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A NEW COUNTEREXAMPLE TO SANGWINE–YAGER'S CONJECTURE

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Abstract. Sangwine-Yager conjectured in [9] that if $r_1 \leqslant \ldots \leqslant r_n$ are the real parts of the roots of the (formal) alternating Steiner polynomial of $V(K-tE)$, then $0 < r_1 \leqslant r(K;E) \leqslant R(K;E) \leqslant r_n$, where $r(K;E)$ and $R(K;E)$ are the inradius, respectively, outradius, or circumradius, of K relative to E. We present here a new counterexample to this conjecture in dimension 3 when none of the bodies is a Euclidean ball. Previous examples due to Henk and Hernández Cifre, and, respectively, to Hernández Cifre and Saorín, were constructed with fairly technical tools. Our example is non-trivial in the sense that both K and E are top dimensional convex bodies, yet it is easy to present.

1. Introduction

A staple of Euclidean planar geometry, Bonnesen's inequality [1], [2] states that if $\mathscr{C}$ is a Jordan curve of length L which bounds a domain of area A, then

$$tL - A - t^2\pi \geqslant 0, \quad \text{for any } t \in [r,R],$$

where r, called inradius, is the radius of the largest disk included in the domain K bounded by $\mathscr{C}$

$$r = \sup\{\rho \geqslant 0 : \exists p \in \mathbb{R}^2 \text{ such that } \rho B_2 + p \subset K\}$$

and R, called circumradius or outradius, is the radius of the smallest disk containing $\mathscr{C}$

$$R = \inf\{\rho \geqslant 0 : \exists p \in \mathbb{R}^2 \text{ such that } K \subset \rho B_2 + p\}.$$

The power of this inequality is more evident when considering its immediate corollary

$$L^2 - 4\pi A \geqslant \pi^2(R-r)^2$$

which provides a qualitative version of the planar isoperimetric inequality, a characterization of its equality case, and stability estimates for the planar isoperimetric inequality. For a nice survey on Bonnesen's inequality, we direct the reader to [8].

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Restricting K to be convex, there exists an anisotropic version of the Bonnesen inequality which one can regard as a version of Bonnesen inequality in the two-dimensional plane whose unit ball is an arbitrary compact convex set E containing the origin, but not necessarily symmetric with respect to the origin. We call the classical case, when the unit ball is the Euclidean unit ball centred at the origin, isotropic. Below, we will state this generalized (anisotropic) inequality following Flanders [4].

Let $\mathscr{C}_0$ and $\mathscr{C}_1$ be two Jordan curves bounding convex domains E and K with areas $A(E)$ and $A(K)$ respectively. Let $r(K;E)$ and $R(K;E)$ be the inradius, respectively circumradius, of K relative to E, more precisely,

$$r(K;E) = \sup\{\rho \geqslant 0 : \exists p \in \mathbb{R}^2 \text{ such that } \rho E + p \subset K\}, \tag{1}$$

respectively,

$$R(K;E) = \inf\{\rho \geqslant 0 : \exists p \in \mathbb{R}^2 \text{ such that } K \subset \rho E + p\}. \tag{2}$$

Let $V(K,E)$ denote the mixed volume of K and E defined by

$$V(K,E) = \lim_{\varepsilon \searrow 0} \frac{A(K+\varepsilon E) - A(K)}{\varepsilon},$$

where $+$ denotes the usual vector addition.

Then, the anisotropic Bonnesen inequality states that

$$tV(K,E) - A(K) - t^2 A(E) \geqslant 0, \quad \forall t \in [r(K;E), R(K;E)].$$

Bonnesen's inequality, isotropic and anisotropic alike, does not hold for, even regular, domains in dimension $n > 2$ when L is replaced by surface area (or $V(K,E)$ is replaced by a weighted surface area in the anisotropic case) and A is replaced by the volume as the Lebesgue measure in $\mathbb{R}^n$. However, there are several results in the literature on estimating the (relative) inradius and circumradius in higher dimension via certain Bonnesen-type inequalities and we mention in particular those of Sangwine-Yager [9], [10], and Zhou et al. [16]. Jiazu Zhou, alone or with collaborators, has in fact obtained also many generalizations of the planar Bonnesen inequality, isotropic and anisotropic, including some for domains in planes of constant curvature, among which we mention [14], [15].

This reinforced the interest in having a result relating the roots of a polynomial to the values of the inradius and circumradius of convex bodies in the n-dimensional Euclidean space. We will start by presenting an early conjecture in this direction. Let K and E be two convex bodies in $\mathbb{R}^n$, thus two compact convex sets of $\mathbb{R}^n$ with non-empty interior. It was shown by Steiner in early 1800's that, in any dimension $n \geqslant 2$, the volume (area if $n = 2$) of the set $K + tE$, $t > 0$, is a polynomial in t:

$$V(K+tE) = \sum_{i=0}^{n} \binom{n}{i} t^i V(\underbrace{K,\ldots,K}_{n-i}, \underbrace{E,\ldots,E}_{i}), \tag{3}$$

where the quantities $V(K,\dots,K,E,\dots,E)$, sometime denoted by $V_i(K,E)$, are mixed volumes of K and E defined precisely as coefficients of the Steiner polynomial, see [12].

In reference to Bonnesen's inequality, a first conjecture in higher dimension stated that for $n>2$ *the alternating Steiner polynomial*

$$\sum_{i=0}^{n}\binom{n}{k}(-t)^i V(K,\dots,K,E,\dots,E), \tag{4}$$

formally $V(K-tE)$, is negative on the closed interval $[r(K;E),R(K;E)]$. This conjecture was shown to be false by a counterexample constructed by Sangwine-Yager in [10] employing cap bodies.

Inspired by a problem proposed by Teissier in the realm of algebraic geometry, [13], Sangwine-Yager stated in [9] the following conjecture: *if $r_1 \leqslant \dots \leqslant r_n$ are the real parts of the roots of the alternating Steiner polynomial as above in (4), then* $0<r_1 \leqslant r(K;E) \leqslant R(K;E) \leqslant r_n$.

Unfortunately, also this conjecture has been shown recently to be false. Henk and Hernández Cifre have shown that, in any dimension, the conjecture holds for bodies E and cap bodies of E which are also known as 1-tangential bodies of E - result known to Sangwine-Yager in dimension 3, [11], but it does not hold for 2-tangential bodies, [6]. Their examples are (in the isotropic case): planar bodies (a symmetric planar lens) in $\mathbb{R}^3$, a 2-tangential body of the Euclidean ball B_{15} in dimension 15 and B_{15}, and a 3-tangential body of B_{12} in dimension 12 and B_{12}. For definitions of p-tangential bodies ($1 \leqslant p \leqslant n-1$), we direct the reader to [12]. As a side remark, note that the roots of the Steiner polynomial are the opposite of the roots of the alternating Steiner polynomial which has been the context in which Henk and Hernández Cifre worked on. Until more recently, there was hope for a partial validity of the isotropic version of Sangwine-Yager's conjecture for the inequality $0<r_1 \leqslant r(K;E=B_n)$. However, this has been disproved also in 2014 by Hernández Cifre and Saorín, [7].

In this paper, we present a new counterexample to Sangwine-Yager's conjecture in dimension 3. The counterexample is fairly simple and is a result of trying to find a Bonnesen-type inequality for a class of polyhedra. We have started investigating the inradius and circumradius for pairs of polyhedra with parallel corresponding sides (same set of outernormals to the top dimensional faces). We have quickly reached the conclusion that such an inequality cannot hold in general, but, in the process, we have shown that if K is a cube with a cut-off corner and E is a parallelepiped whose outernormals to the faces are those of the cube, the inradius $r(K;E)$ violates the conjecture. Switching K and E between them, the circumradius $R(E;K)$ will violate the conjecture as well.

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2. Main results

We call a convex body in $\mathbb{R}^n$ a compact convex set in $\mathbb{R}^n$ with non-empty interior. The support function of a convex body K as a function on the unit sphere $\mathbb{S}^{n-1} = \{(u_1,\dots,u_n) \in \mathbb{R}^n : \sum_{i=1}^n u_i^2 = 1\}$ assigns to each unit vector $u \in \mathbb{S}^{n-1}$ the number

$$h_K(u) = \max_{x\in K}(x\cdot u),$$

where $\cdot$ denotes the usual scalar product in $\mathbb{R}^n$. Note that, if K contains the origin of $\mathbb{R}^n$ in its interior, then the support function h_K is positive on its domain.

Now, let us consider two convex bodies in $\mathbb{R}^3$ as follows. Let the first convex body, denoted by K, be a rectangular parallelepiped centred at the origin whose 2-dimensional faces are parallel to the coordinate planes. Let K have length L, width w, and height h and assume, without loss of generality, that $L \geqslant w \geqslant h$. Let the second convex body, denoted by E, be the unit cube, centred at the origin and having the 2-dimensional faces parallel to the coordinate planes, with a corner cut off by the affine plane $\mathscr{P}$ intersecting the cube at the points $(1/2,0,-1/2),(0,1/2,-1/2)$ and $(1/2,1/2,0)$, so that the origin remains in the interior of E.

Our immediate goal is to compute the coefficients of the alternating Steiner polynomial of K relative to the body E:

$$\mathscr{S}(t) = V_0(K,E) - 3tV_1(K,E) + 3t^2V_2(K,E) - t^3V_3(K,E),$$

where $V_i(K,E)$ is the i-th mixed volume of K with respect to E, $0 \leqslant i \leqslant 3$, as in (3).

2.1. Alternating Steiner polynomial of K relative to E

We note from the definition of mixed volumes, see also [12], that $V_0(K,E) = V(K)$, the volume of K, and that $V_3(K,E) = V(E)$, the volume of E.

Let $u_1,\dots,u_6$ be the outer unit normals to the 2-dimensional faces F_i of K. We denote the 2-dimensional faces $\overline{F}_i$ of E so that $u_1,\dots,u_6$ are normals to the faces of E in this order as well, while $\overline{F}_7$ is face of E contained in the plane $\mathscr{P}$ of unit outer normal $u_7 = \frac{1}{\sqrt{3}}(1,1,-1)$. Finally, $h_i = h_K(u_i)$ and $\overline{h}_i = h_E(u_i)$ for each corresponding i.

Then, following, for example, [12] formula (5.23), the first and second mixed volumes are given by, respectively,

$$V_1(K,E) = \frac{1}{3}\sum_{i=1}^{6}\overline{h}_iA(F_i) \quad \text{and} \quad V_2(K,E) = \frac{1}{3}\sum_{i=1}^{7}h_iA(\overline{F}_i).$$

Clearly, $V(K) = Lwh$ and, as E is the unit cube without a pyramidal corner, we have that $V(E) = 1 - \frac{1}{3}\cdot\left(\frac{1}{2}\right)^4 = \frac{47}{48}$.

Furthermore, assuming the ordering of the faces due to the relation between their outernormals as follows $u_1 = -u_6$, $u_2 = -u_4$, $u_3 = -u_5$, we have that

$$h_1 = \frac{h}{2} = h_6,\ h_2 = \frac{L}{2} = h_4,\ h_3 = \frac{w}{2} = h_5,\ \overline{h}_1 = \ldots = \overline{h}_6 = \frac{1}{2},$$

and

$$A(F_1) = Lw = A(F_6),\ A(F_2) = wh = A(F_4),\ A(F_3) = Lh = A(F_5).$$

Moreover,

$$A(\overline{F}_i) = 1, \text{ for } i = 1,2,3, \text{ and } A(\overline{F}_i) = \frac{7}{8}, \text{ for } i = 4,5,6,$$

while $\overline{F}_7$ is an equilateral triangle with side $\frac{\sqrt{2}}{2}$, thus height $\frac{\sqrt{6}}{4}$ and area $A(\overline{F}_7) = \frac{\sqrt{3}}{8}$.

Finally, to proceed with the calculation of $V_2(K,E)$, we still need the value h_7, which is the distance from the origin to the plane $\overline{\mathscr{P}}$, which has the same normal as $\mathscr{P}$ but passes through the vertex of coordinates $(\frac{w}{2}, \frac{L}{2}, \frac{-h}{2})$ of K.

The plane $\overline{\mathscr{P}}$ has equation $x+y-z = \frac{L+w+h}{2}$, so by minimizing $\sqrt{x^2+y^2+z^2}$ subject to $(x,y,z) \in \overline{P}$, we get $h_7 = \frac{\sqrt{3}}{6}(L+w+h)$.

We can now compute the values of the mixed volumes. Using (5), we get

$$V_1(K,E) = \frac{Lw+Lh+wh}{3} \quad \text{and} \quad V_2(K,E) = \frac{(L+w+h)}{3}.$$

The alternating Steiner polynomial of K relative to E is then:

$$\mathscr{S}(t) = Lwh - (Lw+Lh+wh)t + (L+w+h)t^2 - \frac{47}{48}t^3. \tag{5}$$

From the fundamental theorem of algebra, this polynomial may have three real roots or one real root and two complex conjugate roots.

2.2. A specific counter-example to Sangwine-Yager's conjecture concerning the inradius

Let us consider K as the body described earlier with the specific values $L = 4$, $w = 3$, and $h = 2$.

Using the definitions of the inradius, respectively outradius, of K relative to E, (1), (2), we have that $r(K;E) = 2$ and $R(K;E) = 4$.

From (5), the alternating Steiner polynomial for the bodies K and E is

$$\mathscr{S}(t) = 24 - 26t + 9t^2 - \frac{47}{48}t^3.$$

This polynomial has three real roots: $r_1 \approx 2.1200$, $r_2 \approx 2.5663$ and $r_3 \approx 4.5052$ (where exact formulas can be found in [3] and approximations were performed with *WolframAlpha*).

Thus, for our pair of convex bodies K and E as before, we have

$$r(K;E) < r_1 < R(K;E) < r_3,$$

contradicting Sangwine-Yager's conjecture in that $r(K;E) < r_1$, a result that we will state formally below.

THEOREM 1. *There exist non-spherical convex bodies K and E in $\mathbb{R}^3$ such that the inradius $r(K;E)$ is smaller than the smallest real part of the roots of the alternating Steiner polynomial of K with respect to E.*

2.3. Reversing the roles. A specific counter-example to Sangwine-Yager's conjecture concerning the circumradius

We are interested in confirming that we get a counter-example in the opposite direction by taking now K as the body of reference, and by noting that, for any two convex bodies E and K, we have

$$r(E;K)\cdot R(K;E) = 1. \tag{6}$$

Thus, $r(E;K) = \frac{1}{4}$ and $R(E;K) = \frac{1}{2}$ and the alternating Steiner polynomial is:

$$\mathscr{S}(t) = \frac{47}{48} - 9t + 26t^2 - 24t^3.$$

This polynomial has also three real roots: $r_1 \approx 0.22197$, $r_2 \approx 0.38966$ and $r_3 \approx 0.47170$.

In this case, we see that

$$r_1 < r(E;K) < r_3 < R(E;K),$$

where the fact that $r_3 < R(E;K)$ contradicts the right-side inequality of Sangwine-Yager's conjecture. We have thus proved:

THEOREM 2. *There exist non-spherical convex bodies E and K in $\mathbb{R}^3$ such that the circumradius $R(E;K)$ is larger than the largest real part of the roots of the alternating Steiner polynomial of E with respect to K.*

Note that the key argument here is related to the reciprocity condition (6) that holds for any two convex bodies K and E. This is precisely the argument that led to the anisotropic result by Henk and Hernández Cifre which makes the second body still being a ball.

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