Mathematical
Inequalities
& Applications
Volume 22, Number 2 (2019), 601–617 doi:10.7153/mia-2019-22-42

MONOTONICITY AND INEQUALITIES INVOLVING ZERO–BALANCED HYPERGEOMETRIC FUNCTION

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(*Communicated by J. Pečarić*)

Abstract. In the article, we present a monotonicity property involving the zero-balanced hypergeometric function $F(a,b;a+b;x)$ for all $a,b>0$, and establish several sharp inequalities for $F(a,b;a+b;x)$ in the first quadrant of ab-plane, which are the generalizations of the previously results.

1. Introduction

For real numbers a,b, and c with $c\neq 0,-1,-2,\ldots$, the Gaussian hypergeometric function $F(a,b;c;x)$ [49, 51, 52, 53, 64, 80, 88] is defined by

$$F(a,b;c;x) = {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}$$

for $x\in(-1,1)$, where $(a)_n$ is the Pochhammer symbol given by

$$(a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(n+a)}{\Gamma(a)}$$

for $n=1,2,\ldots$, and $(a)_0=1$ for $a\neq 0$, $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$ is the classical Euler Gamma function [3, 15, 36, 38, 40, 69, 72, 79, 81, 83, 84, 85, 87, 90, 91, 92]. The function $F(a,b;c;x)$ is said to be zero-balanced if $c=a+b$. The asymptotic properties for $F(a,b;c;x)$ as $x\to 1$ are as follows (see [9, Theorems 1.19 and 1.48])

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad a+b<c, \tag{1.1}$$

$$F(a,b;c;x) = (1-x)^{c-a-b}F(c-a,c-b;c;x), \qquad a+b>c, \tag{1.2}$$

Mathematics subject classification (2010): 33C05, 33E05.
Keywords and phrases: Gaussian hypergeometric function, generalized elliptic integral, monotonicity, inequality.
This work was supported by the Natural Science Foundation of China (Grant Nos. 11701176, 11626101, 11601485) and the Science and Technology Research Program of Zhejiang Educational Committee (Grant no. Y201635325)
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and when $a+b=c$,

$$B(a,b)F(a,b;c;x)+\log(1-x)=R(a,b)+O((1-x)\log(1-x)), \tag{1.3}$$

where $B(z,w)=\Gamma(z)\Gamma(w)/[\Gamma(z+w)]$ $(\Re(z)>0, \quad \Re(w)>0)$ is the classical Beta function, and

$$R(a,b)=-\psi(a)-\psi(b)-2\gamma,$$

$\psi(z)=\Gamma'(z)/\Gamma(z)$ $(\Re(z)>0)$ and γ is the Euler-Mascheroni constant. Equation (1.3) was established by Ramanujan [12, pp. 33–34].

It is well known that the Gaussian hypergeometric function $F(a,b;c;x)$ has many important applications in other branches of mathematics [14, 27, 28, 30, 31, 32, 33, 34, 42, 66], and a lot of special functions and elementary functions are the particular cases or limiting cases [2, 4, 29, 48, 63, 70]. For example, for $r\in[0,1]$ and $a\in(0,1/2]$, the complete elliptic integrals $\mathscr{K}(r)$ [1, 6, 7, 16, 17, 18, 20, 23, 25, 35, 41, 58, 59, 62, 68, 82] and $\mathscr{E}(r)$ [19, 21, 22, 24, 26, 50, 54, 57, 65, 75, 76, 77, 78] of the first and second kinds, and their generalizations $\mathscr{K}_a(r)$ and $\mathscr{E}_a(r)$ [8, 13, 55, 67, 89] can be expressed by $F(a,b;c;x)$ as follows:

$$\mathscr{K}(r)=\frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;r^2\right), \quad \mathscr{K}(0)=\frac{\pi}{2}, \quad \mathscr{K}(1)=+\infty,$$

$$\mathscr{K}_a(r)=\frac{\pi}{2}F(a,1-a;1;r^2), \quad \mathscr{K}_a(0)=\frac{\pi}{2}, \quad \mathscr{K}_a(1)=+\infty,$$

$$\mathscr{E}(r)=\frac{\pi}{2}F\left(-\frac{1}{2},\frac{1}{2};1;r^2\right), \quad \mathscr{E}(0)=\frac{\pi}{2}, \quad \mathscr{E}(1)=1,$$

$$\mathscr{E}_a(r)=\frac{\pi}{2}F(a-1,1-a;1;r^2), \quad \mathscr{E}_a(0)=\frac{\pi}{2}, \quad \mathscr{E}_a(1)=\frac{\sin(\pi a)}{2(1-a)}.$$

In the past few years, $F(a,b;c;x)$ has been extensively studied by many authors in geometric function theory and modular equations. Numerous remarkable properties and inequalities for this function have been obtained. In [11, 37, 39, 86] the authors studied Legendre's relation of hypergeometric function and related $\mathscr{M}$-function, the Landen inequalities for zero-balanced hypergeometric function can be found in the literature [43, 47, 74], and the quotient of two hypergeometric functions as the generalization of the modulus of the plane Grötzsch ring in conformal geometry was introduced and investigated in [44, 45, 61]. For the above, or more properties see the Anderson-Vamanamurthy-Vuorinen book "Conformal Invariants, Inequalities, and Quasiconformal Mappings" [9] or a survey "Topics in special functions" [10].

Since the hypergeometric series $F(a,b;c;x)$ converges for all $x\in(-1,1)$, the asymptotic properties and inequalities at $x=1$ for $F(a,b;c;x)$ have been the subject of intensive research. Especially when $c=a+b$, equation (1.3) shows that the zero-balanced hypergeometric function $F(a,b;a+b;x)$, as well as its special cases $\mathscr{K}$ and $\mathscr{K}_a$, has a logarithmic singularity at $x=1$, namely,

$$F(a,b;a+b;x)\sim-\frac{1}{B(a,b)}\log(1-x), \quad x\to 1. \tag{1.4}$$

Thus it is very interesting to establish some asymptotic formulas or sharp inequalities for $F(a,b;a+b;x)$ as $x\to 1$.

Qiu and Vuorinen [43] proved the monotonicity properties for the functions $x\to xF(a,b;a+b;x)/\log[1/(1-x)]$, $x\to B(a,b)F(a,b;a+b;x)+(1/x)\log(1-x)$ and $x\to [B(a,b)F(a,b;a+b;x)+\log(1-x)-R(a,b)]/\left([(1-x)/x]\log[1/(1-x)]\right)$. As applications, some sharp inequalities for $F(a,b;a+b;x)$ were derived. Recently, Wang, Chu and Song [60] refined Qiu and Vuorinen's results, in which the authors gave a complete answer to the monotonicity properties of the above functions for arbitrary $(a,b)\in\{(a,b)|a>0,b>0\}$.

For the complete elliptic integral $\mathscr{K}(r)$, Alzer [5] proved that the double inequality

$$1+\alpha(1-r^2)<\frac{\mathscr{K}(r)}{\log\left(4/\sqrt{1-r^2}\right)}<1+\beta(1-r^2) \tag{1.5}$$

holds for all $r\in(0,1)$ if and only if $\alpha\leqslant\pi/(4\log 2)-1$ and $\beta\geqslant 1/4$. In 2015, Wang, Chu and Qiu [56] generalized inequality (1.5) and obtained

$$1+\alpha(1-r^2)<\frac{\mathscr{K}_a(r)}{\sin(\pi a)\log\left[e^{R(a,1-a)/2}/\sqrt{1-r^2}\right]}<1+\beta(1-r^2) \tag{1.6}$$

for all $a\in(0,1/2]$ and $r\in(0,1)$ if and only if $\alpha\leqslant\pi/[R(a,1-a)\sin\pi a]-1$ and $\beta\geqslant a(1-a)$.

Very recently, making use of the following two-side inequality established in [56]

$$\frac{R(a,1-a)^2}{(1+a-a^2)R(a,1-a)-1}<\frac{\pi}{\sin(\pi a)}<(1+a-a^2)R(a,1-a),\ a\in(0,1/2], \tag{1.7}$$

the authors [73] proved that the function

$$\begin{aligned} Y(r)=&\frac{2\mathscr{K}_a(r)}{\sin(\pi a)(1-r^2)\log[e^{R(a,1-a)}/(1-r^2)]}-\frac{1}{1-r^2}\\ =&\frac{B(a,1-a)F(a,1-a;1;r^2)-\log[e^{R(a,1-a)}/(1-r^2)]}{(1-r^2)\log[e^{R(a,1-a)}/(1-r^2)]} \end{aligned} \tag{1.8}$$

is strictly increasing from $(0,1)$ onto $(\pi/[R(a,1-a)\sin\pi a]-1,a(1-a))$ for all $a\in(0,1/2]$, and consequently inequality (1.6) can be also derived. Actually, in order to search for beautiful inequalities for the zero-balanced hypergeometric function, or $\mathscr{K}(r)$ and $\mathscr{K}_a(r)$, Qiu and Vuorinen [43] raised the following open problem about a generalization of $Y(r)$.

PROBLEM 1.1. *Let $a,b\in(0,1)$ with $a+b<1$ and define F^* on $(0,1)$ by*

$$F^*(x)=\frac{B(a,b)F(a,b;a+b;x)-\log[e^{R(a,b)}/(1-x)]}{(1-x)\log[e^{R(a,b)}/(1-x)]}. \tag{1.9}$$

Is it true that the function F^* has a Maclaurin expansion $\sum_{n=0}^{\infty}d_nx^n$ with non-negative coefficients d_n.

Problem 1.1 is very difficult, and until now, it is still open. The main purpose of this paper is to prove the monotonicity property of $F^*(x)$ for arbitrary $(a,b) \in \{(a,b)|a>0,b>0\}$. This result lead to some sharp inequalities for $F(a,b;a+b;x)$, which extend inequality (1.6).

2. Main results

Throughout this paper, for $a,b>0$, we denote

$$M_1(a,b) = \frac{1-abB(a,b)/(a+b)}{1-ab(a+1)(b+1)B(a,b)/[(a+b)(a+b+1)]} \tag{2.1}$$

and

$$M_2(a,b) = R(a,b)-1-\frac{R(a,b)}{B(a,b)-R(a,b)}\left[1-B(a,b)\frac{ab}{a+b}\right]. \tag{2.2}$$

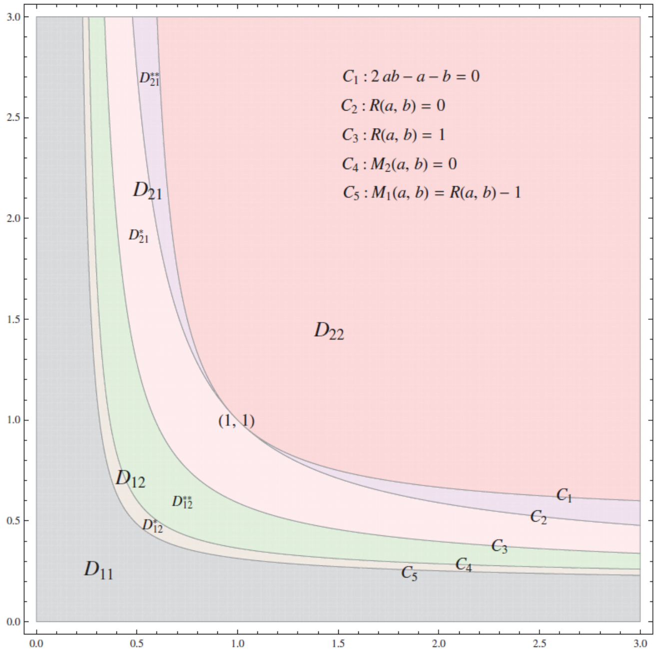

Figure 1: *The regions* D_{11}, D_{12}, D_{21}, D_{22}, D_{12}^*, D_{12}^{**}, D_{21}^* *and* D_{21}^{**}, *where* $C_1 : 2ab-a-b=0$, $C_2 : R(a,b)=0$, $C_3 : R(a,b)=1$, $C_4 : M_2(a,b)=0$, $C_5 : M_1(a,b)=R(a,b)-1$.

Let

$$D_1 = \{(a,b)|a,b>0, R(a,b) \geqslant 1\},$$

$$D_2 = \{(a,b)|a,b>0, R(a,b) < 1\},$$

$$D_{11} = \{(a,b)|a,b>0, R(a,b)\geqslant 1, R(a,b)-1\geqslant M_1(a,b)\},$$

$$D_{12} = \{(a,b)|a,b>0, R(a,b)\geqslant 1, R(a,b)-1< M_1(a,b)\},$$

$$D_{21} = \{(a,b)|a,b>0, R(a,b)<1, 2ab-a-b<0\},$$

$$D_{22} = \{(a,b)|a,b>0, R(a,b)<1, 2ab-a-b\geqslant 0\},$$

$$D_{12}^{*} = \{(a,b)|a,b>0, R(a,b)\geqslant 1, R(a,b)-1< M_1(a,b), M_2(a,b)\geqslant 0\},$$

$$D_{12}^{**} = \{(a,b)|a,b>0, R(a,b)\geqslant 1, R(a,b)-1< M_1(a,b), M_2(a,b)< 0\},$$

$$D_{21}^{*} = \{(a,b)|a,b>0, 0\leqslant R(a,b)<1, 2ab-a-b<0\}$$

and

$$D_{21}^{**} = \{(a,b)|a,b>0, R(a,b)<0, 2ab-a-b<0\}.$$

Then $D_{12}^{*}\cup D_{12}^{**}=D_{12}$, $D_{21}^{*}\cup D_{21}^{**}=D_{21}$, $D_{11}\cup D_{12}=D_1$, $D_{21}\cup D_{22}=D_2$ and $D_1\cup D_2=\{a,b|a,b>0\}$ (see Figure 1).

REMARK 2.1. According to Lemma 3.4 in Section 3, inequalities $1-abB(a,b)/(a+b)>0$ and $1-ab(a+1)(b+1)B(a,b)/(a+b)/(a+b+1)>0$ hold for all $a,b>0$. Hence $M_1(a,b)$ in (2.1) is positive for each $(a,b)\in\{(a,b)|a>0,b>0\}$. On the other hand, Theorem 1.52(2) in [9] shows that the function $x\to B(a,b)F(a,b;a+b;x)+\log(1-x)(a,b>0)$ is strictly decreasing from $(0,1)$ onto $(R(a,b),B(a,b))$. Thus $B(a,b)>R(a,b)$ for all $a,b>0$, and thereby $M_2(a,b)$ is well-defined.

THEOREM 2.2. *Let*

$$F(x)=\frac{(1-x)\log[e^{R(a,b)}/(1-x)]}{B(a,b)F(a,b;a+b;x)-\log[e^{R(a,b)}/(1-x)]},\quad x\in(0,1).$$

Then the following statements hold

(1) *If* $(a,b)\in D_{11}\cup D_{12}^{*}$, *then the function* $F(x)$ *is strictly decreasing from* $(0,1)$ *onto* $(1/(ab), R(a,b)/[B(a,b)-R(a,b)])$;

(2) *If* $(a,b)\in D_{22}$, *then the function* $F(x)$ *is strictly increasing from* $(0,1)$ *onto* $(R(a,b)/[B(a,b)-R(a,b)], 1/(ab))$;

(3) *If* $(a,b)\in D_{12}^{**}\cup D_{21}$, *then there exists* $x_0\in(0,1)$ *such that* $F(x)$ *is strictly increasing on* $(0,x_0)$, *and strictly decreasing on* $(x_0,1)$.

Using Theorem 2.2, we can derive the monotonicity property of F^{*} immediately.

COROLLARY 2.3. *Let*

$$F^{*}(x)=\frac{1}{F(x)}=\frac{B(a,b)F(a,b;a+b;x)-\log[e^{R(a,b)}/(1-x)]}{(1-x)\log[e^{R(a,b)}/(1-x)]}.$$

Then the following statements hold

(1) $(a,b)\in\{(a,b)|R(a,b)\geqslant 0\}$. *Then* F^{*} *is strictly increasing from* $(0,1)$ *onto* $(B(a,b)/R(a,b)-1, ab)$ *if* $(a,b)\in D_{11}\cup D_{12}^{*}$, *and if* $(a,b)\in D_{12}^{**}\cup D_{21}^{*}$, *then there*

exists $x_0 \in (0,1)$ *such that* F^* *is strictly decreasing on* $(0,x_0)$*, and strictly increasing on* $(x_0,1)$*. Moreover, for* $(a,b) \in D_{11} \cup D_{12}^*$*, the double inequality*

$$1+\frac{B(a,b)-R(a,b)}{R(a,b)}(1-x)<\frac{B(a,b)F(a,b,;a+b;x)}{\log\left[e^{R(a,b)}/(1-x)\right]}<1+ab(1-x) \qquad (2.3)$$

holds for all $x \in (0,1)$ *with the best possible constants* $[B(a,b)-R(a,b)]/R(a,b)$ *and* ab*, and for* $(a,b) \in D_{12}^{**} \cup D_{21}^*$*, inequality*

$$\frac{B(a,b)F(a,b,;a+b;x)}{\log\left[e^{R(a,b)}/(1-x)\right]}<1+\max\left\{ab,\frac{B(a,b)-R(a,b)}{R(a,b)}\right\}(1-x) \qquad (2.4)$$

holds for all $x \in (0,1)$*.*

(2) $(a,b) \in \{(a,b)|R(a,b)<0\}$*. Denote* $x_0^* = x_0^*(a,b)$ *by the solution of equation* $R(a,b) = \log(1-x)$*, one has*

(i) If $(a,b) \in D_{21}^{**}$*, then* F^* *is strictly decreasing from* $(0,x_0^*)$ *onto* $(-\infty,[B(a,b)-R(a,b)]/R(a,b))$*, and there exist* $x_0^{**} \in (x_0^*,1)$ *such that* F^* *is strictly decreasing on* (x_0^*,x_0^{**}) *and strictly increasing on* $(x_0^{**},1)$*;*

(ii) If $(a,b) \in D_{22}$*, then* F^* *is strictly decreasing from* $(0,x_0^*)$ *onto* $(-\infty,[B(a,b)-R(a,b)]/R(a,b))$*, and strictly decreasing from* $(x_0^*,1)$ *onto* $(ab,+\infty)$*.*

REMARK 2.4. If $b = 1-a > 0$, then $B(a,1-a) = \pi/\sin(\pi a)$, and

$$M_2(a,b) = \frac{B(a,1-a)[(1+a-a^2)R(a,1-a)-1]-R(a,1-a)^2}{B(a,1-a)-R(a,1-a)}.$$

It follows (1.7) that $\{(a,b)|a,b>0,a+b=1\} \subset D_{11} \cup D_{12}^*$, so that $Y(r)$ in (1.8), which is equal to $F^*(r^2)$, is strictly increasing on $(0,1)$ by Corollary 2.3. Besides, Substituting r^2 for x in inequality (2.3), we obtain inequality (1.6).

3. Lemmas

In order to prove Theorem 2.2 we need several lemmas, which we present in this section.

LEMMA 3.1. ([9, Theorem 1.25]) *For* $-\infty < a < b < \infty$*, let* $f,g : [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$*, and be differentiable on* (a,b)*, let* $g'(x) \neq 0$ *on* (a,b)*. If* $f'(x)/g'(x)$ *is increasing (decreasing) on* (a,b)*, then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \text{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ *is strictly monotone, then the monotonicity in the conclusion is also strict.*

LEMMA 3.2. ([74, Theorem 2.1]) *Suppose that the power series* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $g(x) = \sum_{n=0}^{\infty} b_n x^n$ *have the radius of convergence* $r > 0$ *with* $b_n > 0$ *for all* $n \in$

$\{0,1,2,\cdots\}$. *Let* $h(x)=f(x)/g(x)$ *and* $H_{f,g}=(f'/g')g-f$, *then the following statements] are true:*

(1) *If the non-constant sequence* $\{a_n/b_n\}_{n=0}^{\infty}$ *is increasing (decreasing), then* $h(x)$ *is strictly increasing (decreasing) on* $(0,r)$;

(2) *If the non-constant sequence* $\{a_n/b_n\}$ *is increasing (decreasing) for* $0<n\leqslant n_0$ *and decreasing (increasing) for* $n>n_0$, *then the function* h *is strictly increasing (decreasing) on* $(0,r)$ *if and only if* $H_{f,g}(r^-)\geqslant(\leqslant)0$. *While if* $H_{f,g}(r^-)<(>)0$, *then there exists* $x_0\in(0,r)$ *such that* $h(x)$ *is strictly increasing (decreasing) on* $(0,x_0)$ *and strictly decreasing (increasing) on* (x_0,r).

LEMMA 3.3. ([71, Theorem 9]) *For* $-\infty\leqslant a<b\leqslant\infty$, *let* f *and* g *be differentiable functions on* (a,b) *with* $g'\neq 0$ *on* (a,b), $\operatorname{sgn}(\cdot)$ *be signum function, and* $H_{f,g}=(f'/g')g-f$. *Suppose that (i)* $g'\neq 0$ *on* (a,b); *(ii)* $f(b^-)=g(b^-)=0$; *(iii) there exits* $c\in(a,b)$ *such that* f'/g' *is increasing (decreasing) on* (a,c) *and decreasing (increasing) on* (c,b). *Then*

(1) *when* $\operatorname{sgn}(g')\operatorname{sgn}H_{f,g}(a^+)\leqslant(\geqslant)0$, f/g *is decreasing (increasing) on* (a,b);

(2) *when* $\operatorname{sgn}(g')\operatorname{sgn}H_{f,g}(a^+)>(<)0$, *there is a unique number* $x_b\in(a,b)$ *such that* f/g *is increasing (decreasing) on* (a,x_b) *and decreasing (increasing) on* (x_b,b).

LEMMA 3.4. ([9, Lemma 1.50 (1)]) *For* $a,b>0$, *the sequence*

$$f(n)=\frac{(a)_n(b)_n}{(a+b)_n(n-1)!}$$

is strictly increasing to the limit $1/B(a,b)$.

LEMMA 3.5. *For* $a,b>0$ *with* $1/a+1/b\leqslant 2$, *then*

$$R(a,b)\leqslant 0,$$

with equality if and only if $a=b=1$.

Proof. Since $R(a,b)=-\Psi(a)-\Psi(b)-2\gamma$ is strictly decreasing in a and b, we have

$$\begin{aligned}R(a,b)&\leqslant R\left(a,\frac{a}{2a-1}\right)=-\Psi(a)-\Psi\left(\frac{a}{2a-1}\right)-2\gamma\\&=\frac{1}{a}+\frac{1}{\frac{a}{2a-1}}-\sum_{k=1}^{\infty}\left(\frac{2}{k}-\frac{1}{k+a}-\frac{1}{k+\frac{a}{2a-1}}\right)\\&=2-\sum_{k=1}^{\infty}\left[\frac{\frac{2a^2}{2a-1}(k+1)}{k\left(k^2+\frac{2a^2}{2a-1}k+\frac{a^2}{2a-1}\right)}\right]\end{aligned}\tag{3.1}$$

by employing $\Psi(x) = -\gamma - 1/x - \sum_{k=1}^{\infty}[1/k - 1/(k+x)]$. It is easy to check that $2a^2/(2a-1) \geqslant 2$ for $a > 1/2$, and $x \to x/(k^2+xk+x/2)(k \in \mathbf{N}^+)$ is strictly increasing on $[2,\infty)$. Thus from (3.1) one has

$$\begin{aligned} R(a,b) &\leqslant 2 - \sum_{k=1}^{\infty}\left[\frac{\frac{2a^2}{2a-1}(k+1)}{k\left(k^2+\frac{2a^2}{2a-1}k+\frac{a^2}{2a-1}\right)}\right] \\ &\leqslant 2 - \sum_{k=1}^{\infty}\frac{2(k+1)}{k(k^2+2k+1)} = 2 - \sum_{k=1}^{\infty}\frac{2}{k(k+1)} = 0. \end{aligned} \tag{3.2}$$

Both inequalities (3.1) and (3.2) become equalities if and only if $a = b = 1$. This completes the proof of Lemma 3.5. $\square$

LEMMA 3.6. *Let $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ be defined by*

$$A_1 = R(a,b) - 1,\ A_n = \frac{1}{n-1}(n \geqslant 2),\ B_n = 1 - \frac{(a)_n(b)_n}{(a+b)_n(n-1)!}B(a,b) \tag{3.3}$$

with $a,b > 0$. Then the sequence $\{A_n/B_n\}$ is strictly decreasing for $n \geqslant 2$.

Proof. By Lemma 3.4, $B_n > 0$ for all $a,b > 0$ and $n \geqslant 1$. Let

$$C_n = \frac{B_n}{A_n} = n - 1 - B(a,b)\frac{(a)_n(b)_n}{(a+b)_n(n-2)!}, \quad n \geqslant 2. \tag{3.4}$$

Then

$$\begin{aligned} C_{n+1} - C_n =& n - B(a,b)\frac{(a)_{n+1}(b)_{n+1}}{(a+b)_{n+1}(n-1)!} - (n-1) + B(a,b)\frac{(a)_n(b)_n}{(a+b)_n(n-2)!} \\ =& 1 - B(a,b)\frac{(a)_n(b)_n}{(a+b)_{n+1}(n-1)!}[(n+a)(n+b) - (n+a+b)(n-1)] \\ =& 1 - B(a,b)\frac{(a)_n(b)_n}{(a+b)_n(n-1)!}\frac{n+ab+a+b}{n+a+b}. \end{aligned} \tag{3.5}$$

Let

$$D_n = \frac{(a)_n(b)_n}{(a+b)_n(n-1)!}\frac{n+ab+a+b}{n+a+b}. \tag{3.6}$$

Then

$$\frac{D_{n+1}}{D_n} - 1 = \frac{ab(1+a+b+ab)}{n(n+1+a+b)(n+ab+a+b)} > 0. \tag{3.7}$$

It follows from Lemma 3.4 that

$$\lim_{n\to\infty} D_n = \frac{1}{B(a,b)}. \tag{3.8}$$

Hence, by (3.6)–(3.8), $D_n < 1/B(a,b)$ for $a,b > 0$, so that $\{C_n\}$ is strictly increasing for $n \geqslant 2$ from (3.5).

Therefore, Lemma 3.6 follows from (3.4) and the monotonicity of the sequence $\{C_n\}$. $\square$

LEMMA 3.7. *For $a,b>0$, let*

$$f_1(x)=R(a,b)-1+\log\left(\frac{1}{1-x}\right) \tag{3.9}$$

and

$$\begin{aligned} g_1(x)=&-B(a,b)\frac{ab}{a+b}F(a+1,b+1;a+b+1;x)+\frac{1}{1-x}\\ =&\frac{1}{1-x}\left[1-B(a,b)\frac{ab}{a+b}F(a,b;a+b+1;x)\right]. \end{aligned} \tag{3.10}$$

Then

$$\lim_{x\to 1^-}H_{f_1,g_1}(x)=\lim_{x\to 1^-}\left[\frac{f_1'(x)}{g_1'(x)}g_1(x)-f_1(x)\right]=\frac{2ab-a-b}{ab}. \tag{3.11}$$

Proof. Differentiating f_1 and g_1 gives

$$f_1'(x)=\frac{1}{1-x}, \tag{3.12}$$

and

$$\begin{aligned} g_1'(x)=&\frac{1}{(1-x)^2}\left[1-\frac{abB(a,b)}{a+b}F(a,b;a+b+1;x)\right]\\ &-\frac{1}{1-x}\frac{a^2b^2B(a,b)}{(a+b)(a+b+1)}F(a+1,b+1;a+b+2;x), \end{aligned} \tag{3.13}$$

where we use the derivative formula of hypergeometric function

$$\frac{dF(a,b;c;x)}{dx}=\frac{ab}{c}F(a+1,b+1;c+1;x).$$

It follows from (3.9), (3.10), (3.12) and (3.13) that

$$\begin{aligned} H_{f_1,g_1}(x)=&\frac{f_1'(x)}{g_1'(x)}g_1(x)-f_1(x)\\ =&\frac{1-\frac{abB(a,b)}{a+b}F(a,b;a+b+1;x)}{h(x)}-\left[R(a,b)-1+\log\left(\frac{1}{1-x}\right)\right]\\ =&\frac{(1-x)^{-1}\left[1-\frac{abB(a,b)}{a+b}F(a,b;a+b+1;x)-h(x)\left(R(a,b)-1+\log\left(\frac{1}{1-x}\right)\right)\right]}{(1-x)^{-1}h(x)} \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} h(x)=&1-\frac{abB(a,b)}{a+b}F(a,b;a+b+1;x)\\ &-(1-x)\frac{a^2b^2B(a,b)}{(a+b)(a+b+1)}F(a+1,b+1;a+b+2;x). \end{aligned} \tag{3.15}$$

Since Gaussian hypergeometric function $F(a,b;c;x)$ has the following asymptotic expansions (see [46, (2.10)])

$$F(a+1,b+1;a+b+2;x)=\frac{1}{B(a+1,b+1)}\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(n!)^2}u_n(1-x)^n,$$

$$\begin{aligned}F(a,b;a+b+1;z)=&\frac{\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)}\\&+\frac{\Gamma(a+b+1)}{\Gamma(a)\Gamma(b)}(x-1)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(n!)(n+1)!}v_n(1-x)^n,\end{aligned}$$

where

$$u_n=2\Psi(n+1)-\Psi(n+a+1)-\Psi(n+b+1)-\log(1-x) \tag{3.16}$$

and

$$v_n=\Psi(n+1)+\Psi(n+2)-\Psi(n+a+1)-\Psi(n+b+1)-\log(1-x), \tag{3.17}$$

one has

$$\begin{aligned}&1-\frac{abB(a,b)}{a+b}F(a,b;a+b+1;x)\\=&1-\frac{abB(a,b)}{a+b}\frac{\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)}-\frac{abB(a,b)}{a+b}\frac{\Gamma(a+b+1)}{\Gamma(a)\Gamma(b)}(x-1)\\&\times\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(n!)(n+1)!}v_n(1-x)^n\\=&ab(1-x)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(n!)(n+1)!}v_n(1-x)^n,\end{aligned}$$

$$\begin{aligned}&(1-x)^{-1}\left[1-\frac{abB(a,b)}{a+b}F(a,b;a+b+1;x)-h(x)\left(R(a,b)-1+\log\left(\frac{1}{1-x}\right)\right)\right]\\=&(1-x)^{-1}\Bigg[ab(1-x)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(n!)(n+1)!}v_n(1-x)^n-(R(a,b)-1-\log(1-x))\\&\times\Bigg(ab(1-x)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(n!)(n+1)!}v_n(1-x)^n\\&-(1-x)\frac{a^2b^2B(a,b)}{(a+b)(a+b+1)}\frac{1}{B(a+1,b+1)}\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(n!)^2}u_n(1-x)^n\Bigg)\Bigg]\\=&ab\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(n!)(n+1)!}v_n(1-x)^n-(R(a,b)-1-\log(1-x))\\&\times\left(ab\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(n!)(n+1)!}v_n(1-x)^n-ab\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(n!)^2}u_n(1-x)^n\right)\end{aligned}$$

$$=ab\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(n!)(n+1)!}(1-x)^n\left\{[v_n-(n+1)u_n]\left[1-R(a,b)+\log(1-x)\right]+v_n\right\}, \tag{3.18}$$

$$\begin{aligned}(1-x)^{-1}h(x)=&\frac{1}{1-x}\left(ab(1-x)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(n!)(n+1)!}v_n(1-x)^n\right)\\&-\frac{a^2b^2B(a,b)}{(a+b)(a+b+1)}F(a+1,b+1;a+b+2;x)\\=&ab\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(n!)(n+1)!}v_n(1-x)^n-ab\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(n!)^2}u_n(1-x)^n\\=&ab\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(n!)(n+1)!}(1-x)^n\left[v_n-(n+1)u_n\right].\end{aligned} \tag{3.19}$$

It follows from (3.14)–(3.19) that

$$\begin{aligned}&\lim_{x\to1^-}H_{f_1,g_1}(x)\\=&\lim_{x\to1^-}\frac{(v_0-u_0)[1-R(a,b)+\log(1-x)]+v_0+o((1-x)\log(1-x))}{v_0-u_0+o((1-x)\log(1-x))}\\=&\lim_{x\to1^-}\frac{1-R(a,b)+\log(1-x)+\Psi(1)+\Psi(2)-\Psi(a+1)-\Psi(b+1)-\log(1-x)}{1+o((1-x)\log(1-x))}\\=&1-R(a,b)+2\Psi(1)+1-\Psi(a)-\Psi(b)-\frac{1}{a}-\frac{1}{b}=\frac{2ab-a-b}{ab}.\quad\square\end{aligned}$$

4. Proof of Theorem 2.2

Obviously

$$\lim_{x\to0^+}F(x)=\frac{R(a,b)}{B(a,b)-R(a,b)}, \tag{4.1}$$

and making use of L'Hôpital's rule we get

$$\begin{aligned}\lim_{x\to1^-}F(x)=&\lim_{x\to1^-}\frac{1-R(a,b)+\log(1-x)}{B(a,b)\frac{ab}{a+b}F(a+1,b+1;a+b+1;x)-\frac{1}{1-x}}\\=&\lim_{x\to1^-}\frac{(1-x)[1-R(a,b)]+(1-x)\log(1-x)}{B(a,b)\frac{ab}{a+b}F(a,b;a+b+1;x)-1}\\=&\lim_{x\to1^-}\frac{R(a,b)-2-\log(1-x)}{B(a,b)\frac{a^2b^2}{(a+b)(a+b+1)}F(a+1,b+1;a+b+2;x)}\\=&\lim_{x\to1^-}\frac{1}{(1-x)\frac{a^2b^2B(a,b)}{(a+b)(a+b+1)}\frac{(a+1)(b+1)}{(a+b+2)}F(a+2,b+2;a+b+3;x)}\end{aligned}$$

$$= \lim_{x\to 1^-} \frac{1}{\frac{a^2b^2B(a,b)}{(a+b)(a+b+1)}\frac{(a+1)(b+1)}{(a+b+2)}F(a+1,b+1;a+b+3;x)}$$
$$= \frac{1}{B(a,b)\frac{a^2b^2(a+1)(b+1)}{(a+b)(a+b+1)(a+b+2)}\frac{\Gamma(a+b+3)\Gamma(1)}{\Gamma(a+2)\Gamma(b+2)}} = \frac{1}{ab}. \tag{4.2}$$

If we let

$$f(x) = (1-x)[R(a,b) - \log(1-x)] \tag{4.3}$$

and

$$g(x) = B(a,b)F(a,b;a+b;x) - \log\left(\frac{e^{R(a,b)}}{1-x}\right), \tag{4.4}$$

then $F(x) = f(x)/g(x)$, $f(1^-) = g(1^-) = 0$,

$$f'(x) = -R(a,b) + \log(1-x) + 1 = -(R(a,b)-1) - \sum_{n=2}^{\infty}\frac{x^{n-1}}{n-1} = -\sum_{n=1}^{\infty} A_n x^{n-1},$$

$$g'(x) = B(a,b)\frac{ab}{a+b}F(a+1,b+1;a+b+1;x) - \frac{1}{1-x}$$
$$= B(a,b)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(a+b)_n(n-1)!}x^{n-1} - \frac{1}{1-x} = -\sum_{n=1}^{\infty} B_n x^{n-1}, \tag{4.5}$$

$$\frac{f'(x)}{g'(x)} = \frac{f_1(x)}{g'(x)} = \frac{\sum_{n=1}^{\infty} A_n x^{n-1}}{\sum_{n=1}^{\infty} B_n x^{n-1}} \tag{4.6}$$

and thereby

$$\lim_{x\to 0^+} H_{f,g}(x) = \lim_{x\to 0^+}\left[\frac{f'(x)}{g'(x)}g(x) - f(x)\right]$$
$$= \frac{R(a,b)-1}{1-\frac{ab}{a+b}B(a,b)}[B(a,b) - R(a,b)] - R(a,b)$$
$$= \frac{B(a,b)-R(a,b)}{1-\frac{ab}{a+b}B(a,b)}M_2(a,b). \tag{4.7}$$

Here

$$A_1 = R(a,b) - 1, \quad A_n = \frac{1}{n-1}(n \geqslant 2), \quad B_n = 1 - \frac{(a)_n(b)_n}{(a+b)_n(n-1)!}B(a,b), \tag{4.8}$$

and $f_1(x)$ and $g_1(x)$ are defined by (3.9) and (3.10) in Lemma 3.7.

Next, we divide the proof into five cases.

Case 1: $(a,b) \in D_{11}$. Then $A_1/B_1 \geqslant A_2/B_2$, and from Lemma 3.6 we know that the non-constant sequence $\{A_n/B_n\}$ is strictly decreasing. Equation (4.6) and Lemma 3.2(1) imply that the function $f_1(x)/g_1(x)$ is strictly decreasing on $(0,1)$, and so is $F(x)$ by applying Lemma 3.1.

Case 2: $(a,b) \in D_{12}^{*}$. Then $A_1/B_1 < A_2/B_2$, and the non-constant sequence $\{A_n/B_n\}$ is strictly increasing for $1 \leqslant n \leqslant 2$, and strictly decreasing for $n \geqslant 2$. By Lemma 3.5, we conclude that $H_{f_1,g_1}(1^-) = (2ab-a-b)/(ab) < 0$. So that equation (4.6) and Lemma 3.2(2) lead to the conclusion that there exists $\xi \in (0,1)$ such that $f'(x)/g'(x)$ is strictly increasing on $(0,\xi)$, and strictly decreasing on $(\xi,1)$.

Equations (2.1), (4.5) and (4.7) show that $\mathrm{sgn}(g') < 0$ and $H_{f,g}(0^+) \geqslant 0$. Thus by application of Lemma 3.3(1) one has that $F(x)$ is strictly decreasing on $(0,1)$.

Case 3: $(a,b) \in D_{12}^{**}$. Then $A_1/B_1 < A_2/B_2$, with the similar argument in Case 2, we conclude that there exists $\eta \in (0,1)$ such that $f'(x)/g'(x)$ is strictly increasing on $(0,\eta)$, and strictly decreasing on $(\eta,1)$. Since $\mathrm{sgn}(g') < 0$ and $H_{f,g}(0^+) > 0$, by Lemma 3.3(2) one has that there exists $x_0 \in (0,1)$ such that $F(x)$ is strictly increasing on $(0,x_0)$, and strictly decreasing on $(x_0,1)$.

Case 4: $(a,b) \in D_{22}$. Then $A_1/B_1 < 0 < A_2/B_2$, and the non-constant sequence $\{A_n/B_n\}$ is strictly increasing for $1 \leqslant n \leqslant 2$, and strictly decreasing for $n \geqslant 2$. Since $H_{f_1,g_1}(1^-) = (2ab-a-b)/(ab) \geqslant 0$, Lemma 3.2(2) and (4.6) lead to the conclusion that $f'(x)/g'(x)$ is strictly increasing on $(0,1)$, and so is $F(x)$ by applying Lemma 3.1.

Case 5: $(a,b) \in D_{21}$. Then $A_1/B_1 < 0 < A_2/B_2$, and the non-constant sequence $\{A_n/B_n\}$ is strictly increasing for $1 \leqslant n \leqslant 2$, and strictly decreasing for $n \geqslant 2$. Lemma 3.2(2) and $H_{f_1,g_1}(1^-) = (2ab-a-b)/(ab) < 0$ show that there exist $\delta \in (0,1)$ such that $f'(x)/g'(x)$ is strictly increasing on $(0,\delta)$, and strictly decreasing on $(\delta,1)$.

Next we claim that $H_{f,g}(0^+) < 0$ for $(a,b) \in D_{21}$. In fact, by Remark 2.1, $B(a,b) > R(a,b)$ and $1 > abB(a,b)/(a+b)$ for all $a,b > 0$, and it is clear to see that $M_2(a,b) < 0$ for $(a,b) \in D_{21}^{*}$, and for $(a,b) \in D_{21}^{**}$,

$$M_2(a,b) = \frac{B(a,b)R(a,b) - B(a,b) - R(a,b)^2 + abB(a,b)R(a,b)/(a+b)}{B(a,b) - R(a,b)} < 0.$$

Finally, inequalities $H_{f,g}(0^+) < 0$ for $(a,b) \in D_{21}$ and $\mathrm{sgn}(g') < 0$ for $x \in (0,1)$ together with Lemma 3.3(2) and the piecewise monotonicity of $f'(x)/g'(x)$ yield that there exists x_1 such that $F(x)$ is strictly increasing on $(0,x_1)$ and strictly decreasing on $(x_1,1)$.

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(Received March 4, 2018)

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