* $\frac{1}{B(x,0)}\int_{B(x,0)} |f(y)| = |f(x)|$.

When $R := dist(x,\partial\Omega)$, M_R is the standard local maximal function M_Ω. The proof of the above result requires a very insightful analysis involving the radius and the position of the point $x \in \Omega$ in (1).

The aim of this article is to study the local one-sided maximal function and its boundedness on fractional Sobolev spaces and weighted fractional Sobolev spaces. The classical one-sided maximal function, for $f \in L^{1,loc}(\mathbb{R}^n)$, is defined as

$$M^+ f(x_1,x_2,\dots,x_n) := \sup_{r>0} \fint_{Q(x,r)} |f(y)|dy$$

where $\fint_E f = \frac{1}{|E|}\int_E f$ for any measurable set E with $0 < |E| < \infty$ and $Q(x,r) = [x_1,x_1+r) \times [x_2,x_2+r) \dots \times [x_n,x_n+r)$. In his remarkable article [13], E. Sawyer characterized weights w *on* $\mathbb{R}$ for which M^+ is bounded on $L^p(w)$. These classes of weights are known as A_p^+ weights and are defined as follows :

For $p > 1$, $w \in A_p^+$ if and only if

$$[w]_{A_p^+} := \sup_{x\in\mathbb{R}} \sup_{h>0} \left(\tfrac{1}{h}\int_{x-h}^{x} w\right) \left(\tfrac{1}{h}\int_{x}^{x+h} w^{-\frac{1}{p-1}}\right)^{p-1} < \infty$$

But, characterization of good weights for M^+ in higher dimensions turns to be a very difficult problem and is quite open to resolve. For more in this direction we refer articles [10, 1] and references therein. Now motivated from [8, 9], we define the local one-sided maximal function as

DEFINITION 1.6. Let $\Omega \subset \mathbb{R}^n$ be any open set and $R : \Omega \to [0,\infty)$ be a measurable function such that $0 \leqslant R(x)\sqrt{n} \leqslant dist(x,\partial\Omega)$ for all $x \in \Omega$. Then the local One-Sided maximal operator M_R^+ is defined for $x = (x_1,\dots,x_n) \in \Omega$ as

$$M_R^+ f(x_1,x_2,\dots,x_n) := \sup_{0\leqslant r\leqslant R(x)} \fint_{Q(x,r)} |f(y)|dy, \tag{3}$$

where $Q(x,r)$ is as defined above and we follow the notation $\fint_{Q(x,0)} |f(y)| = |f(x)|$. When $R(x) = \frac{dist(x,\partial\Omega)}{\sqrt{n}}$, we denote M_R^+ simply by M_Ω^+.

Similarly, M_R^- is defined by

$$M_R^- f(x_1,x_2,\dots,x_n) := \sup_{0\leqslant r\leqslant R(x)} \fint_{Q^-(x,r)} |f(y)|dy,$$

where $Q^-(x,r) = (x_1 - r, x_1] \times \dots \times (x_n - r, x_n]$.

The intent of this paper is to formulate the regularity properties of M_R^+ and M_R^- for functions in fractional Sobolev spaces and the concerning results are stated and proved in details in the next section. Finally, in Section 3, as a consequence of our main result, we obtain a Lebesgue differentiation theorem outside a set of zero Sobolev capacity. Throughout this article, the abbreviation $A \lesssim B$ means there is a constant C(independent of A,B) satisfying $A \leqslant CB$. Unless mentioned otherwise, this implicit constant will depend only on the dimension.

2. A boundedness result for local one-sided maximal function

Here, we describe our main results. The following result establishes the boundedness of local one-sided maximal function on fractional Sobolev spaces. Similar results can be deduced for M_R^- also, but from here on we will only write results about M_R^+.

THEOREM 2.1. *Let $\Omega \subsetneq \mathbb{R}^n$ be an open set, $0 < s < 2$ and $1 < p < \infty$. Then, if w is an A_p weight in $\mathbb{R}^n$, then there exist a constant $C = C(n, p, [w]_{A_p})$*

$$\int_\Omega \int_\Omega \frac{|M_R^+ f(x) - M_R^+ f(y)|^p}{(|x-y| + |R(x) - R(y)|)^{sp}} w(x-y)dydx \leqslant C \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{sp}} w(x-y)dydx \tag{4}$$

holds for all $f \in Z^{s,p,w}(\Omega)$.

Let us define some notations and auxiliary maximal operators before going into the proof of the above theorem. These maximal operators appear very naturally in different contexts. For $i, j \in \{0,1\}$ and for a measurable function F on $\mathbb{R}^{2n}$ we define

$$M_{ij}F(x,y) = \sup_{r>0} \fint_{B(o,r)} |F(x+iz, y+jz)|dz \tag{5}$$

and by M_{ij}^+ and M_{ij}^- we denote analogues maximal functions as in (5) just replacing $B(0,r)$ by $Q(0,r)$ and $Q^-(0,r)$ respectively. For a weight w we denote $w_0(x,y) = w(x-y)^{\frac{1}{p}}$ and $w_1(x,y) = w(y-x)^{\frac{1}{p}}$. $\Omega \subset \mathbb{R}^n$ be an open set and R as in (3), we write

$$L_R(h)(x,y) = L_{R,\Omega,s}(h)(x,y) = \frac{\chi_\Omega(x)\chi_\Omega(y)|h(x) - h(y)|}{(|x-y| + |R(x) - R(y)|)^s}$$

for $(x,y) \in \mathbb{R}^{2n}$ and $L(h) = L_{0,\Omega,s}(h)$ when $R = 0$ identically on Ω. In [5], the authors have shown $|D(M_\Omega f)(x)| \leqslant 2M_\Omega(|Df|)(x)$. In that aspect our next lemma reflects a certain pointwise relation between fractional derivative of $M_R^+ f$ and that of f.

LEMMA 2.2. *Let $\emptyset \neq \Omega \subset \mathbb{R}^n$ be an open set and R as in (3), $0 \leqslant s \leqslant 2$, and $1 < p < \infty$. Let $w \in A_p(\mathbb{R}^n)$, then there exist a constant C_n such that for a.e $(x,y) \in \mathbb{R}^{2n}$*

$$\begin{aligned} w_0(x,y)L_R(M_R^+ f)(x,y) \leqslant C_n[&w_0(x,y)M_{10}(Lf)(x,y) + M_{11}(w_1 M_{01}(Lf))(y,x) \\ &+ M_{11}(w_0 M_{01}(Lf))(x,y)] \end{aligned} \tag{6}$$

holds for all $f \in Z^{s,p,w}(\Omega)$.

Proof. Without loss of generality assume $f \geqslant 0$. By change of variable and the fact that $\tilde{w}(x) = w(-x)$ is again in A_p, so using symmetries we also assume that $M_R^+ f(x) > M_R^+ f(y)$. The above assumptions allow us to find $0 < r_1 \leqslant R(x)$ such that $M_R^+ f(x) = \fint_{Q(x,r_1)} f(z)$, which implies

$$L_R(M_R^+ f)(x,y) \leqslant \frac{|\fint_{Q(x,r_1)} f(z) - \fint_{Q(y,r_2)} f(z)|}{(|x-y| + |R(x) - R(y)|)^s}$$

for any $0 < r_2 \leqslant R(y)$.

Now we consider the following cases. In each case r_2 will be chosen suitably depending on the position of the points and r_1.

Case: $r_1 \leqslant |x-y| + |R(x)-R(y)|$

Take $r_2 = 0$. Now observe that for all $|z| < r_1$ we have $|x+z-y| \leqslant 2(|x-y| + |R(x)-R(y)|)$. Then

$$\begin{aligned} w_0(x,y)L_R(M_R^+ f)(x,y) &\leqslant w_0(x,y)\frac{\fint_{Q(x,r_1)}|f(z)-f(y)|dz}{(|x-y|+|R(x)-R(y)|)^s} \\ &\lesssim w_0(x,y)\fint_{Q(0,r_1)}\frac{|f(x+z)-f(y)|}{|x+z-y|^s}dz \\ &\lesssim w_0(x,y)\fint_{Q(0,r_1)} Lf(x+z,y)dz \\ &\lesssim w_0(x,y)M_{10}(Lf)(x,y) \end{aligned} \tag{7}$$

Case: $r_1 > |x-y| + |R(x)-R(y)|$

Choose $r_2 = r_1 - |x-y| - |R(x)-R(y)| \leqslant R(y)$. Then

$$\begin{aligned} &|\fint_{Q(x,r_1)} f(z) - \fint_{Q(y,r_2)} f(z)| \\ &= |\fint_{Q(0,r_1)}\left(f(x+z) - f(y+\frac{r_2}{r_1}z)\right)| \\ &= |\fint_{Q(0,r_1)} [f(x+z) - \fint_{Q(y+\frac{r_2}{r_1}z,2(|x-y|+|R(x)-R(y)|))\cap G} f(a)da \\ &\quad + \fint_{Q(y+\frac{r_2}{r_1}z,2(|x-y|+|R(x)-R(y)|))\cap G} f(a)da - f(y+\frac{r_2}{r_1}z)]dz| \\ &\leqslant K_1 + K_2 \end{aligned}$$

where,

$$\begin{aligned} K_1 &= \fint_{Q(0,r_1)}\fint_{Q(y+\frac{r_2}{r_1}z,2(|x-y|+|R(x)-R(y)|))\cap\Omega} |f(x+z)dz - f(a)|dadz \\ K_2 &= \fint_{Q(0,r_1)}\fint_{Q(y+\frac{r_2}{r_1}z,2(|x-y|+|R(x)-R(y)|))\cap\Omega} |f(y+\frac{r_2}{r_1}z) - f(a)|dadz \\ &= \fint_{Q(0,r_2)}\fint_{Q(y+z,2(|x-y|+|R(x)-R(y)|))\cap\Omega} |f(y+z) - f(a)|dadz \end{aligned}$$

Let's estimate K_1 first.

$$\begin{aligned} &(y_i + \tfrac{r_2}{r_1}z_i) - (x_i + z_i - 3|x-y| - 3|R(x)-R(y)|) \\ &= 3|x-y| + 3|R(x)-R(y)| + (y_i - x_i) + (\tfrac{r_2}{r_1} - 1)z_i > 0 \end{aligned}$$

and

$$\begin{aligned} & x_i+z_i+3|x-y|+3|R(x)-R(y)|-(y_i+\tfrac{r_2}{r_1}z_i+2|x-y|+2|R(x)-R(y)|) \\ & =|x-y|+|R(x)-R(y)|+(1-\tfrac{r_2}{r_1})z_i>0 \text{ for all } |z_i|<r_1 \text{ and } i\in\{1,\ldots,n\}. \end{aligned}$$

which implies $Q(y+\frac{r_2}{r_1}z,2(|x-y|+|R(x)-R(y)|))\subset Q[x+z,3(|x-y|+|R(x)-R(y)|)]$ where $Q[t,r]$ denotes the cube centred at t with side-length $2r$. Using these estimates we obtain

$$\begin{aligned} & w_0(x,y)\,\frac{K_1}{(|x-y|+|R(x)-R(y)|)^s} \\ \leqslant\ & \frac{w_0(x,y)}{(|x-y|+|R(x)-R(y)|)^s} \fint_{Q(0,r_1)} \fint_{Q[x+z,3(|x-y|+|R(x)-R(y)|)]\cap\Omega} |f(x+z)-f(a)|dadz \\ \leqslant\ & \fint_{Q(0,r_1)} w_0(x,y) \fint_{Q[x+z,3|x-y|+3|R(x)-R(y)|]} \frac{\chi_\Omega(x+z)\chi_\Omega(a)|f(x+z)-f(a)|}{|x+z-a|^s}dadz \\ \lesssim\ & \fint_{Q(0,r_1)} w_0(x,y) \fint_{Q[y+z,4|x-y|+4|R(x)-R(y)|]} Lf(x+z,a)dadz \\ \lesssim\ & \fint_{Q(0,r_1)} w_0(x+z,y+z)M_{01}(Lf)(x+z,y+z)dz \\ \lesssim\ & M_{11}(w_0M_{01}(Lf))(x,y). \end{aligned} \tag{8}$$

where we have used $|x+z-a|\leqslant C_n(|x-y|+|R(x)-R(y)|)$ and $Q[x+z,3|x-y|+3|R(x)-R(y)|]\subset Q[y+z,4|x-y|+4|R(x)-R(y)|]$.

Now let's estimate K_2. As

$$\begin{aligned} & (y_i+z_i)-(x_i+z_i-4|x-y|-4|R(x)-R(y)|) \\ & =4|x-y|+4|R(x)-R(y)|+(y_i-x_i)>0 \end{aligned}$$

and

$$\begin{aligned} & (x_i+z_i+4|x-y|+4|R(x)-R(y)|)-(y_i+z_i+2|x-y|+2|R(x)-R(y)|>0 \\ & =2|x-y|+2(x_i-y_i)+2|R(x)-R(y)|>0 \end{aligned}$$

we have $Q(y+z,2(|x-y|+|R(x)-R(y)|))\subset Q[x+z,4(|x-y|+|R(x)-R(y)|]$ for all i. Hence,

$$\begin{aligned} & w_0(x,y)\frac{K_2}{(|x-y|+|R(x)-R(y)|)^s} \\ \lesssim\ & \fint_{Q(0,r_2)} w_0(x,y) \fint_{Q[y+z,2(|x-y|+|R(x)-R(y)|)]} \frac{\chi_\Omega(y+z)\chi_\Omega(a)|f(y+z)-f(a)|}{(|y+z-a|)^s}dadz \\ \lesssim\ & \fint_{Q(0,r_2)} w_0(x,y) \fint_{Q(x+z,4(|x-y|+|R(x)-R(y)|))} Lf(y+z,s)dsdz \\ \lesssim\ & \fint_{Q(0,r_2)} w_1(y+z,x+z)M_{01}(Lf)(y+z,x+z)dz \\ \lesssim\ & M_{11}(w_1M_{01}(Lf))(y,x). \end{aligned} \tag{9}$$

where we have used $|y+z-s| \leqslant C_n(|x-y|+|R(x)-R(y)|)$. Now combining the estimates (7), (8), and (9), we obtain the desired inequality (6). This completes the proof of the lemma. □

Now we are in a position to prove Theorem 2.1.

Proof. Take $f \in Z^{s,p,w}(\Omega)$. Then by Lemma 2.2,

$$w(x-y)^{\frac{1}{p}} L_R(M_R^+ f)(x,y) \leqslant C_n[w(x-y)^{\frac{1}{p}} M_{10}(Lf)(x,y) + M_{11}(w(y-x)^{\frac{1}{p}} M_{01}(Lf))(y,x) + M_{11}(w(x-y)^{\frac{1}{p}} M_{01}(Lf))(x,y)] \tag{10}$$

holds for a.e in $\mathbb{R}^{2n}$. It's easy to observe that M_{11} is bounded on $L^p(\mathbb{R}^{2n})$. And the following weighted inequality holds for the operator M_{10} for A_p weights as an application of Muckenhoupt's Theorem 1.3 and Fubini's theorem. For a complete proof we refer [9]. For $w \in A_p$ and any measurable function F on $\mathbb{R}^{2n}$ there is a constant $C = C(n,p,[w]_{A_p})$ such that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (M_{01}(F)(x,y))^p w(x-y)dxdy \leqslant C \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |F(y,x)|^p w(x-y)dxdy. \tag{11}$$

Eventually the fact if $w \in A_p$ then $\tilde{w}$ is also in A_p. Now our main result (4) is achieved by applying the unweighted boundedness of M_{11} first and then using the result (11) for w and $\tilde{w}$ in (10). This completes the proof of our main Theorem 2.1. □

Our next result, as a consequence of our main Theorem 2.1, yields the boundedness of uncentered local maximal function also in $\mathbb{R}$ on weighted fractional Sobolev spaces. We define the uncentered local Hardy-Littlewood maximal function for an interval Ω in $\mathbb{R}$ as

$$\mathscr{M}_\Omega f(x) := \sup_{x \in I, I \subset \Omega} \fint_I f$$

where the supremum is taken over all intervals I contained in Ω. Then following the relation $\mathscr{M}_\Omega f(x) = \max\{M_\Omega^+(x), M_\Omega^- f(x)\}$ and Theorem 2.1, we obtain the boundedness of uncentered Maximal function on fractional Sobolev spaces in the following theorem.

THEOREM 2.3. *Let $\Omega \subsetneq \mathbb{R}$ be an open set, $0 < s < 2$ and $1 < p < \infty$. Then, if w is an A_p weight in $\mathbb{R}$, then there exist a constant $C = C(n,p,[w]_{A_p})$*

$$\int_\Omega\int_\Omega \frac{|\mathscr{M}_\Omega f(x) - \mathscr{M}_\Omega f(y)|^p}{|x-y|^{sp}} w(x-y)dydx \leqslant C \int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{sp}} w(x-y)dydx \tag{12}$$

holds for all $f \in Z^{s,p,w}(\Omega)$.

Finally, we end this section with the following remark which shows that if we consider the global maximal function then an analogue of Theorem 2.1 can be established without any restriction on the weight.

REMARK 2.4. It is very interesting to observe that when $\Omega = \mathbb{R}^n$, M^+ satisfy the inequality 4 for any weight w. For arbitrary x, y, we assume without loss of generality that $M^+f(x) \geqslant M^+f(y)$. Let $\{r_n\}$ be a sequence such that $\lim_{n\to\infty} A_n(f,x) = M^+f(x)$ where,

$$A_n(f,x) = \fint_{Q(x,r_n)} |f(t)|dt$$

Using the fact that, $|M^+f(x) - M^+f(y)| \leqslant (M^+f(x) - A_n(f,x)) + (A_n(f,x) - A_n(f,y))$, we have

$$\begin{aligned} \frac{|M^+f(x) - M^+f(y)|}{|x-y|^s} w(x-y)^{\frac{1}{p}} &\leqslant w(x-y)^{\frac{1}{p}} \limsup_{n\to\infty} \fint_{Q(0,r_n)} \frac{|f(x+z)-f(y+z)|}{|x-y|^s} dz \\ &\leqslant M_{11}(w_0 Sf)(x,y) \end{aligned}$$

where $w_0(x,y) := w(x-y)^{\frac{1}{p}}$, $Sf(x,y) = \frac{|f(x)-f(y)|}{|x-y|^s}$. The boundedness of M_{11} on $L^p(\mathbb{R}^{2n})$ immediately implies

$$||M^+f||_{Z^{s,p,w}(\mathbb{R}^n)} \leqslant C||f||_{Z^{s,p,w}(\mathbb{R}^n)}$$

for all $f \in Z^{s,p,w}(\mathbb{R}^n)$. This again suggests that the intrinsic nature of the local maximal operator is very different from that of the global maximal operator.

3. Application to Lebesgue differentiation

In this section we study Sobolev Capacity and Lebesgue differentiation for functions in weighted fractional Sobolev spaces. We state the following definition from [3] with standard notations. For $1 < p < \infty$, a function u is called p-quasicontinuous if for every $\varepsilon > 0$ there is a set F such that $C_p(F) < \varepsilon$ and u restricted to $\mathbb{R}^n \setminus F$ is continuous, where $C_p(F)$ denotes the standard Sobolev capacity. In [3], the author has proved the Hardy-Littlewood maximal function of a Sobolev function is quasicontinuous. In this direction we first define Sobolev Capacity

DEFINITION 3.1. ([9]) Suppose $0 < s < 1$ and $p \geqslant 1$. Let w be a weight in $\mathbb{R}^n$. For $E \subset \mathbb{R}^n$, then the Sobolev capacity of E is defined as

$$C_{s,p,w}(E) = \inf_{g \in A(E)} ||g||^p_{Z^{s,p,w}(\mathbb{R}^n)},$$

where $A(E) = \{g \in Z^{s,p,w}(\mathbb{R}^n) : g \geqslant 1 \text{ in an open set containing } E\}$. If $A(E) = \emptyset$, we define $C_{s,p,w}(E) = \infty$.

In [9] the authors have proved the fact that $C_{s,p,w}$ is an outer measure on $\mathbb{R}^n$, see [[9], Lemma 24]. Here we consider $Z^{s,p,w}(\mathbb{R}^n)$ only for those A_p weights w such that $C_0^\infty(\mathbb{R}^n) \subset Z^{s,p,w}(\mathbb{R}^n)$. We obtain a Lebesgue differentiation with respect to the cubes for functions in $Z^{s,p,w}(\mathbb{R}^n)$ which allow us to find quasicontinuous representative of functions in fractional weighted Sobolev spaces. Precisely, we have the following theorem

THEOREM 3.2. *Assume $0<s<1$ and $p>1$. Let $w\in A_p$ be such that $C_0^\infty(\mathbb{R}^n)\subset Z^{s,p,w}(\mathbb{R}^n)$. Then for every $f\in Z^{s,p,w}(\mathbb{R}^n)$, there is a G_δ set $E\subset\mathbb{R}^n$ with $C_{s,p,w}(E)=0$ such that $\lim\limits_{r\to 0^+} \fint_{Q(x,r)} f$ exists for every $x\in\mathbb{R}^n\setminus E$ and if we define*

$$f^*(x):=\lim_{r\to 0^+}\fint_{Q(x,r)} f(y)dy$$

then for every $\varepsilon>0$, there exist an open set $U\subset\mathbb{R}^n$ such that $C_{s,p,w}(U)<\varepsilon$ and $f^*|_{\mathbb{R}^n\setminus U}$ is well-defined and continuous on $\mathbb{R}^n\setminus U$ i.e f^* is a quasicontinuous representative of f.

Proof. The proof mostly follows the ideas of Theorem 21 in [9]. We provide a sketch of the proof for readers convenience. Take $f\in Z^{s,p,w}(\mathbb{R}^n)$ then for each integer $k\geqslant 1$ there exist $f_k\in C_0^\infty(\mathbb{R}^n)$ such that

$$||f-f_k||^p_{Z^{s,p,w}(\mathbb{R}^n)}\leqslant 2^{-k(p+1)}.$$

Define $E_k=\{x\in\mathbb{R}^n : M_1^+(f-f_k)(x)>2^{-k}\}$, where M_1^+ is the local one-sided maximal function with respect to $R=1$ and $\Omega=\mathbb{R}^n$. E_k's are open set and using the boundedness of M^+ and theorem 2.1, there exist $C=C(p,n,[w]_{A_p})$

$$\begin{aligned} C_{s,p,w}(E_k) &\leqslant ||2^k M_1^+(f-f_k)||_{Z^{s,p,w}(\mathbb{R}^n)}\\ &\leqslant C2^{kp}||f-f_k||^p_{Z^{s,p,w}(\mathbb{R}^n)}\leqslant C2^{-k}\end{aligned}$$

Now for all $x\in E_k^c$ and $0<r\leqslant 1$ we observe

$$\begin{aligned} &\limsup_{r\to 0}|f_k(x)-\fint_{Q(x,r)} f|\\ &\leqslant \limsup_{r\to 0}\left(\fint_{Q(x,r)}|f_k(x)-f_k(y)|+\fint_{Q(x,r)}|f_k(y)-f(y)|dy\right)\\ &\leqslant M_1^+(f_k-f)(x)\leqslant C2^{-k}\end{aligned}$$

Let $F_m:=\bigcup\limits_{k\geqslant m} E_k$, then F_m's are open set and using sub-additivity of Sobolev Capacity we obtain $C_{s,p,w}(F_m)\leqslant \sum\limits_{k\geqslant m} C_{s,p,w}(E_k)\leqslant C\sum\limits_{k\geqslant m} 2^{-k}$. Then for $i,j\geqslant m$ and for all $x\in\mathbb{R}^n\setminus F_m$ we have

$$\begin{aligned} |f_j(x)-f_i(x)| &\leqslant \limsup_{r\to 0}|f_j(x)-\fint_{Q(x,r)} f|+\limsup_{r\to 0}|f_i(x)-\fint_{Q(x,r)} f|\\ &\leqslant 2^{-i}+2^{-j}.\end{aligned}$$

hence (f_j) converges uniformly to some continuous function, let's say g_m, on $\mathbb{R}^n\setminus F_m$. It's easy to observe that if $x\in\mathbb{R}^n\setminus F_m$ then

$$\begin{aligned} &\limsup_{r\to 0}|g_m(x)-\fint_{Q(x,r)} f|\\ &\leqslant |g_m(x)-f_i(x)|+\limsup_{r\to 0}|f_i(x)-\fint_{Q(x,r)} f|\to 0 \text{ as } i\to\infty\end{aligned}$$

Hence, we obtain

$$g_m(x) = \lim_{r \to 0^+} \fint_{Q(x,r)} f(y) \text{ for all } x \in \mathbb{R}^n \setminus F_m$$

Let $E := \bigcap_{m \geqslant 1} F_m$, then by construction E is an G_δ set and by monotonicity $C_{s,p,w}(E) = 0$ and

$$f^*(x) = \lim_{r \to 0^+} \fint_{Q(x,r)} f(y) \text{ for all } x \in \mathbb{R}^n \setminus E$$

And for the last part, take any $\varepsilon > 0$ then choose $m \in \mathbb{N}$ such that $C_{s,p,w}(F_m) < \varepsilon$. Take $U := F_m$ and it's easy to see that $f^* = g_m$ on $\mathbb{R}^n \setminus U$ which is continuous on $\mathbb{R}^n \setminus U$. Hence we obtain the theorem. $\square$

Conclusion. Our approach yields the boundedness of M_R^+ on weighted fractional Sobolev spaces provided that the weight belongs to A_p classes. But it's not yet known whether it is possible to extend Theorem 2.1 for a bigger class of weights. For example if $w \in A_p^+$ on $\mathbb{R}$, then we are able to prove a better analogue of Lemma 2.2, namely

$$\begin{aligned} w_0(x,y)L(M_\Omega^+ f)(x,y) \leqslant C[&w_0(x,y)M_{10}^+(Lf)(x,y) + M_{11}(w_1 M_{01}^+(Lf))(y,x) \\ &+ M_{11}(w_0 M_{01}^-(Lf))(x,y)] \end{aligned}$$

everywhere except for the case when $x > y$ together with $M_R^+ f(y) > M_R^+ f(x)$. One major difficulty in this case is that $w \in A_p^+$ implies $\tilde{w} \in A_p^-$ (see[10]) and thus we lose the symmetry.

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REFERENCES

[1] L. FORZANI, F. J. MARTÍN-REYES, S. OMBROSI, *Weighted inequalities for the two-dimensional one-sided Hardy-Littlewood maximal function*, Trans. Amer. Math. Soc. **363** (2011), no. 4, 1699–1719. 42B25 (37A40 47A35).

[2] PIOTR HAJŁASZ, JANI ONNINEN, *On boundedness of maximal functions in Sobolev spaces*, Ann. Acad. Sci. Fenn. Math. **29** (2004), no. 1, 167–176. 42B20 (42B25 46E35).

[3] JUHA KINNUNEN, *The Hardy-Littlewood maximal function of a Sobolev function*, Israel J. Math. **100** (1997), 117–124, 30C65.

[4] JUHA KINNUNEN, EERO SAKSMAN, *Regularity of the fractional maximal function*, Bull. London Math. Soc. **35** (2003), no. 4, 529–535. 42B35 (46E35 47G10).

[5] JUHA KINNUNEN, PETER LINDQVIST, *The derivative of the maximal function*, J. Reine Angew. Math. **503** (1998), 161–167. 42B25.

[6] HANNES LUIRO, *On the regularity of the Hardy-Littlewood maximal operator on subdomains of $\mathbb{R}^n$*, Proc. Edinb. Math. Soc. (2) **53** (2010), no. 1, 211–237. 42B25 (47H09).

[7] HANNES LUIRO, *Continuity of the maximal operator in Sobolev spaces*, Proc. Amer. Math. Soc. **135** (2007), no. 1, 243–251. 42B25 (46E35 47B38).

[8] HANNES LUIRO, ANTTI V. VHKANGAS, *Local maximal operators on fractional Sobolev spaces*, J. Math. Soc. Japan **68** (2016), no. 3, 1357–1368. 42B25 (46E35).

[9] HANNES LUIRO, ANTTI V. VHKANGAS, *Beyond local maximal operators*, Potential Anal. **46** (2017), no. 2, 201–226. 42B25 (46E35 47H99).

[10] F. J. MARTÍN-REYES, *New proofs of weighted inequalities for the one-sided Hardy-Littlewood maximal functions*, Proc. Amer. Math. Soc. **117** (1993), no. 3, 691–698. 42B25.

[11] ONDŘEJ KURKA, *On the variation of the Hardy-Littlewood maximal function*, Ann. Acad. Sci. Fenn. Math. **40** (2015), no. 1, 109–133. 42B25 (42B30 46E35).

[12] B. MUCKENHOUPT, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226.

[13] E. SAWYER, *Weighted inequalities for the one-sided Hardy-Littlewood maximal functions*, Trans. Amer. Math. Soc. **297** (1986), no. 4, 367–374.

[14] TANAKA HITOSHI, *A remark on the derivative of the one-dimensional Hardy-Littlewood maximal function*, Bull. Austral. Math. Soc. **65** (2002), no. 2, 253–258, 42B25.

[15] B. O. TURESSON, *Nonlinear potential theory and weighted Sobolev spaces*, Lecture Notes in Mathematics, **1736**, Springer-Verlag, Berlin, 2000. xiv+173 pp. 31C45 (35J70 46E35).

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