

**BOUNDEDNESS OF GENERALIZED RIESZ
 POTENTIALS OF FUNCTIONS IN MORREY SPACES
 $L^{(1,\varphi;\kappa)}(G)$ OVER NON-DOUBLING MEASURE SPACES**

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Abstract. Our aim in this paper is to deal with the boundedness of generalized Riesz potentials $I_{\rho,\mu,\tau}f$ of functions in Morrey spaces $L^{(1,\varphi;\kappa)}(G)$ over non-doubling measure spaces, as an extension of [4, 6, 9, 12, 19]. The local integrability is assumed to be minimal, so that the results can not be obtained by the Hardy-Littlewood maximal operator. What is new in this paper is that φ depends on $x \in X$ and that the underlying measure μ is not doubling.

1. Introduction

The space introduced by Morrey [11] in 1938 has become a useful tool of the study for the existence and regularity of solutions of partial differential equations. In this paper, we aim to show the boundedness of generalized Riesz potentials from Morrey spaces $L^{(1,\varphi;\kappa)}(G)$ over non-doubling measure spaces (X, d, μ) to Orlicz-Morrey spaces, to generalized Hölder spaces, or, to generalized Campanato spaces, and consequently establish Sobolev embeddings for generalized Riesz potentials, as an extension of Trudinger [24], Serrin [22], Nakai [12], Garcia-Gatto [4] and the authors [6, 8, 9, 19].

Let X be a separable metric space equipped with a non-negative Radon measure μ . By $B(x, r)$ we denote the open ball centered at x of radius $r > 0$. We write $d(x, y)$ for the distance of the points x and y in X . We assume that

$$\mu(\{x\}) = 0 \tag{1}$$

and that

$$0 < \mu(B(x, r)) < \infty \tag{2}$$

for $x \in X$ and $r > 0$ for simplicity. In the present paper, we do not postulate on μ the “so called” doubling condition. Recall that a Radon measure μ is said to be doubling, if there exists a constant $C > 0$ such that

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)) \tag{3}$$

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for all $x \in \text{supp}(\mu)(= X)$ and $r > 0$. Otherwise μ is said to be non-doubling. In connection with the $5r$ -covering lemma, the doubling condition had been a key condition in harmonic analysis. See [15] and [26].

Let ρ be a function from $(0, \infty)$ to itself and satisfy the Dini condition

$$\int_0^r \frac{\rho(t)}{t} dt < +\infty \tag{4}$$

for all sufficiently small $r > 0$. Note that in the classical case of Euclidean space (4) is equivalent for $I_\rho \chi(x) < \infty$ for almost everywhere, where χ denotes the indicator function of the unit ball. We do not postulate the doubling condition on ρ . See Example 1 for an example which fails the doubling condition. We define the generalized Riesz potential $I_{\rho, \mu, \tau} f$ of f by

$$I_{\rho, \mu, \tau} f(x) = \int_G \frac{\rho(d(x, y))}{\mu(B(x, \tau d(x, y)))} f(y) d\mu(y),$$

where $f \in L^1(G)$. If $\rho(r) = r^\alpha$, $0 < \alpha < n$ and X is the Euclidean space \mathbb{R}^n with the usual distance and the Lebesgue measure, then $I_{\rho, \mu, \tau} f$ is the usual Riesz potential $I_\alpha f$ of order α defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} dy.$$

Let G be a bounded open set in X . Our aim in this paper is to show that $I_{\rho, \mu, \tau}$, which generalizes I_α , is bounded from Morrey spaces $L^{(1, \varphi; \kappa)}(G)$ to Morrey spaces $L^{(1, \psi; 2\kappa)}(G)$ (Theorem 1), to Orlicz-Morrey space $L^{(\Phi, \Psi; 2\kappa)}(G)$ (Theorem 2) or, to generalized Hölder space $\mathcal{L}^{(1, \psi; 9\kappa)}(G)$ (Theorem 3) in the non-doubling setting, as an extension of [4, 8, 9, 19, 22, 24] (see Section 2 for the definitons). Here κ is an auxiliary modified parameter, which arises naturally in the setting of the general metric measure spaces. Here as it turns out, Φ generalizes the parameter which corresponds to the local integrability and φ generalizes the parameter which corresponds to the global integrability. In this paper, we do not assume the doubling condition on φ as well as in [23]. What is new in this paper is that φ depends on $x \in X$ and that the underlying measure μ is not doubling. See Example 2 of such φ .

We also show that the modified fractional integral operator $\tilde{I}_{\rho, \mu, \tau}$ is bounded from the generalized Morrey space $L^{(1, \varphi; \kappa)}(X; \mu)$ to the generalized Campanato space $\mathcal{L}^{(1, \psi; 9\kappa)}(X; \mu)$ (Theorem 4) in the non-doubling setting, as an extension of [4, 6, 12] (see Section 2 for the definitions of $\tilde{I}_{\rho, \mu, \tau}$ and $\mathcal{L}^{(1, \psi; 9\kappa)}(X; \mu)$).

Throughout this paper, let C denote various constants independent of the variables in question and $C(a, b, \dots)$ be a constant that depends on a, b, \dots . The symbol $g \sim h$ means that $C^{-1}h \leq g \leq Ch$ for some constant $C > 0$.

2. Notation and terminologies

Let \mathcal{G} be the set of all almost decreasing functions with respect to the order by ball inclusion from $X \times (0, \infty)$ to $(0, \infty)$, that is, $\varphi \in \mathcal{G}$ if and only if there exists a

constant $c_\varphi > 0$ such that

$$\varphi(x, r) \geq c_\varphi \varphi(y, s) \tag{5}$$

whenever $B(x, r) \subset B(y, s)$. When we are placing ourselves in the setting of Euclidean space, (5) is natural in that this condition entails many other important assumptions on φ ; see [1, Section 2]. As is pointed out by Sihwaningrum, Gunawan and Nakai [23], $\varphi(x, r) = \mu(B(x, \kappa r))^{\frac{\lambda-1}{p}}$ did not fall under the scope of the existing results. See Example 2. In this paper, we do not assume the doubling condition on φ : we do not suppose there exists a constant $c \geq 1$ such that

$$\frac{1}{c} \leq \frac{\varphi(x, r)}{\varphi(x, s)} \leq c \quad \text{for } x \in X, r, s > 0 \quad \text{with } \frac{1}{2} \leq \frac{r}{s} \leq 2. \tag{6}$$

Let G be a bounded open set in X . For $\varphi \in \mathcal{G}$ and $\kappa \geq 1$, we define the generalized modified Morrey space $L^{(1, \varphi; \kappa)}(G)$ as follows:

$$L^{(1, \varphi; \kappa)}(G) \equiv \left\{ f \in L^1_{\text{loc}}(G) : \|f\|_{L^{(1, \varphi; \kappa)}(G)} < \infty \right\}$$

with the norm

$$\|f\|_{L^{(1, \varphi; \kappa)}(G)} = \sup_{z \in G, 0 < r \leq d_G} \frac{1}{\varphi(z, \kappa r)} \cdot \frac{1}{\mu(B(z, \kappa r))} \int_{B(z, r)} |f(x)| d\mu(x)$$

(see [23]). Then a routine argument shows that $L^{(1, \varphi; \kappa)}(G)$ is a Banach space. One of the thrusts to consider $\varphi = \varphi(z, t)$ depending on $z \in X$ is that φ depends on $z \in X$ in the case of variable exponent and that this definition unifies some examples in [10, 17] together with some model cases [7, 25]. Here and below a tacit understanding is that any function $f \in L^1_{\text{loc}}(G)$ is defined to be zero outside of G .

We deal with the following class of Young functions: Let us consider the family \mathcal{Y} of all continuous, increasing, convex and bijective functions from $[0, \infty)$ to itself. For $\Phi \in \mathcal{Y}$, the Orlicz space $L^\Phi(G)$ is defined by

$$L^\Phi(G) \equiv \left\{ f \in L^1_{\text{loc}}(G) : \|f\|_{L^\Phi(G)} < \infty \right\},$$

where

$$\|f\|_{L^\Phi(G)} \equiv \inf \left\{ \lambda > 0 : \int_G \Phi \left(\frac{|f(x)|}{\lambda} \right) d\mu(x) \leq 1 \right\}.$$

If $\Phi_1, \Phi_2 \in \mathcal{Y}$ are equivalent in the sense that there exists a constant $C \geq 1$ with

$$\Phi_1(C^{-1}r) \leq \Phi_2(r) \leq \Phi_1(Cr)$$

for all $r > 0$, then we see easily that

$$L^{\Phi_1}(G) = L^{\Phi_2}(G)$$

with equivalent norms. If

$$\Phi(r) = \exp(r^p) - 1, \quad \exp(\exp(r^p)) - e$$

for $r > 0$, or

$$r^p(\log(3+r))^\lambda, \quad r^p(\log(3+r))^q(\log(3+(\log(3+r))))^\lambda$$

for $r > 0$ such that $|\log r|$ is large, then $L^\Phi(G)$ will be denoted by

$$\exp(L^p)(G), \text{ expexp}(L^p)(G), L^p(\log L)^\lambda(G) \text{ or } L^p(\log L)^q(\log \log L)^\lambda(G),$$

respectively.

For $\Phi \in \mathcal{Y}$, $\varphi \in \mathcal{G}$ and $\kappa \geq 1$, the generalized modified Orlicz-Morrey space $L^{(\Phi, \varphi; \kappa)}(G)$ (of the first kind) is defined by

$$L^{(\Phi, \varphi; \kappa)}(G) \equiv \left\{ f \in L^1_{\text{loc}}(G) : \|f\|_{L^{(\Phi, \varphi; \kappa)}(G)} < \infty \right\},$$

where

$$\begin{aligned} & \|f\|_{L^{(\Phi, \varphi; \kappa)}(G)} \\ & \equiv \sup_{z \in G, 0 < r \leq d_G} \inf \left\{ \lambda > 0 : \frac{1}{\mu(B(z, \kappa r))} \int_{B(z, r)} \Phi \left(\frac{|f(x)|}{\lambda} \right) d\mu(x) \leq \varphi(z, \kappa r) \right\} \end{aligned}$$

(cf. [23]). Note that this definition is of the first kind in terms of [3]. See also [13]. Then, again it is routine to prove that $\|\cdot\|_{L^{(\Phi, \varphi; \kappa)}(G)}$ is a norm and that $L^{(\Phi, \varphi; \kappa)}(G)$ is a Banach space. Note that the classical Orlicz-space $L^\Phi(G)$ is a special case of Orlicz-Morrey spaces when $d\mu = dx$ and $\varphi(z, \kappa r)\mu(B(z, \kappa r)) \equiv 1$.

For $\varphi \in \mathcal{G}$ and $\kappa \geq 1$, let the generalized Campanato space $\mathcal{L}^{(1, \varphi; \kappa)}(X; \mu)$ be the set of all functions $f \in L^1_{\text{loc}}(X; \mu)$ such that

$$\|f\|_{\mathcal{L}^{(1, \varphi; \kappa)}(X; \mu)} \equiv \sup_{z \in X, r > 0} \inf_{c \in \mathbb{C}} \frac{1}{\varphi(z, \kappa r)\mu(B(z, \kappa r))} \int_{B(z, r)} |f(x) - c_{B(z, r)}| d\mu(x) < \infty.$$

We consider the modified fractional integral operator of generalized order ρ with modification parameter τ by:

$$\tilde{I}_{\rho, \mu, \tau} f(x) \equiv \int_{X \setminus \{x\}} K(x, y) f(y) d\mu(y) = \int_X K(x, y) f(y) d\mu(y), \tag{7}$$

where a ball $\mathbb{B} = B(x_0, 1)$ with the basepoint $x_0 \in X$ is fixed and the integral kernel is given by:

$$K(x, y) \equiv \frac{\rho(d(x, y))}{\mu(B(x, \tau d(x, y)))} - \frac{\rho(d(x_0, y))}{\mu(B(x_0, \tau d(x_0, y)))} \chi_{X \setminus \mathbb{B}}(y) \quad (x, y \in X).$$

3. Main results

In this section, we state our main theorems, whose proofs are given in Section 5.

Let us begin with the following result, which is the one of Gunawan type [5] and extends [19, Theorem 2], [9, Theorem 3.1] and [12, Theorem 3.2].

THEOREM 1. Let $\kappa \geq 1$, $0 < k_1 < k_2 < \infty$ and $\tau \geq 4 \max(\kappa, k_2)$. Let $\rho : (0, \infty) \rightarrow (0, \infty)$ be a measurable function satisfying

$$\sup_{r/2 \leq s \leq r} \rho(s) \leq C_\rho \int_{k_1 r}^{k_2 r} \frac{\rho(s)}{s} ds \quad (r > 0) \tag{8}$$

for some $C_\rho > 0$. Let $\varphi \in \mathcal{G}$, and define

$$\psi(x, r) \equiv \left(\int_0^{2\kappa^{-1}k_2 r} \frac{\rho(t)}{t} dt \right) \varphi(x, r) + \int_{\kappa^{-1}k_1 r}^{4k_2 d_G} \frac{\rho(t)\varphi(x, t)}{t} dt \tag{9}$$

for $0 < r \leq d_G$. Then there exists a constant $C > 0$ such that

$$\frac{1}{\mu(B(z, 2\kappa r))} \int_{B(z, r)} |I_{\rho, \mu, \tau} f(x)| d\mu(x) \leq C \psi(z, 2\kappa r) \|f\|_{L^{(1, \varphi; \kappa)}(G)} \tag{10}$$

for $z \in G$, $0 < r \leq d_G$ and $f \in L^{(1, \varphi; \kappa)}(G)$, where $C > 0$ is a constant depending only on C_ρ , c_φ , k_1 and k_2 .

We note that (8) goes back to [16]. By defining ψ by (9), we can describe a boundedness property of I_ρ for an arbitrary measurable function ρ satisfying (8). We present an example of ρ and then two examples of φ .

EXAMPLE 1. If ρ is increasing, then ρ satisfies (8) with $k_1 = 1$ and $k_2 = 2$. If $\alpha > 0$ such that

$$\rho(r) = \begin{cases} r^\alpha & (0 < r < 1) \\ e^{-(r-1)} & (r \geq 1), \end{cases}$$

then ρ satisfies (8) with $k_1 = 1/4$ and $k_2 = 1/2$. See also [14, Lemma 2.5], [16, 18, 23] and [20, Remark 2.2].

EXAMPLE 2. (i) Let $\varphi(x, r) = \mu(B(x, \kappa r))^{\frac{\lambda-1}{p}}$ and $\lambda < 1$. Then $\varphi \in \mathcal{G}$.
 (ii) Let

$$\varphi(x, r) = \begin{cases} r^{\beta(x)} & (0 < r \leq 1/e) \\ r^{\beta_*} & (r > 1/e), \end{cases}$$

where $\sup_{x \in X} \beta(x) \leq 0$ and $\beta_* \leq 0$. If $\beta(\cdot)$ satisfies the local log-Hölder condition, that is, there exists a constant $C > 0$ such that

$$|\beta(x) - \beta(y)| \leq \frac{C}{\log(1/d(x, y))} \quad (d(x, y) \leq 1/e),$$

then $\varphi \in \mathcal{G}$. See [23].

We now state a result for Orlicz-Morrey spaces, which is an extension of [19, Theorem 5] and [9, Theorem 3.2].

THEOREM 2. Let $\kappa \geq 1$, $0 < k_1 < k_2 < \infty$ and $\tau \geq 4 \max(\kappa, k_2)$. Let also $\rho, \tilde{\rho} : (0, \infty) \rightarrow (0, \infty)$ be measurable functions satisfying (8) and

$$\sup_{r/2 \leq s \leq r} \tilde{\rho}(s) \leq C_{\tilde{\rho}} \int_{k_1 r}^{k_2 r} \frac{\tilde{\rho}(s)}{s} ds \quad (r > 0).$$

Let $\varphi \in \mathcal{G}$. Assume

$$\int_0^1 \frac{\rho(t)\varphi(x,t)}{t} dt = \infty \quad (x \in G) \tag{11}$$

and that the function $t \in (0, d_X] \mapsto \tilde{\rho}(t)/\rho(t)$ is continuous and decreasing.

We abbreviate:

$$\psi_1(x, r) \equiv \int_{2k_1 r}^{4k_2 d_G} \frac{\rho(t)\varphi(x,t)}{t} dt, \tag{12}$$

$$\Theta(x, r) \equiv \frac{\psi_1(x, r)\tilde{\rho}(4k_2 r)}{\rho(4k_2 r)}, \tag{13}$$

$$\psi(x, r) \equiv \left(\int_0^{2k_2 r} \frac{\tilde{\rho}(t)}{t} dt \right) \varphi(x, r) + \int_{2k_1 r}^{4k_2 d_G} \frac{\tilde{\rho}(t)\varphi(x,t)}{t} dt \tag{14}$$

and

$$\Psi(x, r) \equiv \psi(x, r) + \frac{1}{\mu(B(z, r))} \int_{B(z, (2\kappa)^{-1}r)} \psi(x, r) d\mu(x)$$

for $x \in G$ and $0 < r \leq d_G$. If $\Phi \in \mathcal{B}$ satisfies

$$C_G = \sup \left\{ \frac{(\psi_1(x, (\Theta(x, \cdot))^{-1}(s)))}{\Phi^{-1}(s)} : x \in G, \Theta(x, d_G) \leq s < \infty \right\} < \infty, \tag{15}$$

then there exists a constant $A > 0$ such that

$$\frac{1}{\mu(B(z, 2\kappa r))} \int_{B(z, r)} \Phi \left(\frac{|I_{\rho, \mu, \tau} f(x)|}{A \|f\|_{L^{(1, \varphi, \kappa)}(G)}} \right) d\mu(x) \leq \Psi(z, 2\kappa r) \tag{16}$$

for $z \in G$, $0 < r \leq d_G$ and $f \in L^{(1, \varphi; \kappa)}(G) \setminus \{0\}$, where $A > 0$ is a constant depending only on $C_\rho, C_{\tilde{\rho}}, c_\varphi, k_1, k_2$ and C_G .

REMARK 1. Let $x \in G$ be fixed. Then $\Theta(x, \cdot)$ is a bijection from $(0, d_G]$ to $[\Theta(x, d_G), \infty)$ by the assumptions in the theorem. Indeed, by the definition of Θ above, Θ is a decreasing function. In addition, $\lim_{r \downarrow 0} \Theta(x, r) = \infty$ by (11) and the decreasing property of $\frac{\tilde{\rho}}{\rho}$, showing that $\Theta(x, \cdot) : (0, d_G] \rightarrow [\Theta(x, d_G), \infty)$ is bijective.

We shall show a result of Gunawan type on continuity of $I_{\rho, \mu, \tau}$, which is an extension of [19, Theorem 7] and [9, Theorem 3.4].

THEOREM 3. *Let $0 < \theta \leq 1$, $\kappa \geq 1$, $0 < k_1 < k_2 < \infty$ and $\tau \geq 2 \max(\kappa, k_2)$. Let $\rho : (0, \infty) \rightarrow (0, \infty)$ be a measurable function satisfying (8). Let $\varphi \in \mathcal{G}$. Assume the following condition on ρ : There exists $C'_\rho > 0$ such that*

$$\left| \frac{\rho(d(x,y))}{\mu(B(x, \tau d(x,y)))} - \frac{\rho(d(z,y))}{\mu(B(z, \tau d(z,y)))} \right| \leq C'_\rho \left(\frac{d(x,z)}{d(x,y)} \right)^\theta \frac{\rho(d(x,y))}{\mu(B(x, \tau d(x,y)))} \tag{17}$$

whenever $d(x,z) \leq d(x,y)/2$. Assume in addition the Dini condition:

$$\int_0^1 \frac{\rho(t)\varphi(x,t)}{t} dt < \infty \quad (x \in G). \tag{18}$$

Abbreviate

$$\psi(x,r) \equiv \int_0^{3k_2 r} \frac{\rho(t)\varphi(x,t)}{t} dt + r^\theta \int_{2k_1 r}^{4k_2 d_G} \frac{\rho(t)\varphi(x,t)}{t^{1+\theta}} dt \tag{19}$$

for $x \in G$ and $0 < r \leq d_G$. Then

$$|I_{\rho,\mu,\tau} f(x) - I_{\rho,\mu,\tau} f(z)| \leq C(\psi(x, 2d(x,z)) + \psi(z, 2d(x,z))) \|f\|_{L^{(1,\varphi;\kappa)}(G)}$$

for $x, z \in G$ and $f \in L^{(1,\varphi;\kappa)}(G)$.

Concerning the assumption (18) and $\bar{I}_{\rho,\mu,\tau} f$, two helpful remarks may be in order.

REMARK 2. Based on Theorem 1, we investigate the well-definedness of the operators.

1. The integral defining $\bar{I}_{\rho,\mu,\tau} f(x)$ converges absolutely for μ -almost all $x \in G$. To prove this, we set

$$\bar{I}_{\rho,\mu,\tau} f(x) \equiv \int_{X \setminus \{x\}} |K(x,y)f(y)| d\mu(y)$$

and prove $\bar{I}_{\rho,\mu,\tau} f(x)$ for μ -almost every $x \in B(x_0, j)$ for each $j \in \mathbb{N}$. In fact, since $\varphi(x, \cdot)$ is almost decreasing, we have

$$\frac{1}{\mu(B(z, 2\kappa r))} \int_{B(z,r)} \bar{I}_{\rho,\mu,\tau} [\chi_{\mathbb{B}} f](x) d\mu(x) \leq C\psi(z, 2\kappa r) \|f\|_{L^{(1,\varphi;\kappa)}(G)} \tag{20}$$

according to Theorem 1. Here ψ is given by (23) below. It follows from (8) that

$$\bar{I}_{\rho,\mu,\tau} [\chi_{(4j\mathbb{B}) \setminus \mathbb{B}} f](x) < \infty$$

for μ -almost all $x \in X$. Indeed, similar to (20), we have

$$\frac{1}{\mu(B(z, 2\kappa r))} \int_{B(z,r)} |\bar{I}_{\rho,\mu,\tau} [\chi_{4j\mathbb{B} \setminus \mathbb{B}} f](x)| d\mu(x)$$

$$\begin{aligned}
 &\leq \frac{1}{\mu(B(z, 2\kappa r))} \int_{B(z,r)} |I_{\rho,\mu,\tau}[\chi_{4j\mathbb{B}\setminus\mathbb{B}}f](x)| d\mu(x) \\
 &\quad + \int_{(4j\mathbb{B})\setminus\mathbb{B}} \frac{\rho(d(x_0,y))}{\mu(B(x_0,\tau d(x_0,y)))} |f(y)| d\mu(y) \\
 &\leq C\psi(z, 2\kappa r) \|f\|_{L^{(1,\varphi;\kappa)}(G)} + \int_{(4j\mathbb{B})\setminus\mathbb{B}} \frac{\rho(d(x_0,y))}{\mu(B(x_0,\tau d(x_0,y)))} |f(y)| d\mu(y) \\
 &\leq C\psi(z, 2\kappa r) \|f\|_{L^{(1,\varphi;\kappa)}(G)} + \sup_{1\leq t\leq 4j} \frac{\rho(t)}{\mu(B(x_0,\tau))} \int_{(4j\mathbb{B})\setminus\mathbb{B}} |f(y)| d\mu(y) \\
 &\leq C\psi(z, 2\kappa r) \|f\|_{L^{(1,\varphi;\kappa)}(G)} + \frac{C\rho}{\mu(B(x_0,\tau))} \int_{k_1}^{4jk_2} \frac{\rho(s)}{s} ds \cdot \int_{(4j\mathbb{B})\setminus\mathbb{B}} |f(y)| d\mu(y) \\
 &< \infty.
 \end{aligned}$$

Let $y \in (4j\mathbb{B})^c$. Then $d(x,y)/2 \geq j > d(x,x_0)$. Thus, by (17), we have

$$|\bar{I}_{\rho,\mu,\tau}[\chi_{(4j\mathbb{B})^c}f](x)| \leq j^\theta I_{\tilde{\rho},\mu,\tau}[|f|](x),$$

where $\tilde{\rho}(t) = t^{-\theta}\rho(t)\chi_{(1,\infty)}$. Note that

$$\tilde{\psi}(x,r) \equiv \left(\int_0^{2\kappa^{-1}k_2r} \frac{\tilde{\rho}(t)}{t} dt \right) \varphi(x,r) + \int_{\kappa^{-1}k_1r}^{4k_2d_G} \frac{\tilde{\rho}(t)\varphi(x,t)}{t} dt \leq \psi(x,r). \tag{21}$$

Here ψ is defined by (23) below. Thus, by Theorem 1 once again and by (21) we see that

$$\begin{aligned}
 &\frac{1}{\mu(B(z, 2\kappa r))} \int_{B(z,r)} \bar{I}_{\rho,\mu,\tau}[\chi_{(4j\mathbb{B})^c}f](x) d\mu(x) \\
 &\leq \frac{C}{\mu(B(z, 2\kappa r))} \int_{B(z,r)} j^\theta I_{\tilde{\rho},\mu,\tau}[|\chi_{(4j\mathbb{B})^c}f|](x) d\mu(x) \\
 &\leq Cj^\theta \tilde{\psi}(z, 2\kappa r) \|f\|_{L^{(1,\varphi;\kappa)}(G)} \\
 &\leq Cj^\theta \psi(z, 2\kappa r) \|f\|_{L^{(1,\varphi;\kappa)}(G)},
 \end{aligned}$$

as was to be shown.

2. Note that, if (18) holds and $0 < \theta \leq 1$, then

$$r \in (0, d_G] \mapsto r^\theta \int_{2k_1r}^{4k_2d_G} \frac{\rho(t)\varphi(x,t)}{t^{1+\theta}} dt \in [0, \infty)$$

is bounded for each fixed $x \in G$, since

$$r^\theta \int_{2k_1r}^{4k_2d_G} \frac{\rho(t)\varphi(x,t)}{t^{1+\theta}} dt \leq C \int_{2k_1r}^{4k_2d_G} \frac{\rho(t)\varphi(x,t)}{t} dt.$$

The linear operator $\tilde{I}_{\rho,\mu,\tau}$ acts on the generalized Morrey space $L^{(1,\varphi;\kappa)}(X;\mu)$.

We will give the boundedness of $\tilde{I}_{\rho,\mu,\tau}$ from $L^{(1,\varphi;\kappa)}(X;\mu)$ to $\mathcal{L}^{(1,\psi;9\kappa)}(X;\mu)$, as an extension of [4, Theorem 5.2], [6, Theorem 1] and [12, Theorem 3.3]. Unlike other theorems in this paper, we can work on the whole space X .

THEOREM 4. *Let $0 < k_1 < k_2 < \infty$. Let $\varepsilon > 0$, $\tau \geq \max(4, 2\kappa, 2k_2)$ and $\kappa \geq 1$. Let ρ satisfy (8) and $\varphi \in \mathcal{G}$. Assume that there exists a constant $C > 0$ such that*

$$\left| \frac{\rho(d(z,y))}{\mu(B(z,\tau d(z,y)))} - \frac{\rho(d(x,y))}{\mu(B(x,\tau d(x,y)))} \right| \leq C \left(\frac{d(x,z)}{d(y,z)} \right)^\varepsilon \frac{\rho(d(z,y))}{\mu(B(z,\tau d(z,y)))} \tag{22}$$

for all $x, y, z \in X$ with $2d(x,z) < d(y,z)$. Abbreviate

$$\begin{aligned} \psi(x,r) \equiv & \left(\int_0^{2\kappa^{-1}k_2r} \frac{\rho(t)}{t} dt \right) \varphi(x,r) + r^\varepsilon \int_{8\kappa^{-1}k_1r/9}^\infty \frac{\rho(t)\varphi(x,t)}{t^{1+\varepsilon}} dt \\ & + \int_{\kappa^{-1}k_1r}^{4k_2d_G} \frac{\rho(t)\varphi(x,t)}{t} dt \left(\int_1^{\max(1,2\kappa^{-1}k_2r)} \frac{\rho(t)}{t^{1+\theta}} dt \right) \varphi(x,r) \\ & + \int_{\max(1,\kappa^{-1}k_1r)}^{\max(1,4k_2d_G)} \frac{\rho(t)\varphi(x,t)}{t^{1+\theta}} dt \quad (x \in X, r > 0). \end{aligned} \tag{23}$$

Assume that $\psi(x,r) < \infty$ for all $x \in X$ and $r > 0$ and that

$$\int_0^1 \frac{\rho(t)\varphi(x,t)}{t} dt < \infty \tag{24}$$

for all $x \in X$.

Then $\tilde{I}_{\rho,\mu,\tau}$ is bounded from $L^{(1,\varphi;\kappa)}(X;\mu)$ to $\mathcal{L}^{(1,\psi;9\kappa)}(X;\mu)$.

REMARK 3. Condition (24), which guarantees that, $\bar{I}\chi_{B(z,r)}(x) < \infty$ for μ -almost all $x \in X$, is natural in view of the classical case. In fact, in the classical case $\varphi(x,t) = t^{-\frac{n}{p}}$ and $\rho(t) = t^\alpha$, this assumption reads $\alpha > n/p$, while condition “ $\psi(x,r) < \infty$ ” reads $\alpha < n/p + \varepsilon$.

4. Preliminary lemmas

LEMMA 1. *Let $\kappa \geq 1$, $k \geq 0$, $0 < k_1 < k_2 < \infty$ and $\tau \geq 2\max(\kappa, k_2)$. Let also $\rho : (0, \infty) \rightarrow (0, \infty)$ be a measurable function satisfying (8). Let $\varphi \in \mathcal{G}$ and $f \in L^{(1,\varphi;\kappa)}(G)$. Then*

$$\begin{aligned} & \int_{B(x,r)} \frac{\rho(d(x,y))}{\mu(B(x,\tau d(x,y)))} |f(y)| d\mu(y) \\ & \leq C \left(\int_0^{2k_2r} \frac{\rho(t)\varphi(x,t)}{t} dt \right) \|f\|_{L^{(1,\varphi;\kappa)}(G)} \quad (x \in G, r \leq d_G) \end{aligned} \tag{25}$$

and

$$\begin{aligned} & \int_{B(x,d_G) \setminus B(x,r)} \frac{\rho(d(x,y))}{\mu(B(x,\tau d(x,y)))d(x,y)^k} |f(y)| d\mu(y) \\ & \leq C \left(\int_{2k_1 r}^{4k_2 d_G} \frac{\rho(t)\varphi(x,t)}{t^{1+k}} dt \right) \|f\|_{L^{(1,\varphi;\kappa)}(G)} \quad (x \in G, r \leq d_G), \end{aligned} \quad (26)$$

where $C > 0$ is a constant depending only on C_ρ , c_φ , k_1, k_2 and k .

Proof. We will first prove two auxiliary estimates (27) and (28). If $y \in B(x, 2^j r) \setminus B(x, 2^{j-1} r)$ and $j \in \mathbb{Z}$, then a geometric observation shows

$$\frac{\rho(d(x,y))}{\mu(B(x,\tau d(x,y)))d(x,y)^k} \leq \frac{1}{\mu(B(x, 2^{j-1} \tau r))(2^{j-1} r)^k} \sup_{2^{j-1} r \leq s \leq 2^j r} \rho(s),$$

so that

$$\frac{\rho(d(x,y))}{\mu(B(x,\tau d(x,y)))d(x,y)^k} \leq \frac{C_\rho}{\mu(B(x, 2^{j-1} \tau r))(2^{j-1} r)^k} \int_{2^{j-1} r}^{2^j r} \frac{\rho(s)}{s} ds. \quad (27)$$

Meanwhile using, $\tau \geq 2 \max(\kappa, k_2)$, we obtain

$$\begin{aligned} & \int_{B(x, 2^j r) \setminus B(x, 2^{j-1} r)} \frac{\rho(d(x,y))}{\mu(B(x,\tau d(x,y)))d(x,y)^k} |f(y)| d\mu(y) \\ & \leq \frac{C_\rho}{(2^{j-1} r)^k} \int_{2^{j-1} r}^{2^j r} \frac{\rho(s)}{s} ds \times \frac{1}{\mu(B(x, 2^{j-1} \tau r))} \int_{B(x, 2^j r)} |f(y)| d\mu(y) \\ & \leq C_\rho \frac{\varphi(x, 2^j \max(\kappa, k_2) r)}{(2^{j-1} r)^k} \int_{2^{j-1} r}^{2^j r} \frac{\rho(s)}{s} ds \\ & \quad \times \frac{1}{\varphi(x, 2^j \max(\kappa, k_2) r) \mu(B(x, 2^j \max(\kappa, k_2) r))} \int_{B(x, 2^j \max(\kappa, k_2) r)} |f(y)| d\mu(y) \\ & \leq C_\rho \frac{\varphi(x, 2^j \max(\kappa, k_2) r)}{(2^{j-1} r)^k} \int_{2^{j-1} r}^{2^j r} \frac{\rho(s)}{s} ds \times \|f\|_{L^{(1,\varphi;\kappa)}(G)}. \end{aligned}$$

Set $d \equiv [1 + \log_2(k_2/k_1)]$, the integer part of $1 + \log_2(k_2/k_1)$. Since $\varphi(x, \cdot)$ is assumed to be almost decreasing uniformly over $x \in X$, we have

$$\begin{aligned} & \frac{\varphi(x, 2^j \max(\kappa, k_2) r)}{(2^{j-1} r)^k} \int_{2^{j-1} r}^{2^j r} \frac{\rho(s)}{s} ds \leq 2^k \frac{1}{c_\varphi (2^j r)^k} \int_{2^{j-1} r}^{2^{j+d} k_1 r} \frac{\varphi(x, s) \rho(s)}{s} ds \\ & = 2^k \sum_{l=1}^d \frac{1}{c_\varphi (2^j r)^k} \int_{2^{j+l-1} k_1 r}^{2^{j+l} k_1 r} \frac{\varphi(x, s) \rho(s)}{s} ds \\ & = \frac{1}{c_\varphi} (2^{d+1} k_1)^k \sum_{l=1}^d \int_{2^{j+l-1} k_1 r}^{2^{j+l} k_1 r} \frac{\varphi(x, s)}{(2^{j+d} k_1 r)^k} \cdot \frac{\rho(s)}{s} ds \\ & \leq C_1 \sum_{l=1}^d \int_{2^{j+l-1} k_1 r}^{2^{j+l} k_1 r} \frac{\varphi(x, s) \rho(s)}{s^{1+k}} ds, \end{aligned}$$

so that

$$\frac{\varphi(x, 2^j \max(\kappa, k_2)r)}{(2^{j-1}r)^k} \int_{2^j k_1 r}^{2^j k_2 r} \frac{\rho(s)}{s} ds \leq C_1 \int_{2^j k_1 r}^{2^{j+d} k_1 r} \frac{\varphi(x, s)\rho(s)}{s^{1+k}} ds, \tag{28}$$

where $C_1 > 0$ is a constant depending only on c_φ, k, k_1 and k_2 .

Based on (27) and (28), we prove (25) and (26).

Using (1), we decompose

$$\begin{aligned} & \int_{B(x,r)} \frac{\rho(d(x,y))}{\mu(B(x, \tau d(x,y)))} |f(y)| d\mu(y) \\ &= \sum_{j=0}^{\infty} \int_{B(x, 2^{-j}r) \setminus B(x, 2^{-j-1}r)} \frac{\rho(d(x,y))}{\mu(B(x, \tau d(x,y)))} |f(y)| d\mu(y). \end{aligned}$$

Due to (27) and (28) with $k = 0$, we obtain

$$\begin{aligned} & \int_{B(x,r)} \frac{\rho(d(x,y))}{\mu(B(x, \tau d(x,y)))} |f(y)| d\mu(y) \\ & \leq \sum_{j=0}^{\infty} C_\rho C_1 \int_{2^{-j} k_1 r}^{2^{-j+d} k_1 r} \frac{\varphi(x, s)\rho(s)}{s} ds \times \|f\|_{L^{(1, \varphi; \kappa)}(G)} \\ & \leq dC_\rho C_1 \left(\int_0^{2k_2 r} \frac{\rho(t)\varphi(x, t)}{t} dt \right) \|f\|_{L^{(1, \varphi; \kappa)}(G)}, \end{aligned}$$

which proves (25).

We move on to the proof of (26). We choose $j_0 \in \mathbb{Z}$ so that $d_G \leq 2^{j_0}r < 2d_G$ and we decompose

$$\begin{aligned} & \int_{B(x, d_G) \setminus B(x,r)} \frac{\rho(d(x,y))}{\mu(B(x, \tau d(x,y)))d(x,y)^k} |f(y)| d\mu(y) \\ & \leq \sum_{j=1}^{j_0} \int_{B(x, 2^j r) \setminus B(x, 2^{j-1}r)} \frac{\rho(d(x,y))}{\mu(B(x, \tau d(x,y)))d(x,y)^k} |f(y)| d\mu(y). \end{aligned}$$

If we use (27) and (28), we have

$$\begin{aligned} & \int_{B(x, d_G) \setminus B(x,r)} \frac{\rho(d(x,y))}{\mu(B(x, \tau d(x,y)))d(x,y)^k} |f(y)| d\mu(y) \\ & \leq C_\rho C_1 \sum_{j=1}^{j_0} \int_{2^j k_1 r}^{2^{j+d} k_1 r} \frac{\varphi(x, s)\rho(s)}{s^{1+k}} ds \times \|f\|_{L^{(1, \varphi; \kappa)}(G)} \\ & \leq dC_\rho C_1 \left(\int_{2k_1 r}^{4k_2 d_G} \frac{\rho(t)\varphi(x, t)}{t^{1+k}} dt \right) \|f\|_{L^{(1, \varphi; \kappa)}(G)}. \end{aligned}$$

Thus, (26) follows. \square

This lemma is needed for the $L^{(1, \varphi; \kappa)}(G) - L^{(1, \psi; 2\kappa)}(G)$ estimate.

LEMMA 2. Let $\kappa \geq 1$. Let $0 < k_1 < k_2 < \infty$ and $\tau \geq 4$. Let $\rho : (0, \infty) \rightarrow (0, \infty)$ be a measurable function satisfying (8). Then, for all $f \in L^{(1, \varphi; \kappa)}(G)$, $z \in X$ and $r > 0$,

$$\begin{aligned} & \frac{1}{\mu(B(z, \kappa r))} \int_{B(z, r)} \left(\int_{B(z, r)} \frac{\rho(d(x, y))}{\mu(B(x, \tau d(x, y)))} |f(y)| d\mu(y) \right) d\mu(x) \\ & \leq C\varphi(z, \kappa r) \left(\int_0^{2k_2 r} \frac{\rho(t)}{t} dt \right) \|f\|_{L^{(1, \varphi; \kappa)}(G)}. \end{aligned}$$

Proof. By Fubini’s theorem and the dyadic decomposition of the ball, we have

$$\begin{aligned} & \int_{B(z, r)} \left(\int_{B(z, r)} \frac{\rho(d(x, y))}{\mu(B(x, \tau d(x, y)))} |f(y)| d\mu(y) \right) d\mu(x) \\ & = \int_{B(z, r)} |f(y)| \left(\int_{B(z, r)} \frac{\rho(d(x, y))}{\mu(B(x, \tau d(x, y)))} d\mu(x) \right) d\mu(y) \\ & \leq \int_{B(z, r)} |f(y)| \left(\int_{B(y, 2r)} \frac{\rho(d(x, y))}{\mu(B(x, \tau d(x, y)))} d\mu(x) \right) d\mu(y) \\ & \leq \int_{B(z, r)} |f(y)| \left(\sum_{j=0}^{\infty} \int_{B(y, 2^{-j+1}r) \setminus B(y, 2^{-j}r)} \frac{\sup_{2^{-j}r \leq s \leq 2^{-j+1}r} \rho(s)}{\mu(B(x, 2^{-j}\tau r))} d\mu(x) \right) d\mu(y) \\ & \leq \int_{B(z, r)} |f(y)| \left(\sum_{j=0}^{\infty} \int_{B(y, 2^{-j+1}r)} \frac{\sup_{2^{-j}r \leq s \leq 2^{-j+1}r} \rho(s)}{\mu(B(y, 2^{-j-1}\tau r))} d\mu(x) \right) d\mu(y), \end{aligned}$$

where we used $\tau \geq 4$ for the penultimate line. Since ρ satisfies (8) and $\tau \geq 4$, we have

$$\begin{aligned} & \int_{B(z, r)} \left(\int_{B(z, r)} \frac{\rho(d(x, y))}{\mu(B(x, \tau d(x, y)))} |f(y)| d\mu(y) \right) d\mu(x) \\ & \leq C_\rho \int_{B(z, r)} |f(y)| \left(\sum_{j=0}^{\infty} \int_{2^{-j+1}k_1 r}^{2^{-j+1}k_2 r} \frac{\rho(s)}{s} dt \right) d\mu(y) \\ & \leq C_\rho \left(\int_0^{2k_2 r} \frac{\rho(t)}{t} dt \right) \int_{B(z, r)} |f(y)| d\mu(y) \\ & \leq C_\rho \varphi(z, \kappa r) \mu(B(z, \kappa r)) \left(\int_0^{2k_2 r} \frac{\rho(t)}{t} dt \right) \|f\|_{L^{(1, \varphi; \kappa)}(G)}, \end{aligned}$$

as required. \square

LEMMA 3. Let $\kappa \geq 1$, $0 < k_1 < k_2 < \infty$ and $\tau \geq 4 \max(\kappa, k_2)$. Let $\rho : (0, \infty) \rightarrow (0, \infty)$ be a measurable function satisfying (8). Let $\varphi \in \mathcal{G}$ and $f \in L^{(1, \varphi; \kappa)}(G)$. Then

$$\begin{aligned} & \frac{1}{\mu(B(z, 2\kappa r))} \int_{B(z, r)} \left(\int_{B(x, d_G) \setminus B(x, r)} \frac{\rho(d(x, y))}{\mu(B(x, \tau d(x, y)))} |f(y)| d\mu(y) \right) d\mu(x) \\ & \leq C\psi(z, 2\kappa r) \|f\|_{L^{(1, \varphi; \kappa)}(G)} \end{aligned}$$

for all $z \in X$ and $r > 0$, where $C > 0$ is a constant depending only on C_ρ , c_φ , k_1, k_2 and k .

Proof. Set $d \equiv [1 + \log_2(k_2/k_1)]$, the integer part of $1 + \log_2(k_2/k_1)$. We choose $j_0 \in \mathbb{Z}$ so that $d_G \leq 2^{j_0}r < 2d_G$. We claim

$$\frac{\rho(d(x,y))}{\mu(B(x,\tau d(x,y)))} \leq \frac{C_\rho}{\mu(B(x,2^{j-1}\tau r))} \int_{2^j k_1 r}^{2^{j+1} k_2 r} \frac{\rho(s)}{s} ds \tag{29}$$

and

$$\varphi(z, (2^j + 1) \max(\kappa, k_2)r) \int_{2^j k_1 r}^{2^{j+1} k_2 r} \frac{\rho(s)}{s} ds \leq C_1 \int_{2^j k_1 r}^{2^{j+d} k_1 r} \frac{\varphi(z,s)\rho(s)}{s} ds, \tag{30}$$

for $x, z \in X$, $j \in \mathbb{Z}$ and $r > 0$.

If $y \in B(x, 2^j r) \setminus B(x, 2^{j-1}r)$, then a geometric observation shows

$$\frac{\rho(d(x,y))}{\mu(B(x,\tau d(x,y)))} \leq \frac{1}{\mu(B(x,2^{j-1}\tau r))} \sup_{2^{j-1}r \leq s \leq 2^j r} \rho(s).$$

If we combine this with (8), then we have (29).

Note that using $\tau \geq 4 \max(\kappa, k_2)$, we learn

$$B(z, (2^j + 1) \max(\kappa, k_2)r) \subset B(x, 2^{j+1} \max(\kappa, k_2)r) \subset B(x, 2^{j-1} \tau r)$$

for $x \in B(z, r)$. Hence, we obtain for $x \in B(z, r)$

$$\begin{aligned} & \int_{B(x,2^j r) \setminus B(x,2^{j-1} r)} \frac{\rho(d(x,y))}{\mu(B(x,\tau d(x,y)))} |f(y)| d\mu(y) \\ & \leq C_\rho \int_{2^j k_1 r}^{2^{j+1} k_2 r} \frac{\rho(s)}{s} ds \times \frac{1}{\mu(B(x,2^{j-1}\tau r))} \int_{B(x,2^j r)} |f(y)| d\mu(y) \\ & \leq C_\rho \varphi(z, (2^j + 1) \max(\kappa, k_2)r) \int_{2^j k_1 r}^{2^{j+1} k_2 r} \frac{\rho(s)}{s} ds \\ & \quad \times \frac{1}{\varphi(z, (2^j + 1) \max(\kappa, k_2)r) \mu(B(z, (2^j + 1) \max(\kappa, k_2)r))} \\ & \quad \times \int_{B(z, (2^j+1) \max(\kappa, k_2)r)} |f(y)| d\mu(y) \\ & \leq C \varphi(z, (2^j + 1) \max(\kappa, k_2)r) \int_{2^j k_1 r}^{2^{j+1} k_2 r} \frac{\rho(s)}{s} ds \times \|f\|_{L^{(1,\varphi;\kappa)}(G)}. \end{aligned}$$

Since $\varphi(x, \cdot)$ is assumed almost decreasing uniformly over $x \in X$ and

$$(2^j + 1) \max(\kappa, k_2)r \geq 2^j k_2 r > s,$$

we have and hence $\varphi(z, (2^j + 1) \max(\kappa, k_2)r) \leq \frac{1}{c_\varphi} \varphi(z, s)$

$$\varphi(z, (2^j + 1) \max(\kappa, k_2)r) \int_{2^j k_1 r}^{2^{j+1} k_2 r} \frac{\rho(s)}{s} ds \leq \frac{1}{c_\varphi} \int_{2^j k_1 r}^{2^{j+1} k_2 r} \frac{\varphi(z,s)\rho(s)}{s} ds$$

$$\begin{aligned}
 &\leq \frac{1}{c_\varphi} \int_{2^j k_1 r}^{2^{j+d} k_1 r} \frac{\varphi(z, s) \rho(s)}{s} ds \\
 &= \sum_{l=1}^d \frac{1}{c_\varphi} \int_{2^{j+l-1} k_1 r}^{2^{j+l} k_1 r} \frac{\varphi(z, s) \rho(s)}{s} ds \\
 &= C_1 \int_{2^j k_1 r}^{2^{j+d} k_1 r} \frac{\varphi(z, s) \rho(s)}{s} ds,
 \end{aligned}$$

where $C_1 = \frac{1}{c_\varphi}$, which proves (30).

We decompose

$$\begin{aligned}
 &\int_{B(x, d_G) \setminus B(x, r)} \frac{\rho(d(x, y))}{\mu(B(x, \tau d(x, y)))} |f(y)| d\mu(y) \\
 &\leq \sum_{j=1}^{j_0} \int_{B(x, 2^j r) \setminus B(x, 2^{j-1} r)} \frac{\rho(d(x, y))}{\mu(B(x, \tau d(x, y)))} |f(y)| d\mu(y).
 \end{aligned}$$

If we use (29) and (30), we have

$$\begin{aligned}
 &\int_{B(x, d_G) \setminus B(x, r)} \frac{\rho(d(x, y))}{\mu(B(x, \tau d(x, y)))} |f(y)| d\mu(y) \\
 &\leq C_\rho C_1 \sum_{j=1}^{j_0} \int_{2^j k_1 r}^{2^{j+d} k_1 r} \frac{\varphi(z, s) \rho(s)}{s} ds \times \|f\|_{L^{(1, \varphi; \kappa)}(G)} \\
 &\leq d C_\rho C_1 \left(\int_{2k_1 r}^{4k_2 d_G} \frac{\rho(t) \varphi(z, t)}{t} dt \right) \|f\|_{L^{(1, \varphi; \kappa)}(G)}.
 \end{aligned}$$

Thus, Lemma 3 is proved. \square

5. Proofs of the theorems

We are now ready to prove our theorems.

5.1. Proof of Theorem 1

Let $z \in G$ and $r \in (0, d_G]$. By the positivity of the kernel, we may assume that $f \geq 0$. We decompose

$$\begin{aligned}
 &\frac{1}{\mu(B(z, 2\kappa r))} \int_{B(z, r)} I_{\rho, \mu, \tau} f(x) d\mu(x) \\
 &= \frac{1}{\mu(B(z, 2\kappa r))} \int_{B(z, r)} \left(\int_{B(x, r)} \frac{\rho(d(x, y))}{\mu(B(x, \tau d(x, y)))} f(y) d\mu(y) \right) d\mu(x) \\
 &\quad + \frac{1}{\mu(B(z, 2\kappa r))} \int_{B(z, r)} \left(\int_{B(x, d_G) \setminus B(x, r)} \frac{\rho(d(x, y))}{\mu(B(x, \tau d(x, y)))} f(y) d\mu(y) \right) d\mu(x).
 \end{aligned}$$

If $d(x, z) \leq r$, then $B(x, r) \subset B(z, 2r)$. Thus,

$$\begin{aligned} & \frac{1}{\mu(B(z, 2\kappa r))} \int_{B(z, r)} I_{\rho, \mu, \tau} f(x) d\mu(x) \\ \leq & \frac{1}{\mu(B(z, 2\kappa r))} \int_{B(z, 2r)} \left(\int_{B(z, 2r)} \frac{\rho(d(x, y))}{\mu(B(x, \tau d(x, y)))} f(y) d\mu(y) \right) d\mu(x) \\ & + \frac{1}{\mu(B(z, 2\kappa r))} \int_{B(z, r)} \left(\int_{B(x, d_G) \setminus B(x, r)} \frac{\rho(d(x, y))}{\mu(B(x, \tau d(x, y)))} f(y) d\mu(y) \right) d\mu(x) \\ = & I_1 + I_2 \end{aligned}$$

for $z \in G$ and $0 < r \leq d_G$. By Lemma 2 with r replaced by $2r$, we have

$$I_1 \leq C_1 \varphi(z, 2\kappa r) \left(\int_0^{4k_2 r} \frac{\rho(t)}{t} dt \right) \|f\|_{L^{(1, \varphi; \kappa)}(G)} \leq C_1 \psi(z, 2\kappa r) \|f\|_{L^{(1, \varphi; \kappa)}(G)}. \tag{31}$$

Meanwhile, by Lemma 3 we have

$$I_2 \leq C_2 \psi(z, 2\kappa r) \|f\|_{L^{(1, \varphi; \kappa)}(G)}. \tag{32}$$

Hence it follows from (31) and (32) that

$$\frac{1}{\mu(B(z, 2\kappa r))} \int_{B(z, r)} I_{\rho, \mu, \tau} f(x) d\mu(x) = I_1 + I_2 \leq C \psi(z, 2\kappa r) \|f\|_{L^{(1, \varphi; \kappa)}(G)},$$

where $C > 0$ depends only on C_ρ, c_φ, k_1 and k_2 . \square

5.2. Proof of Theorem 2

By Theorem 1, we have

$$\frac{1}{\mu(B(z, 2\kappa r))} \int_{B(z, r)} |I_{\tilde{\rho}, \mu, \tau} f(x)| d\mu(x) \leq C_1 \psi(z, 2\kappa r) \|f\|_{L^{(1, \varphi; \kappa)}(G)} \tag{33}$$

for $z \in G$ and $0 < r \leq d_G$.

Let $g \equiv |f|/|f|_{L^{(1, \varphi; \kappa)}(G)}$. For $x \in G$ and $0 < \delta \leq d_G$, since $\tilde{\rho}/\rho$ is decreasing, we have by (26) with $k = 0$

$$\begin{aligned} I_{\rho, \mu, \tau} g(x) &= \int_{B(x, \delta)} \frac{\rho(d(x, y))}{\mu(B(x, \tau d(x, y)))} g(y) d\mu(y) \\ &\quad + \int_{B(x, d_G) \setminus B(x, \delta)} \frac{\rho(d(x, y))}{\mu(B(x, \tau d(x, y)))} g(y) d\mu(y) \\ &\leq \frac{\rho(\delta)}{\tilde{\rho}(\delta)} \int_{B(x, \delta)} \frac{\tilde{\rho}(d(x, y))}{\mu(B(x, \tau d(x, y)))} g(y) d\mu(y) + C_2 \int_{2k_1 \delta}^{4k_2 d_G} \frac{\rho(t)\varphi(x, t)}{t} dt \\ &\leq \frac{\rho(\delta)}{\tilde{\rho}(\delta)} I_{\tilde{\rho}, \mu, \tau} g(x) + C_2 \psi_1(x, \delta) \end{aligned}$$

$$\leq \frac{\rho(4k_2\delta)}{\tilde{\rho}(4k_2\delta)} I_{\tilde{\rho},\mu,\tau}g(x) + C_2\psi_1(x, \delta).$$

Hence, in view of the definition of Θ , we have

$$I_{\rho,\mu,\tau}g(x) \leq \frac{\psi_1(x, \delta)}{\Theta(x, \delta)} I_{\tilde{\rho},\mu,\tau}g(x) + C_2\psi_1(x, \delta). \tag{34}$$

Now let

$$\delta \equiv \begin{cases} (\Theta(x, \cdot))^{-1}(I_{\tilde{\rho},\mu,\tau}g(x)) & \text{when } I_{\tilde{\rho},\mu,\tau}g(x) \geq \Theta(x, d_G), \\ d_G & \text{when } I_{\tilde{\rho},\mu,\tau}g(x) < \Theta(x, d_G). \end{cases}$$

Observe that

$$\psi_1(x, \delta) = \begin{cases} \psi_1(x, (\Theta(x, \cdot))^{-1}(I_{\tilde{\rho},\mu,\tau}g(x))) & \text{when } I_{\tilde{\rho},\mu,\tau}g(x) \geq \Theta(x, d_G), \\ \psi_1(x, d_G) & \text{when } I_{\tilde{\rho},\mu,\tau}g(x) < \Theta(x, d_G), \end{cases}$$

by definition.

We claim that

$$\frac{\psi_1(x, \delta)}{\Theta(x, \delta)} I_{\tilde{\rho},\mu,\tau}g(x) \leq \begin{cases} \psi_1(x, (\Theta(x, \cdot))^{-1}(I_{\tilde{\rho},\mu,\tau}g(x))) & \text{when } I_{\tilde{\rho},\mu,\tau}g(x) \geq \Theta(x, d_G), \\ \psi_1(x, d_G) & \text{when } I_{\tilde{\rho},\mu,\tau}g(x) < \Theta(x, d_G). \end{cases} \tag{35}$$

Indeed, when $I_{\tilde{\rho},\mu,\tau}g(x) < \Theta(x, d_G)$, we have $\delta = d_G$. Hence,

$$\frac{\psi_1(x, \delta)}{\Theta(x, \delta)} I_{\tilde{\rho},\mu,\tau}g(x) = \psi_1(x, d_G) \times \frac{1}{\Theta(x, d_G)} I_{\tilde{\rho},\mu,\tau}g(x) \leq \psi_1(x, d_G).$$

When $I_{\tilde{\rho},\mu,\tau}g(x) \geq \Theta(x, d_G)$, we have $\delta = (\Theta(x, \cdot))^{-1}(I_{\tilde{\rho},\mu,\tau}g(x))$. Hence,

$$\begin{aligned} \frac{\psi_1(x, \delta)}{\Theta(x, \delta)} I_{\tilde{\rho},\mu,\tau}g(x) &= \frac{\psi_1(x, (\Theta(x, \cdot))^{-1}(I_{\tilde{\rho},\mu,\tau}g(x)))}{I_{\tilde{\rho},\mu,\tau}g(x)} I_{\tilde{\rho},\mu,\tau}g(x) \\ &= \psi_1(x, (\Theta(x, \cdot))^{-1}(I_{\tilde{\rho},\mu,\tau}g(x))). \end{aligned}$$

Consequently our claim (35) is justified.

It follows from (34) and (35) that

$$I_{\rho,\mu,\tau}g(x) \leq (1 + C_2) \max \{ \psi_1(x, (\Theta(x, \cdot))^{-1}(I_{\tilde{\rho},\mu,\tau}g(x))), \psi_1(x, d_G) \}. \tag{36}$$

By (15), we obtain

$$\psi_1(x, (\Theta(x, \cdot))^{-1}(s)) \leq C_G \Phi^{-1}(s) \quad \text{for } x \in G, \Theta(x, d_G) \leq s < \infty. \tag{37}$$

Hence, taking $A \equiv C_G(C_1 + 1)(1 + C_2)$ and using (36) and (37), we obtain

$$\frac{|I_{\rho,\mu,\tau}f(x)|}{A \|f\|_{L^{(1,\varphi;\kappa)}(G)}} \leq \frac{I_{\rho,\mu,\tau}g(x)}{A}$$

$$\begin{aligned}
 &\leq \frac{\max \{ \psi_1(x, (\Theta(x, \cdot))^{-1}(I_{\tilde{\rho}, \mu, \tau} g(x))), \psi_1(x, d_G) \}}{C_G(C_1 + 1)} \\
 &= \frac{\max \{ \psi_1(x, (\Theta(x, \cdot))^{-1}(I_{\tilde{\rho}, \mu, \tau} g(x))), \psi_1(x, (\Theta(x, \cdot))^{-1}(\Theta(x, d_G))) \}}{C_G(C_1 + 1)} \\
 &\leq \frac{\max \{ \Phi^{-1}(I_{\tilde{\rho}, \mu, \tau} g(x)), \Phi^{-1}(\Theta(x, d_G)) \}}{C_1 + 1}.
 \end{aligned} \tag{38}$$

Since $\tilde{\rho}/\rho$ is decreasing and $\frac{\tilde{\rho}(4k_2d_G)}{\rho(4k_2d_G)}\psi_1(x, d_G) = \Theta(x, d_G)$, we see that

$$\begin{aligned}
 \psi(x, 2\kappa r) &\geq \int_{2k_1d_G}^{4k_2d_G} \frac{\tilde{\rho}(t)\varphi(x, t)}{t} dt \\
 &\geq \frac{\tilde{\rho}(4k_2d_G)}{\rho(4k_2d_G)} \int_{2k_1d_G}^{4k_2d_G} \frac{\rho(t)\varphi(x, t)}{t} dt \\
 &= \frac{\tilde{\rho}(4k_2d_G)}{\rho(4k_2d_G)}\psi_1(x, d_G) \\
 &= \Theta(x, d_G)
 \end{aligned} \tag{39}$$

for all $0 < r \leq d_G$. If we use (38) and the convexity of Φ , we have

$$\begin{aligned}
 &\frac{1}{\mu(B(z, 2\kappa r))} \int_{B(z, r)} \Phi \left(\frac{|I_{\rho, \mu, \tau} f(x)|}{A\|f\|_{L^{(1, \varphi; \kappa)}(G)}} \right) d\mu(x) \\
 &\leq \frac{1}{C_1 + 1} \times \frac{1}{\mu(B(z, 2\kappa r))} \int_{B(z, r)} \max \{ I_{\tilde{\rho}, \mu, \tau} g(x), \Theta(x, d_G) \} d\mu(x).
 \end{aligned}$$

Hence, with the aid of (33) and (39), we have

$$\begin{aligned}
 &\frac{1}{\mu(B(z, 2\kappa r))} \int_{B(z, r)} \Phi \left(\frac{|I_{\rho, \mu, \tau} f(x)|}{A\|f\|_{L^{(1, \varphi; \kappa)}(G)}} \right) d\mu(x) \\
 &\leq \frac{1}{(C_1 + 1)\mu(B(z, 2\kappa r))} \int_{B(z, r)} \max \{ I_{\tilde{\rho}, \mu, \tau} g(x), \Theta(x, d_G) \} d\mu(x) \\
 &\leq \frac{1}{(C_1 + 1)\mu(B(z, 2\kappa r))} \int_{B(z, r)} I_{\tilde{\rho}, \mu, \tau} g(x) d\mu(x) \\
 &\quad + \frac{1}{\mu(B(z, 2\kappa r))} \int_{B(z, r)} \Theta(x, d_G) d\mu(x) \\
 &\leq \frac{C_1}{C_1 + 1} \psi(z, 2\kappa r) + \frac{1}{(C_1 + 1)\mu(B(z, 2\kappa r))} \int_{B(z, r)} \psi(x, 2\kappa r) d\mu(x) \leq \Psi(z, 2\kappa r),
 \end{aligned}$$

which proves (16). \square

5.3. Proof of Theorem 3

Write

$$I_{\rho, \mu, \tau} f(x) - I_{\rho, \mu, \tau} f(z)$$

$$\begin{aligned}
 &= \int_{B(x,2d(x,z))} \frac{\rho(d(x,y))}{\mu(B(x,\tau d(x,y)))} f(y) d\mu(y) - \int_{B(x,2d(x,z))} \frac{\rho(d(z,y))}{\mu(B(z,\tau d(z,y)))} f(y) d\mu(y) \\
 &\quad + \int_{G \setminus B(x,2d(x,z))} \left(\frac{\rho(d(x,y))}{\mu(B(x,\tau d(x,y)))} - \frac{\rho(d(z,y))}{\mu(B(z,\tau d(z,y)))} \right) f(y) d\mu(y).
 \end{aligned}$$

By (19) and (25), we have

$$\int_{B(x,2d(x,z))} \frac{\rho(d(x,y))}{\mu(B(x,\tau d(x,y)))} |f(y)| d\mu(y) \leq C_1 \psi(x, 2d(x,z)) \|f\|_{L^{(1,\varphi;\kappa)}(G)} \tag{40}$$

and

$$\begin{aligned}
 &\int_{B(x,2d(x,z))} \frac{\rho(d(z,y))}{\mu(B(z,\tau d(z,y)))} |f(y)| d\mu(y) \\
 &\leq \int_{B(z,3d(x,z))} \frac{\rho(d(z,y))}{\mu(B(z,\tau d(z,y)))} |f(y)| d\mu(y) \\
 &\leq C'_1 \psi(z, 2d(x,z)) \|f\|_{L^{(1,\varphi;\kappa)}(G)}
 \end{aligned} \tag{41}$$

for $x, z \in G$. On the other hand, we have by (17) and (26)

$$\begin{aligned}
 &\int_{G \setminus B(x,2d(x,z))} \left| \frac{\rho(d(x,y))}{\mu(B(x,\tau d(x,y)))} - \frac{\rho(d(z,y))}{\mu(B(z,\tau d(z,y)))} \right| |f(y)| d\mu(y) \\
 &\leq C'_\rho d(x,z)^\theta \int_{G \setminus B(x,2d(x,z))} \frac{\rho(d(x,y))}{\mu(B(x,\tau d(x,y)))} d(x,y)^\theta |f(y)| d\mu(y) \\
 &\leq C_2 d(x,z)^\theta \left(\int_{4k_1 d(x,z)}^{4k_2 d_G} \frac{\rho(t)\varphi(x,t)}{t^{1+\theta}} dt \right) \|f\|_{L^{(1,\varphi;\kappa)}(G)} \\
 &\leq C_2 \psi(x, 2d(x,z)) \|f\|_{L^{(1,\varphi;\kappa)}(G)}.
 \end{aligned} \tag{42}$$

Now from (40), (41) and (42), we establish

$$|I_{\rho,\mu,\tau} f(x) - I_{\rho,\mu,\tau} f(z)| \leq C(\psi(x, 2d(x,z)) + \psi(z, 2d(x,z))) \|f\|_{L^{(1,\varphi;\kappa)}(G)}$$

for $x, z \in G$, as required. \square

5.4. Proof of Theorem 4

Let $z \in X$, $r > 0$, and $f \in L^{(1,\varphi;\kappa)}(X; \mu)$ be fixed. We have to show that

$$\frac{1}{\mu(B(z, 9\kappa r))} \int_{B(z,r)} |\tilde{I}_{\rho,\mu,\tau} f(x) - c_B| d\mu(x) \leq C\psi(z, 9\kappa r) \|f\|_{L^{(1,\varphi;\kappa)}(X;\mu)} \tag{43}$$

for some constant $c_B = c_{B(z,r)}$.

To this end, we let $f_1 \equiv \chi_{B(z,4r)} f$ and $f_2 \equiv f - f_1$. Define

$$c_{B,1} \equiv - \int_{B(z,4r)} \frac{\rho(d(x_0,y))}{\mu(B(x_0,\tau d(x_0,y)))} \chi_{X \setminus \mathbb{B}}(y) f(y) d\mu(y),$$

$$c_{B,2} \equiv \tilde{I}_{\rho,\mu,\tau} f_2(z) \text{ and } c_B \equiv c_{B,1} + c_{B,2}.$$

We claim that c_B does the job by proving that

$$\frac{1}{\mu(B(z, 9\kappa r))} \int_{B(z,r)} |\tilde{I}_{\rho,\mu,\tau} f_1(x) - c_{B,1}| d\mu(x) \leq C\psi(z, 9\kappa r) \|f\|_{L^{(1,\varphi;\kappa)}(X;\mu)}, \tag{44}$$

and that

$$\frac{1}{\mu(B(z, 9\kappa r))} \int_{B(z,r)} |\tilde{I}_{\rho,\mu,\tau} f_2(x) - c_{B,2}| d\mu(x) \leq C\psi(z, 9\kappa r) \|f\|_{L^{(1,\varphi;\kappa)}(X;\mu)}. \tag{45}$$

First we deal with f_1 . We have

$$\begin{aligned} & \frac{1}{\mu(B(z, 9\kappa r))} \int_{B(z,r)} |\tilde{I}_{\rho,\mu,\tau} f_1(x) - c_{B,1}| d\mu(x) \\ & \leq \frac{1}{\mu(B(z, 9\kappa r))} \int_{B(z,r)} \left(\int_{B(z,4r)} \frac{\rho(d(x,y))}{\mu(B(x, \tau d(x,y)))} |f(y)| d\mu(y) \right) d\mu(x) \\ & \leq \frac{1}{\mu(B(z, 9\kappa r))} \int_{B(z,r)} \left(\int_{B(x,5r)} \frac{\rho(d(x,y))}{\mu(B(x, \tau d(x,y)))} |f(y)| d\mu(y) \right) d\mu(x) \end{aligned}$$

from the triangle inequality for integrals and a geometric observation. Let us concentrate on the inner integral. We decompose the integration domain dyadically using (1) and then estimate the kernel crudely to have

$$\begin{aligned} & \int_{B(x,5r)} \frac{\rho(d(x,y))}{\mu(B(x, \tau d(x,y)))} |f(y)| d\mu(y) \tag{46} \\ & \leq \int_{B(x,8r)} \frac{\rho(d(x,y))}{\mu(B(x, \tau d(x,y)))} |f(y)| d\mu(y) \\ & = \sum_{k=1}^{\infty} \int_{B(x,2^{4-k}r) \setminus B(x,2^{3-k}r)} \frac{\rho(d(x,y))}{\mu(B(x, \tau d(x,y)))} |f(y)| d\mu(y) \\ & \leq \sum_{k=1}^{\infty} \left(\sup_{t \in [2^{3-k}r, 2^{4-k}r]} \rho(t) \right) \frac{1}{\mu(B(x, 2^{3-k}\tau r))} \int_{B(x,2^{4-k}r)} |f(y)| d\mu(y). \end{aligned}$$

If we insert the above estimate into the integral above and then use the Fubini theorem, we have

$$\begin{aligned} & \int_{B(z,r)} \left(\frac{1}{\mu(B(x, 2^{3-k}\tau r))} \int_{B(x,2^{4-k}r)} |f(y)| d\mu(y) \right) d\mu(x) \\ & = \int_{B(z,r)} \left(\frac{1}{\mu(B(x, 2^{3-k}\tau r))} \int_X \chi_{B(x,2^{4-k}r)}(y) |f(y)| d\mu(y) \right) d\mu(x) \\ & = \int_X \left(\int_{B(z,r)} \frac{1}{\mu(B(x, 2^{3-k}\tau r))} \chi_{B(x,2^{4-k}r)}(y) |f(y)| d\mu(x) \right) d\mu(y). \end{aligned}$$

Note from $\tau \geq 4$ that if $d(x,y) < 2^{4-k}r$, then $B(x, 2^{3-k}\tau r) \supset B(y, 2^{4-k}r)$. Thus

$$\int_{B(z,r)} \left(\frac{1}{\mu(B(x, 2^{3-k}\tau r))} \int_{B(x,2^{4-k}r)} |f(y)| d\mu(y) \right) d\mu(x)$$

$$\begin{aligned}
 &\leq \int_X \left(\frac{1}{\mu(B(y, 2^{4-k}r))} \int_{B(z,r)} \chi_{B(x, 2^{4-k}r)}(y) |f(y)| d\mu(x) \right) d\mu(y) \\
 &= \int_X \left(\frac{1}{\mu(B(y, 2^{4-k}r))} \int_{B(z,r)} \chi_{B(y, 2^{4-k}r)}(x) |f(y)| d\mu(x) \right) d\mu(y) \\
 &= \int_X \frac{\mu(B(z,r) \cap B(y, 2^{4-k}r))}{\mu(B(y, 2^{4-k}r))} |f(y)| d\mu(y).
 \end{aligned}$$

Notice that $y \in B(z, 9r)$ in order that $B(z,r) \cap B(y, 2^{4-k}r) \neq \emptyset$ for some $k = 1, 2, \dots$. Therefore,

$$\begin{aligned}
 &\int_{B(z,r)} \left(\frac{1}{\mu(B(x, 2^{3-k}r))} \int_{B(x, 2^{4-k}r)} |f(y)| d\mu(y) \right) d\mu(x) \\
 &\leq \int_{B(z, 9r)} |f(y)| d\mu(y).
 \end{aligned} \tag{47}$$

Thus, it follows from (8), (23), (46) and (47) that

$$\begin{aligned}
 &\frac{1}{\mu(B(z, 9\kappa r))} \int_{B(z,r)} |\tilde{I}_{\rho, \mu, 4} f_1(x) - c_{B,1}| d\mu(x) \\
 &\leq \sum_{k=1}^{\infty} \left(\sup_{t \in [2^{3-k}r, 2^{4-k}r]} \rho(t) \right) \frac{1}{\mu(B(z, 9\kappa r))} \int_{B(z, 9r)} |f(y)| d\mu(y) \\
 &\leq C \sum_{k=1}^{\infty} \int_{k_1 2^{4-k}r}^{k_2 2^{4-k}r} \frac{\rho(s)}{s} ds \frac{1}{\mu(B(z, 9\kappa r))} \int_{B(z, 9r)} |f(y)| d\mu(y) \\
 &\leq C \varphi(z, 9\kappa r) \int_0^{8k_2r} \frac{\rho(s)}{s} ds \cdot \|f\|_{L^{(1, \varphi, \kappa)}(X; \mu)} \\
 &\leq C \psi(z, 9\kappa r) \|f\|_{L^{(1, \varphi, \kappa)}(X; \mu)}.
 \end{aligned}$$

In summary, we obtain (44).

We deal with f_2 . We shall consider

$$\begin{aligned}
 &\tilde{I}_{\rho, \mu, \tau} f_2(x) - c_{B,2} \\
 &= \int_{X \setminus B(z, 4r)} \left(\frac{\rho(d(x, y))}{\mu(B(x, \tau d(x, y)))} - \frac{\rho(d(z, y))}{\mu(B(z, \tau d(z, y)))} \right) f(y) d\mu(y) \quad (x \in B(z, r)).
 \end{aligned}$$

By the triangle inequality and (22), we have

$$\begin{aligned}
 &|\tilde{I}_{\rho, \mu, \tau} f_2(x) - c_{B,2}| \\
 &\leq \int_{X \setminus B(z, 4r)} \left| \frac{\rho(d(x, y))}{\mu(B(x, \tau d(x, y)))} - \frac{\rho(d(z, y))}{\mu(B(z, \tau d(z, y)))} \right| |f(y)| d\mu(y) \\
 &\leq C \int_{X \setminus B(z, 4r)} \left(\frac{d(x, z)}{d(y, z)} \right)^\varepsilon \frac{\rho(d(z, y))}{\mu(B(z, \tau d(z, y)))} |f(y)| d\mu(y).
 \end{aligned} \tag{48}$$

By the dyadic decomposition, using (1) and (8), we obtain

$$\int_{X \setminus B(z, 4r)} \left(\frac{d(x, z)}{d(y, z)} \right)^\varepsilon \frac{\rho(d(z, y))}{\mu(B(z, \tau d(z, y)))} |f(y)| d\mu(y)$$

$$\begin{aligned}
 &= \sum_{k=1}^{\infty} \int_{B(z, 2^{k+2}r) \setminus B(z, 2^{k+1}r)} \left(\frac{d(x, z)}{d(y, z)} \right)^{\varepsilon} \frac{\rho(d(z, y))}{\mu(B(z, \tau d(z, y)))} |f(y)| d\mu(y) \\
 &\leq \sum_{k=1}^{\infty} \frac{1}{2^{k\varepsilon}} \frac{1}{\mu(B(z, 2^{k+1}\tau r))} \sup_{t \in [2^{k+1}r, 2^{k+2}r]} \rho(t) \int_{B(z, 2^{k+2}r)} |f(y)| d\mu(y) \\
 &\leq C \sum_{k=1}^{\infty} \int_{2^{k+2}k_1r}^{2^{k+2}k_2r} \frac{\varphi(z, 2^{k+2} \max(\kappa, k_2)r)}{2^{k\varepsilon}} \rho(s) \frac{ds}{s} \\
 &\quad \times \frac{1}{\varphi(z, 2^{k+2} \max(\kappa, k_2)r) \mu(B(z, 2^{k+1}\tau r))} \int_{B(z, 2^{k+2} \max(1, k_2/\kappa)r)} |f(y)| d\mu(y).
 \end{aligned}$$

Since $\tau \geq 2 \max(\kappa, k_2)$, we have

$$\begin{aligned}
 &\int_{X \setminus B(z, 4r)} \left(\frac{d(x, z)}{d(y, z)} \right)^{\varepsilon} \frac{\rho(d(z, y))}{\mu(B(z, \tau d(z, y)))} |f(y)| d\mu(y) \\
 &\leq C \|f\|_{L^{(1, \varphi, \kappa)}(X; \mu)} \sum_{k=1}^{\infty} \int_{2^{k+2}k_1r}^{2^{k+2}k_2r} \frac{\varphi(z, 2^{k+2} \max(\kappa, k_2)r)}{2^{k\varepsilon}} \rho(s) \frac{ds}{s}. \tag{49}
 \end{aligned}$$

Note that (23), together with the almost decreasingness of φ , yields

$$\begin{aligned}
 &\sum_{k=1}^{\infty} \int_{2^{k+2}k_1r}^{2^{k+2}k_2r} \frac{\varphi(z, 2^{k+2} \max(\kappa, k_2)r)}{2^{k\varepsilon}} \rho(s) \frac{ds}{s} \\
 &\leq Cr^{\varepsilon} \sum_{k=1}^{\infty} \int_{2^{k+2}k_1r}^{2^{k+2}k_2r} \frac{\varphi(z, s)}{s^{\varepsilon}} \rho(s) \frac{ds}{s} \leq C\psi(z, 9\kappa r).
 \end{aligned}$$

Thus if we insert the above estimate into (49), we have

$$\int_{X \setminus B(z, 4r)} \left(\frac{d(x, z)}{d(y, z)} \right)^{\varepsilon} \frac{\rho(d(z, y))}{\mu(B(z, \tau d(z, y)))} |f(y)| d\mu(y) \leq C\psi(z, 9\kappa r) \|f\|_{L^{(1, \varphi, \kappa)}(X; \mu)}. \tag{50}$$

From (48) and (50), it follows that

$$|\tilde{I}_{\rho, \mu, \tau} f_2(x) - c_{B, 2}| \leq C\psi(z, 9\kappa r) \|f\|_{L^{(1, \varphi, \kappa)}(X; \mu)} \quad (x \in B(z, r)). \tag{51}$$

If we integrate (51) over $B(z, r)$, then we obtain (45). Combining (44) and (45), we obtain (43). \square

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