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DUAL COMPLEMENTS FOR DOMAINS OF $\mathbb{C}^n$

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Abstract. Let $\Omega \subset \mathbb{C}^n$ be a bounded, strictly convex domain and $\widetilde{\Omega}$ be its dual complement. Very few such domains with fully described dual complements have been known. We present new types of domains for which their dual complements can be completely described.

1. Introduction

In the Grothendieck-Köthe-da Silva duality theory for the spaces of holomorphic functions defined in a convex domain $\Omega \subset \mathbb{C}^n$ containing the origin $0 \in \mathbb{C}^n$ the notion of dual domain is one of the important basic facts.

If $0 \in \Omega$, then its dual complement (or generalized complement, in [3], [4] it is called the conjugate set of Ω)

$$\widetilde{\Omega} = \{w \in \mathbb{C}^n : w_1 z_1 + \ldots + w_n z_n \neq 1, z \in \Omega\}$$

is the set of hyperplanes that do not intersect the domain Ω. Thus $0 \in \widetilde{\Omega}$. It is also a known fact that for the domain Ω the dual complement of its closure $\overline{\Omega}$ is the interior of the set $\widetilde{\Omega}$, that is, $\widetilde{\overline{\Omega}} = int(\widetilde{\Omega})$. In particular, when Ω has a smooth ($\mathscr{C}^2$) boundary then $\widetilde{\Omega} = int(\widetilde{\Omega}) \cup \partial\widetilde{\Omega}$. Furthermore, if the bounded domain Ω is convex and $0 \in \Omega$, then $\lambda\overline{\Omega} \subset \Omega$, for every $0 < \lambda < 1$. Thus the closed domain $\overline{\Omega}$ and the open domain $\widetilde{\overline{\Omega}}$ are starlike ([3]). In general, it is not an easy task to describe the dual complement of the domain Ω, however for the case of Reinhardt domains with center at the origin we have very precise results. Recall that an open subset Ω of $\mathbb{C}^n$ is called Reinhardt domain if $(z_1, \ldots, z_n) \in \Omega$ implies $(e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n) \in \Omega$ for all real numbers $\theta_1, \ldots, \theta_n$. Actually, if Ω is a Reinhardt domain centered at the origin, then $F(\Omega) \subset \mathbb{R}^n_+$, where $\mathbb{R}^n_+ = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i \geqslant 0\}$, $F(z_1, z_2, \ldots, z_n) = (|z_1|, |z_2|, \ldots, |z_n|)$. For any $B \subset \mathbb{R}^n_+$, its inverse image by F^{-1} is defined to be the set $F^{-1}(B) = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : F(z_1, z_2, \ldots, z_n) \in B\}$. It is straightforward to verify that domain $\Omega \subset \mathbb{C}^n$ is Reinhardt if and only if $\Omega = F^{-1}(F(\Omega))$. Hence, any Reinhardt domain Ω is determined completely by its absolute image $F(\Omega)$. Thus we have the following definition

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DEFINITION 1. Let $\Omega \subset \mathbb{C}^n$ be a Reinhardt domain centered at the origin $0 \in \mathbb{C}^n$. We say that the point $(y_1,\dots,y_n) \in \widetilde{F(\Omega)} \subset \mathbb{R}^n_+$ if and only if $\sum\limits_{i=1}^{n} x_i y_i < 1$ for every $(x_1,x_2,\dots,x_n) \in F(\Omega)$. Then the dual complement of Ω is the set $\widetilde{\Omega} = F^{-1}(\widetilde{F(\Omega)})$.

The most recent example of the dual domain is given in the following statement [5, Lemma 1.1].

LEMMA 1. *For $r > 0$, $p > 1$ and $k_i \in \mathbb{R}_+ \setminus \{0\}$, $i = 1,2,\dots,n$, fixed numbers, let*

$$\Omega = \Big\{ z \in \mathbb{C}^n : \sum_{i=1}^{n} k_i |z_i|^p < r^p \Big\}, \tag{1}$$

be a Reinhardt domain centered at the origin. Then, for $q = \frac{p}{p-1}$,

$$\widetilde{\Omega} = \Big\{ \zeta \in \mathbb{C}^n : \sum_{i=1}^{n} (k_i)^{\frac{1}{1-p}} |\zeta_i|^q \leqslant \frac{1}{r^q} \Big\}. \tag{2}$$

Recall that the known cases of $k_i = 1$, $(i = 1,\dots,n)$ to be found in ([2, 3]). The appropriate case where $p = 1$ corresponds to that where Ω is a hypercone, whose dual complement $\widetilde{\Omega}$ is the closed polydisk. When $p = 2$, the appropriate domain Ω is a ball about the origin of radius r, whose dual complement is the closed ball about the origin of radius $\frac{1}{r}$. In the case $p = \infty$, the dual complement $\widetilde{\Omega}$ is the closed hypercone. So the above lemma gave new results for $1 < p < \infty$ and $p \neq 2$, or for all p, with some $k_i \neq 1$.

Our goal is to seriously extend the collection of domains for which certain description of their dual complements is possible. What is of additional interest is that not only convex or linearly convex (which is the same for Reinhardt domains) are considered.

2. General norms

Say that $\|\cdot\|$ is a Reinhardt type norm on $\mathbb{C}^n$ if there exists a norm $\|\cdot\|_1$ on $\mathbb{R}^n$ such that $\|z\| = \|\, |z| \,\|_1$, $z = (z_1,z_2,\dots,z_n) \in \mathbb{C}^n$, $|z| = (|z_1|,|z_2|,\dots,|z_n|) \in \mathbb{R}^n$. In general, a domain generated by such a norm need not be Reinhardt, say take all z_j to be real. However, for $r > 0$, denote

$$\Omega_r = \{ z \in \mathbb{C}^n : \|z\| < r \}.$$

Evidently, Ω is a Reinhardt domain. Define a dual (associate, see [8]) norm $\|\cdot\|^*$ by

$$\|w\|^* = \sup\Big\{ \Big| \sum_{k=1}^{n} z_k w_k \Big| : \|z\| \leqslant 1 \Big\}$$

and set

$$G_r = \{ z \in \mathbb{C}^n : \|z\|^* \leqslant r \}.$$

Recall some well-known facts on norms.

LEMMA 2. *Let* $z, w \in \mathbb{C}^n$. *Then the Hölder inequality*

$$\Big|\sum_{k=1}^{n} z_k w_k\Big| \leqslant \|z\|\,\|w\|^*$$

holds. Moreover, this inequality is saturated, i.e., for the $v \in \mathbb{C}^n$ *there exists a* $u \in \mathbb{C}^n$ *with* $\|u\| = 1$ *such that*

$$\|v\|^* = \Big|\sum_{k=1}^{n} u_k v_k\Big|.$$

Proof. The Hölder inequality is an easy consequence of the definition of the dual norm. Indeed,

$$\Big|\sum_{k=1}^{n} z_k w_k\Big| = \|z\|\Big|\sum_{k=1}^{n} \frac{z_k}{\|z\|} w_k\Big| \leqslant \|z\| \sup\Big\{\sum_{k=1}^{n} c_k w_k; \|c\| \leqslant 1\Big\} = \|z\|\,\|w\|^*.$$

Now assume $v \in \mathbb{C}^n$. By the definition of the dual norm we have

$$\|v\|^* = \sup\Big\{\Big|\sum_{k=1}^{n} u_k v_k\Big|; \|u\| \leqslant 1\Big\}$$

which can be easily rewritten as

$$\|v\|^* = \sup\Big\{\Big|\sum_{k=1}^{n} u_k v_k\Big|; \|u\| = 1\Big\}.$$

From the definition of supremum we can find a sequence $u^{(m)} = (u_1^{(m)}, u_2^{(m)}, \dots, u_n^{(m)}) \in \mathbb{C}^n$, $m = 1, 2, \dots$ with $\|u^{(m)}\| = 1$ and

$$\|v\|^* - \frac{1}{m} < \Big|\sum_{k=1}^{n} u_k^{(m)} v_k\Big|.$$

Choose a subsequence $u^{(m_s)}$ of $u^{(m)}$ such that $u_k^{(m_s)} \to w_k$. Then $\|w\| = 1$ and

$$\|v\|^* \leqslant \Big|\sum_{k=1}^{n} w_k v_k\Big|,$$

which finishes the proof. $\square$

THEOREM 1. *Let* $\|\cdot\|$ *be a Reinhardt type norm and* $r > 0$. *Then* $\widetilde{\Omega_r} = G_{\frac{1}{r}}$.

Proof. Prove first $G_{\frac{1}{r}} \subset \widetilde{\Omega_r}$. Let $w \in G_{\frac{1}{r}}$. Then for each $z \in \Omega_r$, we have, by the Hölder inequality,

$$\Big|\sum_{k=1}^{n} z_k w_k\Big| \leqslant \|z\|\,\|w\|^* < r\,\frac{1}{r} = 1,$$

therefore $w \in \widetilde{\Omega_r}$.

Let us now prove the converse inequality. Suppose $w \notin G_{\frac{1}{r}}$. Rewriting it we have $\|w\|^* > \frac{1}{r}$. It follows from the saturation of the Hölder inequality that there exists $a \in \mathbb{C}^n$ with $\|a\| = 1$ and

$$\Big|\sum_{k=1}^{n} a_k w_k\Big| = \|w\|^*.$$

Set $\rho = \frac{1}{\|w\|^*}$, $c_k = \rho a_k$, $k = 1,2,\dots,n$. Then $\|c\| = \rho = \frac{1}{\|w\|^*} < r$ and

$$\Big|\sum_{i=1}^{n} c_k w_k\Big| = \rho\Big|\sum_{k=1}^{n} a_k w_k\Big| = \frac{1}{\|w\|^*}\|w\|^* = 1.$$

Find $0 \leqslant \varphi < 2\pi$ with

$$\mathrm{e}^{\mathrm{i}\varphi} = \sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{c_k w_k}.$$

Define $b_k = \mathrm{e}^{-\mathrm{i}\varphi}\mathrm{c_k}$ for $k = 1,2,\dots,n$. Then $\|b\| = \|c\| < r$ and $b \in \Omega_r$. Moreover,

$$\sum_{k=1}^{n} b_k w_k = \mathrm{e}^{-\mathrm{i}\varphi}\sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{c_k w_k} = \mathrm{e}^{-\mathrm{i}\varphi}\mathrm{e}^{\mathrm{i}\varphi} = 1$$

and so, $w \notin \widetilde{\Omega_r}$. We have proved an implication $w \notin G_{\frac{1}{r}} \Rightarrow w \notin \widetilde{\Omega_r}$, which, in turn, proves $\widetilde{\Omega_r} \subset G_{\frac{1}{r}}$ and thus completes the proof of the theorem. □

3. Variable powers case

We are going to consider domains, more general than those in (1), generated by a variable exponent norm. We refer to the book [7] or its textbook embodiment [8], though we deal not with functions or sequences but with a simpler finite-dimensional case. The needed prerequisites are as follows.

For $a = (a_1,\dots,a_n)$ and $p(\cdot) = (p_1,\dots,p_n)$, with $1 \leqslant p_k < \infty$, $k = 1,2,\dots,n$, its variable Luxemburg norm is

$$\|a\|_{p(\cdot)} = \inf_{\lambda}\{\lambda > 0 : \sum_{k=1}^{n}\left(\frac{|a_k|}{\lambda}\right)^{p_k} \leqslant 1\}.$$

For $b = (b_1,\dots,b_n)$, its dual norm is

$$\|b\|^*_{p(\cdot)} = \sup_{\|a\|_{p(\cdot)}\leqslant 1}\left|\sum_{k=1}^{n} a_k b_k\right|. \tag{3}$$

Obviously, $\|a\|_{p(\cdot)}$ is a Reinhardt type norm. Therefore, we are in a position to establish a new variety of dual domains.

THEOREM 2. *For $r > 0$ and $p(\cdot) = (p_1, \dots, p_n)$, with $1 \leqslant p_k < \infty,\ k = 1, 2, \dots, n$, let*

$$\Omega = \{z \in \mathbb{C}^n : \|z\|_{p(\cdot)} < r\}, \tag{4}$$

be a Reinhardt domain centered at the origin. Then,

$$\widetilde{\Omega} = \left\{\zeta \in \mathbb{C}^n : \|\zeta\|^*_{q(\cdot)} \leqslant \frac{1}{r}\right\}. \tag{5}$$

REMARK 1. Of course, for p and q constant, Lemma 1 follows from Theorem 2 as a particular case.

In the general case, we would like to express the dual complement via the $\|\cdot\|_{q(\cdot)}$ norm, with $q(\cdot) = (q_1, \dots, q_n),\ q_k = \frac{p_k}{p_k - 1},\ k = 1, 2, \dots, n$. We have

$$\sum_{k=1}^{n} \left(\frac{|b_k|}{\lambda}\right)^{q_k} \leqslant 1$$

if and only if $\lambda \geqslant \|b\|_{q(\cdot)}$. This also means that

$$\sum_{k=1}^{n} \left(\frac{|b_k|}{\lambda}\right)^{q_k - 1} |b_k| \leqslant \lambda.$$

On the other hand, for the same λ,

$$\sum_{k=1}^{n} \left(\frac{|b_k|}{\lambda}\right)^{q_k - 1} |b_k| \leqslant \|b\|^*_{q(\cdot)},$$

since for a with $a_k = \left(\frac{|b_k|}{\lambda}\right)^{q_k - 1}$ we obtain $\|a\|_{p(\cdot)} \leqslant 1$. This leads to

$$\|b\|_{q(\cdot)} \leqslant \|b\|^*_{q(\cdot)}, \tag{6}$$

and, in accordance with this, we cannot in general replace $\|\zeta\|^*_{q(\cdot)}$ by $\|\zeta\|_{q(\cdot)}$ in the definition of $\widetilde{\Omega}$ in (5).

All these are well illustrated by the following example.

EXAMPLE 1. Let $p_1 = \frac{3}{2}$ and $p_2 = 3$. Correspondingly, $q_1 = 3$ and $q_2 = \frac{3}{2}$. If $r = 1$, then $\Omega = \{(z_1, z_2) : |z_1|^{\frac{3}{2}} + |z_2|^3 < 1\}$. However, for $r \neq 1$, we have to calculate the norm λ from the equation

$$\left(\frac{x}{\lambda}\right)^{\frac{3}{2}} + \left(\frac{y}{\lambda}\right)^3 = 1,$$

with $x = |z_1|$ and $y = |z_2|$. Solving the corresponding quadratic equation, we arrive at

$$\lambda^{\frac{3}{2}} = \frac{x^{\frac{3}{2}}}{2} + \sqrt{\frac{x^3}{4} + y^3},$$

which allows us to write

$$\Omega = \{(z_1,z_2) : \frac{|z_1|^{\frac{3}{2}}}{2} + \sqrt{\frac{|z_1|^3}{4} + |z_2|^3} < r^{\frac{2}{3}}\}.$$

It follows from the symmetry of the exponents that

$$\|\zeta\|_{q(\cdot)}^{\frac{3}{2}} = \frac{|\zeta_2|^{\frac{3}{2}}}{2} + \sqrt{\frac{|\zeta_2|^3}{4} + |\zeta_1|^3}.$$

We calculate the norm $\|\zeta\|^*_{q(\cdot)}$ as a solution of the extremal problem $\sup(xu+yv)$, where the sup is taken over (x,y) such that $x^{\frac{3}{2}} + y^3 = 1$. Equivalently, we are searching for $\sup((1-y^3)^{\frac{2}{3}}u + yv)$ as a function of y. Routine calculations again lead to a quadratic equation with the solution

$$y^3 = -\frac{v^3}{16u^3} + \frac{\sqrt{v^6 + 32u^3v^3}}{16u^3}.$$

By this,

$$\widetilde{\Omega} = \{(\zeta_1,\zeta_2) : \left(1 + \frac{|\zeta_2|^3}{16|\zeta_1|^3} - \frac{\sqrt{|\zeta_2|^6 + 32|\zeta_1|^3|\zeta_2|^3}}{16|\zeta_1|^3}\right)^{\frac{2}{3}} |\zeta_1|$$
$$+ \left(-\frac{|\zeta_2|^3}{16|\zeta_1|^3} + \frac{\sqrt{|\zeta_2|^6 + 32|\zeta_1|^3|\zeta_2|^3}}{16|\zeta_1|^3}\right)^{\frac{1}{3}} |\zeta_2| \leqslant \frac{1}{r^{\frac{2}{3}}}\}.$$

Taking $\zeta_2 = 0$, we arrive at the comparison of the inequalities $|\zeta_1| \leqslant \frac{1}{r^{\frac{2}{3}}}$ for $\widetilde{\Omega}$ and $|\zeta_1| \leqslant \frac{1}{r^{\frac{2}{9}}}$ for the Luxemburg $q(\cdot)$ norm. Taking, say, $r = 8$ gives for $|\zeta_1|$ two different intervals: $[0, \frac{1}{4}]$ for $\widetilde{\Omega}$ and the wider interval $[0, \frac{1}{4^{\frac{1}{3}}}]$ for the Luxemburg $q(\cdot)$ norm.

4. Weighted anisotropic case

Let $p = (p_1, p_2, \ldots, p_n)$, $1 < p_i < \infty$, and $w = (w_1, w_2, \ldots, w_n)$, $w_i > 0$. Define for $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$ a sequence A_k, $k = 1, 2, \ldots, n-1$, by

$$A_1(z) = (|z_1|^{p_1} w_1)^{\frac{1}{p_1}},$$
$$A_{k+1}(z) = (A_k^{p_{k+1}}(z) + |z_{k+1}|^{p_{k+1}} w_{k+1})^{\frac{1}{p_{k+1}}}$$

and set

$$\|z\|_{p,w} = A_n(z).$$

THEOREM 3. *Let* $p=(p_1,p_2,\ldots,p_n)$ *and* $w=(w_1,w_2,\ldots,w_n)$ *be given. Define*

$$
\begin{aligned}
A_1^*(z) &= (|z_1|^{p'_1} w_1^{-\frac{p'_1}{p_1}})^{\frac{1}{p'_1}},\\
A_{k+1}^*(z) &= ((A_k^*(z))^{p'_{k+1}} + |z_{k+1}|^{p'_{k+1}} w_{k+1}^{-\frac{p'_{k+1}}{p_{k+1}}})^{\frac{1}{p'_{k+1}}}.
\end{aligned}
$$

Then $\|z\|_{p,w}^* = A_n^*(z)$.

Proof. Prove first by mathematical induction for $1 \leqslant k \leqslant n$ an inequality

$$
\Big|\sum_{i=1}^{k} u_i v_i\Big| \leqslant A_k(u)A_k^*(v).
$$

Since

$$
|u_1 v_1| = (|u_1|^{p_1} w_1)^{\frac{1}{p_1}} (|v_1|^{p'_1} w_1^{-\frac{p'_1}{p_1}})^{\frac{1}{p}'_1} = A_1(u)A_1^*(v),
$$

the inequality holds for $k=1$. Assume that the inequality holds for $k-1$. Then we have, by the Hölder inequality for sequences,

$$
\begin{aligned}
\Big|\sum_{i=1}^{k} u_i v_i\Big| &= \Big|\sum_{i=1}^{k-1} u_i v_i + u_k v_k\Big| \leqslant \Big|\sum_{i=1}^{k-1} u_i v_i\Big| + |u_k||v_k|\\
&\leqslant |A_{k-1}(u)A_{k-1}^*(v) + u_k v_k| = |A_{k-1}(u)A_{k-1}^*(v) + u_k w_k^{\frac{1}{p_k}} v_k w^{-\frac{1}{p_k}}|\\
&\leqslant \big(A_{k-1}^{p_k}(u) + |u_k|^{p_k} w_k\big)^{1 p_k} \big((A_{k-1}^*)^{p'_k}(v) + |v_k|^{p'_k} w_k^{-\frac{p'_k}{p_k}}\big)^{\frac{1}{p'_k}} = A_k(u)A_k^*(v).
\end{aligned}
$$

A special case $k=n$ gives

$$
\Big|\sum_{i=1}^{n} u_i v_i\Big| \leqslant A_k(u)A_k^*(v) = \|u\|_{p,w}\|v\|_{p,w}^*.
$$

Now, we prove that $\|v\|_{p,w}^* = \sup\{\sum_{i=1}^k u_i v_i : \|u\|_{p,w} \leqslant 1\}$. To see this it suffices to find a vector u for a given vector v with

$$
\sum_{i=1}^{n} u_i v_i = A_n(u)A_n^*(v). \tag{7}
$$

Assume that $v \in \mathbb{R}^n$ be fixed. Set for $j \in \{1,2,\ldots,n\}$

$$
\begin{aligned}
\alpha_1 &= 1,\\
\alpha_j &= \big(A_j^*(v)\big)^{p'_{j+1}-p'_j}, \quad j \in \{2,3,\ldots,n-1\},\\
\alpha_n &= 1
\end{aligned}
$$

and define

$$u_j = \Big(\prod_{s=j}^{n} \alpha_s\Big) v_j^{p'_j-1} w_j^{1-p'_j}.$$

It is easy to see

$$\alpha_j \big(A_j^*(v)\big)^{p'_j} + v_{j+1}^{p'_{j+1}} w_{j+1}^{1-p'_{j+1}} = \big(A_{j+1}^*(v)\big)^{p'_{j+1}}. \tag{8}$$

We now wish to prove that for $1 \leqslant k \leqslant n$

$$\sum_{j=1}^{k} u_j v_j = \Big(\prod_{s=k}^{n} \alpha_s\Big) \big(A_k^*(v)\big)^{p'_k}, \tag{9}$$

and again mathematical induction comes to play. Let $k = 1$. Then

$$\begin{aligned} u_1 v_1 &= \Big(\prod_{s=1}^{n} \alpha_s\Big) v_1^{p'_1-1} w_1^{1-p'_1} v_1 = \Big(\prod_{s=1}^{n} \alpha_s\Big) v_1^{p'_1} w_1^{1-p'_1} \\ &= \Big(\prod_{s=1}^{n} \alpha_s\Big) \big(v_1 w_1^{1/p'_1-1}\big)^{p'_1} = \Big(\prod_{s=1}^{n} \alpha_s\Big) \big(v_1 w_1^{-1/p_1}\big)^{p'_1} = \Big(\prod_{s=1}^{n} \alpha_s\Big) \big(A_1^*(v)\big)^{p'_1} \end{aligned}$$

and (9) holds for $k = 1$. Let us prove an induction step. Assume that (9) holds for $1 \leqslant k \leqslant n-1$. Then

$$\begin{aligned} \sum_{j=1}^{k+1} u_j v_j &= \sum_{j=1}^{k} u_j v_j + u_{k+1} v_{k+1} \\ &= \Big(\prod_{s=k}^{n} \alpha_s\Big) \big(A_k^*(v)\big)^{p'_k} + \Big(\prod_{s=k+1}^{n} \alpha_s\Big) v_{k+1}^{p'_{k+1}-1} w_{k+1}^{1-p'_{k+1}} v_{k+1} \\ &= \Big(\prod_{s=k+1}^{n} \alpha_s\Big) \Big(\alpha_k \big(A_k^*(v)\big)^{p'_k} + v_{k+1}^{p'_{k+1}} w_{k+1}^{1-p'_{k+1}}\Big) \overset{(8)}{=} \Big(\prod_{s=k+1}^{n} \alpha_s\Big) \big(A_{k+1}^*(v)\big)^{p'_{k+1}}. \end{aligned}$$

Setting $k = n$ in (9), we obtain

$$\sum_{j=1}^{n} u_j v_j = \big(A_n^*(v)\big)^{p'_n}. \tag{10}$$

Now, prove for $1 \leqslant k \leqslant n$

$$\big(A_k(u)\big)^{p_k} = \Big(\prod_{s=k}^{n} \alpha_s\Big)^{p_k} \big(A_k^*(v)\big)^{p'_k}, \tag{11}$$

again by the mathematical induction. Let $k = 1$. Then

$$\big(A_1(u)\big)^{p_1} = u_1^{p_1} w_1 = \left(\Big(\prod_{s=1}^{n} \alpha_s\Big) v_1^{p'_1-1} w_1^{1-p'_1}\right)^{p_1} w_1$$

$$= \Big(\prod_{s=1}^{n}\alpha_s\Big)^{p_1} v_1^{p_1'} w_1^{1-p_1'} = \Big(\prod_{s=1}^{n}\alpha_s\Big)^{p_1} \big(A_1^*(v)\big)^{p_1'}$$

and hence (11) is satisfied for $k=1$. Let us prove an induction step. It is easy to calculate

$$\begin{aligned}
&(p_{k+1}'-p_k')(p_{k+1}-1)+\frac{p_k' p_{k+1}}{p_k}\\
&=(p_{k+1}'-p_k')(p_{k+1}-1)+p_{k+1}(p_k'-1)\\
&=p_{k+1}'(p_{k+1}-1)-p_k'(p_{k+1}-1)+(p_{k+1}-1)(p_k'-1)+(p_k'-1)\\
&=(p_{k+1}'-1)(p_{k+1}-1)+(p_k'-1)=p_k'.
\end{aligned}\tag{12}$$

Assume that (9) holds for $1\leqslant k\leqslant n-1$. Then

$$\begin{aligned}
\big(A_{k+1}(u)\big)^{p_{k+1}} &= \big(A_k(u)\big)^{p_{k+1}} + u_{k+1}^{p_{k+1}} w_{k+1}\\
&= \Big(\prod_{s=k}^{n}\alpha_s\Big)^{p_{k+1}} \big(A_k^*(v)\big)^{\frac{p_k' p_{k+1}}{p_k}} + \bigg(\Big(\prod_{s=k+1}^{n}\alpha_s\Big) v_k^{p_{k+1}'-1} w_{k+1}^{1-p_{k+1}'}\bigg)^{p_{k+1}} w_{k+1}\\
&= \Big(\prod_{s=k+1}^{n}\alpha_s\Big)^{p_{k+1}} \Big(\alpha_k^{p_{k+1}} \big(A_k^*(v)\big)^{\frac{p_k' p_{k+1}}{p_k}} + v_k^{p_{k+1}'} w_{k+1}^{1-p_{k+1}'}\Big)\\
&= \Big(\prod_{s=k+1}^{n}\alpha_s\Big)^{p_{k+1}} \Big(\alpha_k \alpha_k^{p_{k+1}-1} \big(A_k^*(v)\big)^{\frac{p_k' p_{k+1}}{p_k}} + v_k^{p_{k+1}'} w_{k+1}^{1-p_{k+1}'}\Big)\\
&= \Big(\prod_{s=k+1}^{n}\alpha_s\Big)^{p_{k+1}} \Big(\alpha_k \big(A_k^*(v)\big)^{(p_{k+1}'-p_k')(p_{k+1}-1)} \big(A_k^*(v)\big)^{\frac{p_k' p_{k+1}}{p_k}} + v_k^{p_{k+1}'} w_{k+1}^{1-p_{k+1}'}\Big)\\
&= \Big(\prod_{s=k+1}^{n}\alpha_s\Big)^{p_{k+1}} \Big(\alpha_k \big(A_k^*(v)\big)^{(p_{k+1}'-p_k')(p_{k+1}-1)+\frac{p_k' p_{k+1}}{p_k}} + v_k^{p_{k+1}'} w_{k+1}^{1-p_{k+1}'}\Big)\\
&\overset{(12)}{=} \Big(\prod_{s=k+1}^{n}\alpha_s\Big)^{p_{k+1}} \Big(\alpha_k \big(A_k^*(v)\big)^{p_k'} + v_k^{p_{k+1}'} w_{k+1}^{1-p_{k+1}'}\Big)\\
&\overset{(8)}{=} \Big(\prod_{s=k+1}^{n}\alpha_s\Big)^{p_{k+1}} A_{k+1}^*(v)\big)^{p_{k+1}'}.
\end{aligned}$$

Setting $k=n$ in (11), we obtain

$$(A_n(u))^{p_n} = (A_n^*(v))^{p_n'}$$

which yields with (10)

$$\begin{aligned}
\sum_{j=1}^{n} u_j v_j &= \big(A_n^*(v)\big)^{p_n'} = \big(A_n^*(v)\big)^{p_n'-1} A_n^*(v)\\
&= \big(A_n(u)\big)^{\frac{p_n(p_n'-1)}{p_n'}} A_n^*(v) = A_n(u) A_n^*(v)
\end{aligned}$$

which proves (7). This finishes the proof of the theorem. □

Using Theorem 1, we readily derive from Theorem 3 the following application.

THEOREM 4. *Let* $p = (p_1, p_2, \ldots, p_n)$ *and* $w = (w_1, w_2, \ldots, w_n)$ *be given and* $0 < r < \infty$. *Let* $\Omega_r = \{z \in \mathbb{C}^n : \|z\|_{p,w} < r\}$. *Then* $\widetilde{\Omega_r} = \{z \in \mathbb{C}^n : \|z\|^*_{p,w} \leqslant \frac{1}{r}\}$.

5. The Orlicz case

Considering Orlicz spaces for sequences apparently goes back to J. Lindenstrauss and L. Tsafriri [11]. As above, our case is somewhat simpler, since it is finite-dimensional.

An Orlicz function is a function $M : [0,\infty) \to [0,\infty)$, which is continuous, non-decreasing and convex, $M(0) = 0$ and $M(t) > 0$ if $t > 0$, and $M(t) \to \infty$ as $t \to \infty$. Its conjugate can be defined by

$$N(u) = \sup_{t \geqslant 0} [tu - M(t)].$$

For $a = (a_1, \ldots, a_n)$, its belonging to the Orlicz space generated by M can be defined by the Luxemburg norm (with the convention that the infimum of the empty set is infinite):

$$\|a\|_M = \inf_{\lambda} \left\{ \lambda > 0 : \sum_{k=1}^{n} M\left(\frac{|a_k|}{\lambda} \right) \leqslant 1 \right\} < \infty.$$

We say that an Orlicz function $M(t)$ satisfies the Δ_2 condition (for small t) if for every $a > 1$ there exists a constant $K(a)$ and a positive number $t(a)$ such that $M(at) < K(a)M(t)$ for $0 \leqslant t \leqslant t(a)$. The words "for small x" will be omitted in the sequel.

We say that an Orlicz function $M(t)$ satisfies the Δ_2 condition for large t if there exist such constants $k > 0$ and $t_0 \geqslant 0$ that for $t \geqslant t_0$,

$$M(2t) \leqslant kM(t).$$

It is easy to see that always $k > 2$. All these basics can be found in the classical book [9]. However, for the properties of the sequence Orlicz spaces, a convenient source is [10].

THEOREM 5. *Let* $\Omega_r = \{z \in \mathbb{C}^n : \|z\|_M < r\}$. *Then*

$$\widetilde{\Omega_r} = \left\{ z \in \mathbb{C}^n : \|z\|^*_M \leqslant \frac{1}{r} \right\}. \tag{13}$$

Moreover, if $M(t)$ *satisfies the* Δ_2 *condition and there are two constants* l *and* t_0 *such that for* $t \geqslant t_0$,

$$M(t) \leqslant \frac{1}{2l} M(lt), \tag{14}$$

then

$$\widetilde{\Omega_r} = \left\{ z \in \mathbb{C}^n : \|z\|_N \leqslant \frac{1}{r} \right\}. \tag{15}$$

Proof. The domain Ω_r is definitely defined by means of a Reinhardt type norm $\|\cdot\|_M$. Lemma 2 takes place (cf. [10, Proposition 2.5]). By Theorem 1, (13) follows.

Now, if $M(t)$ satisfies the Δ_2 condition and (14) holds, the associate N-norm coincides with the N-norm (see [9, Ch.II, §9, 5]). Therefore, (15) follows from (13). $\square$

REMARK 2. In [18], a class of sequence spaces is studied that can be considered as a mixture of the two cases from the last two sections. More precisely, we can study the case where the norm is

$$\|a\|_{M,p(\cdot)} = \inf_{\lambda}\left\{\lambda > 0 : \sum_{k=1}^{n}\left(M\left(\frac{|a_k|}{\lambda}\right)\right)^{p_k} \leqslant 1\right\}.$$

Dual complements of the Reinhardt type domains defined by means of such norms can be described along the same lines as above. We omit the details.

6. Concluding remarks

L. Maligranda brought our attention to certain sources where the Köthe duality of ideal function spaces was investigated by several authors. First of all variable l^{p_n} spaces appeared in Orlicz's paper [17], with some generalizations in [6]. A proof of the generalized duality for Orlicz spaces and their generalizations can be found in the paper [16, Thm. 4] and the book [16, Thm. 10.5]. For a simple proof of the duality for Orlicz spaces and their generalizations, see [13, Thms. 1–3]. It is also worth mentioning that a simple description of Köthe duals of Nakano spaces for atomic and atomless measure spaces can be found in [14]. An interesting article on the history of Nakano space and about Nakano as mathematician is given in [15].

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