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HIGHER–DIMENSIONAL WEIGHTED KNOPP TYPE INEQUALITIES

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Abstract. We give a necessary and sufficient condition on weight pairs for a class of higher-dimensional weighted Knopp type inequalities to hold. The corresponding result for the multivariate adjoint Hardy type operator is obtained.

1. Introduction

The classical one-dimensional Knopp inequality [3, Theorem 335, p.250]

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \ln f(t)dt\right)dx \leqslant e\int_0^\infty f(x)dx, \text{ for } f>0 \tag{1}$$

can be regarded as a continuous analogue of Carleman's inequality, where the discrete Carleman inequality [3, Theorem 334, p.249] is of the form

$$\sum_{k=1}^\infty (a_1a_2\cdots a_k)^{1/k} \leqslant e\sum_{k=1}^\infty a_k, \text{ for all } a_k>0, \tag{2}$$

and the constant e in (1) and (2) is the best possible.

The Knopp inequality is closely connected with the classical Hardy inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx \tag{3}$$

provided that $f>0$ and $1<p<\infty$. We usually denote by H the one-dimensional Hardy operator of the form

$$H(f)(x) = \frac{1}{x}\int_0^x f(t)dt, \; x>0.$$

In fact, the one-dimensional Knopp inequality can be obtained when we replace f by the function $f^{1/p}$ in (3) and let p tend to infinity. From this observation, the Knopp inequality may be regarded as a limit case (as $p\to\infty$) of the Hardy inequality.

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There are a whole slew of n-dimensional analogues of the Hardy operator. Here are two natural extensions. We now let $x=(x_1,x_2)\in\mathbb{R}^2_+ := (0,\infty)\times(0,\infty)$ and $dx = dx_1dx_2$. Pachpatte [6] considered the two-dimensional Hardy operator given by

$$P^2(f)(x)=\frac{1}{x_1x_2}\int_0^{x_1}\int_0^{x_2} f(y)dy, \qquad f>0$$

and proved that

$$\int_{\mathbb{R}^2_+}\left(P^2(f)(x)\right)^p dx \leqslant \left(\frac{p}{p-1}\right)^{2p}\int_{\mathbb{R}^2_+} f^p(x)dx \tag{4}$$

for $1<p<\infty$. Taking $p\to\infty$ and replacing f by $f^{1/p}$ in (4), Heinig, Kerman, and Krbec [5] characterized two positive weights (here and in what follows, a weight w will be a nonnegative locally integrable function) u and v for the following inequality

$$\int_{\mathbb{R}^2_+} u(x)\exp\left(P^2(\log f)\right)(x)dx \leqslant C\int_{\mathbb{R}^2_+} v(x)f(x)dx, \tag{5}$$

to hold if and only if for any $\alpha_1,\alpha_2>0$,

$$\sup_{y_1>0,y_2>0} y_1^{\alpha_1}y_2^{\alpha_2}\int_{y_1}^{\infty}\int_{y_2}^{\infty} x_1^{-(1+\alpha_1)}x_2^{-(1+\alpha_2)}w(x)dx<\infty, \tag{6}$$

where $w(x)=u(x)\exp(P^2(\log(1/v)))(x)$. The authors [5] also pointed out that the higher-dimensional result carries over in the same way. Precisely, the result similar to (5) and (6) is also valid for the n-dimensional Hardy operator P^n defined by

$$P^n f(x)=\frac{1}{\prod_{k=1}^n x_k}\int_0^{x_1}\cdots\int_0^{x_n} f(y)dy, \qquad f>0, \tag{7}$$

where $x_k>0$ $(1\leqslant k\leqslant n)$, $x=(x_1,\ldots,x_n)$, $y=(y_1,\ldots,y_n)$ and $dy=dy_1\cdots dy_n$.

As another high-dimensional extension of the one-dimensional Hardy operator, Christ and Grafakos [1] introduced the n-dimensional averaging operator on the ball

$$\mathscr{H}f(x)=\frac{1}{|B(0,|x|)|}\int_{B(0,|x|)} f(y)dy, \quad x\in\mathbb{R}^n\setminus\{0\}, \tag{8}$$

defined for positive functions f on $\mathbb{R}^n$, where $B(0,r)$ is the ball of radius $r>0$ centered at the origin in $\mathbb{R}^n$. In [1] they proved that for $1<p<\infty$

$$\int_{\mathbb{R}^n}(\mathscr{H}f(x))^p dx \leqslant \left(\frac{p}{p-1}\right)^{p}\int_{\mathbb{R}^n} f^p(x)dx. \tag{9}$$

There are many different ways to obtain the inequality (9) and to show the constant $(p/(p-1))^p$ is sharp, see [2] and [4]. Drábek, Heinig and Kufner [2] obtained the characterization for the n-dimensional Knopp inequality

$$\int_{\mathbb{R}^n} u(x)\exp\left(\mathscr{H}(\log f)\right)(x)dx \leqslant C\int_{\mathbb{R}^n} v(x)f(x)dx,$$

with two positive weights u and v on $\mathbb{R}^n$, which holds if and only if

$$\sup_{t>0} t^n \int_{|x|\geqslant t} \frac{w(x)}{|x|^{2n}} < \infty,$$

where $w(x) = u(x)\exp(\mathscr{H}(\log(1/v)))(x)$.

Throughout the paper we shall use the following notations: $m \in \mathbb{Z}_+$, $n_k \in \mathbb{Z}_+$ for $1 \leqslant k \leqslant m$, $x = (x_1, \ldots, x_m) \in \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_m}$, $y = (y_1, \ldots, y_m) \in \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_m}$ and $dy = dy_m \cdots dy_1$. The operators defined by (7) and (8) turn out to be enormously different. For $P^n(f)$, f is defined on the rectangle $[0,x_1] \times [0,x_2] \times \ldots \times [0,x_n]$ in $\mathbb{R}^n$. While, for $\mathscr{H}(f)$, f is defined on the n-dimensional Euclidean ball $B(0,|x|)$. To generalize these two kinds of operators, Lu, Yan and Zhao [7] gave the definition of the Hardy operator $\mathscr{H}_m$ on higher-dimensional product spaces, where

$$\mathscr{H}_m f(x) = \left(\prod_{k=1}^{m} \frac{1}{|B(0,|x_k|)|}\right) \int_{B(0,|x_1|)} \cdots \int_{B(0,|x_m|)} f(y)dy, \ f > 0,$$

and $\prod_{i=1}^{m} |x_i| \neq 0$. They [7] established the boundedness inequality for $\mathscr{H}_m$ on L^p with $1 < p < \infty$:

$$\int_{\mathbb{R}^{n_1}} \cdots \int_{\mathbb{R}^{n_m}} (\mathscr{H}_m f(x))^p \, dx \leqslant \left(\frac{p}{p-1}\right)^{mp} \int_{\mathbb{R}^{n_1}} \cdots \int_{\mathbb{R}^{n_m}} f^p(x)dx.$$

Also, it is easy to check that

$$\lim_{p\to\infty} \left(\mathscr{H}_m(f^{1/p})(x)\right)^p = \exp(\mathscr{H}_m(\log f))(x).$$

Inspired by the above work, it is natural to ask whether we can give a characterization for the weighted multi-dimensional Knopp type inequality on higher-dimensional product spaces. In this paper, we confirm this question and formulate our results as follows.

THEOREM 1. *Let f, U, V be positive measurable functions defined on $\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_m}$. Define W by*

$$W(x) = U(x)\exp\big(\mathscr{H}_m(\log \frac{1}{V})\big)(x).$$

Then the inequality

$$\int_{\mathbb{R}^{n_1}} \cdots \int_{\mathbb{R}^{n_m}} U(x)\exp\left(\mathscr{H}_m\left(\log f\right)\right)(x)dx \leqslant C_1 \int_{\mathbb{R}^{n_1}} \cdots \int_{\mathbb{R}^{n_m}} V(x)f(x)dx \tag{10}$$

holds if and only if for any $\alpha_1, \ldots, \alpha_m > 0$,

$$A_1 := \sup_{t_k>0, 1\leqslant k\leqslant m} \prod_{k=1}^{m} t_k^{n_k\alpha_k} \int_{|x_1|\geqslant t_1} \cdots \int_{|x_m|\geqslant t_m} \frac{W(x)}{\prod_{j=1}^{m} |x_j|^{n_j(\alpha_j+1)}} dx < \infty. \tag{11}$$

Let $\alpha_k > 0$ for $1 \leqslant k \leqslant m$. We define the analogous adjoint Hardy type operator

$$\mathscr{H}_m^*(f)(x) = \prod_{k=1}^{m} \left(\frac{\alpha_k}{n_k} |B(0,|x_k|)|^{\frac{\alpha_k}{n_k}} \right) \int_{\mathbb{R}^{n_1} \setminus B(0,|x_1|)} \cdots \int_{\mathbb{R}^{n_m} \setminus B(0,|x_m|)} \frac{f(y)}{\prod_{j=1}^{m} |B(0,|y_j|)|^{(1+\frac{\alpha_j}{n_j})}} dy.$$

A similar characterization is given for the higher-dimensional Knopp inequality with the operator $\mathscr{H}_m^*$.

THEOREM 2. *Let f, U, V be positive measurable functions defined on $\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_m}$. Define W^* by*

$$W^*(x) = U(x) \exp\left(\mathscr{H}_m^* \left(\log \frac{1}{V} \right) \right)(x).$$

Then the inequality

$$\int_{\mathbb{R}^{n_1}} \cdots \int_{\mathbb{R}^{n_m}} U(x) \exp\left(\mathscr{H}_m^* (\log f)\right)(x) dx \leqslant C_2 \int_{\mathbb{R}^{n_1}} \cdots \int_{\mathbb{R}^{n_m}} V(x) f(x) dx \qquad (12)$$

holds if and only if for any $\alpha_1, \ldots, \alpha_m > 0$

$$A_2 := \sup_{t_k > 0, 1 \leqslant k \leqslant m} \prod_{k=1}^{m} t_k^{-\alpha_k} \int_{|x_1| \leqslant t_1} \cdots \int_{|x_m| \leqslant t_m} \frac{W^*(x)}{\prod_{j=1}^{m} |x_j|^{n_j - \alpha_j}} dx < \infty. \qquad (13)$$

Especially, when $m = 1$ and $n_1 = n$, then $\mathscr{H}_1^*$ is the analogous adjoint Hardy type operator $\mathscr{H}^*$ given by

$$\mathscr{H}^*(f)(x) = \frac{\alpha}{n} |B(0,|x|)|^{\frac{\alpha}{n}} \int_{\mathbb{R}^n \setminus B(0,|x|)} \frac{f(y)}{|B(0,|y|)|^{(1+\frac{\alpha}{n})}} dy, \quad \alpha > 0.$$

From Theorem 2, we have

COROLLARY 1. *Let f, U, V be positive measurable functions defined on $\mathbb{R}^n$. The inequality*

$$\int_{\mathbb{R}^n} U(x) \exp\left(\mathscr{H}^* (\log f)\right)(x) dx \leqslant C_3 \int_{\mathbb{R}^n} V(x) f(x) dx$$

holds if and only if for any $\alpha > 0$,

$$A_3 := \sup_{t > 0} t^{-\alpha} \int_{|x| \leqslant t} \frac{W^*(x)}{|x|^{n-\alpha}} dx < \infty,$$

where

$$W^*(x) = U(x) \exp\left(\mathscr{H}^* \left(\log \frac{1}{V} \right) \right)(x).$$

It should be pointed out that Corollary 1 is new since it does not seem to have previously appeared in the literature. We will give the proof of Theorem 1 in Section 2 and of Theorem 2 in Section 3.

2. Proof of Theorem 1

Proof. For simplicity we only provide the proof in the case $m=2$. The case $m \geqslant 3$ presents only notational differences and does not require any new ideas.

We firstly assume that (11) holds and we shall prove (10). Writing $Vf=g$, we find that (10) is equivalent to

$$\int_{\mathbb{R}^{n_1+n_2}} W(x)\exp\left(\mathscr{H}_2(\log g)\right)(x)dx \leqslant c_1\int_{\mathbb{R}^{n_1+n_2}} g(x)dx, \tag{14}$$

where $x=(x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$, $dx=dx_2dx_1$, and $\mathbb{R}^{n_1+n_2}=\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$. To shorten notation, write $B(r)=B(0,r)$ for $r>0$. By changing variables, we can write the left hand side of (14) as

$$\int_{\mathbb{R}^{n_1+n_2}} W(x)\exp\left(\frac{1}{\nu_{n_1}\nu_{n_2}}\int_{B(1)}\int_{B(1)}\log g(|x_1|z_1,|x_2|z_2)dz_2dz_1\right)dx.$$

Here and in what follows, $\nu_N=\frac{\pi^{N/2}}{\Gamma(1+N/2)}$ is the volume of the unit ball in $\mathbb{R}^N$ and $\omega_{N-1}=N\nu_N$ denotes the area of the unite sphere Σ_{N-1} for any positive integer $N\geqslant 2$.

An easy calculation shows that for any $\alpha_1,\alpha_2>0$

$$\frac{1}{\nu_{n_1}\nu_{n_2}}\int_{B(1)}\int_{B(1)}\log(|z_1|^{n_1\alpha_1}|z_2|^{n_2\alpha_2})dz_2dz_1=-(\alpha_1+\alpha_2).$$

Hence the left hand side of (14) becomes

$$\begin{aligned}
&e^{(\alpha_1+\alpha_2)}\int_{\mathbb{R}^{n_1+n_2}}W(x)\exp\left(\frac{1}{\nu_{n_1}\nu_{n_2}}\int_{B(1)}\int_{B(1)}\log\Big(\prod_{i=1}^{2}|z_i|^{n_i\alpha_i}g(|x_1|z_1,|x_2|z_2)\Big)dz_2dz_1\right)dx\\
&\leqslant\frac{e^{(\alpha_1+\alpha_2)}}{\nu_{n_1}\nu_{n_2}}\int_{\mathbb{R}^{n_1+n_2}}W(x)\left(\int_{B(1)}\int_{B(1)}|z_1|^{n_1\alpha_1}|z_2|^{n_2\alpha_2}g(|x_1|z_1,|x_2|z_2)dz_2dz_1\right)dx,
\end{aligned}$$

where the last estimate follows by Jensen's inequality. Furthermore, the last integral above can be written in the polar coordinates form:

$$\begin{aligned}
&\int_{\Sigma_{n_1-1}}\int_{\Sigma_{n_2-1}}\int_{\Sigma_{n_1-1}}\int_{\Sigma_{n_2-1}}\int_0^\infty\int_0^\infty t_1{}^{n_1-1}t_2{}^{n_2-1}W(t_1x_1',t_2x_2') \\
&\times\int_0^1\int_0^1\prod_{i=1}^{2}s_i{}^{n_i\alpha_i+n_i-1}g(t_1s_1z_1',t_2s_2z_2')ds_2ds_1dt_2dt_1d\sigma(z_2')d\sigma(z_1')d\sigma(x_2')d\sigma(x_1').
\end{aligned}\tag{15}$$

By Fubini's Theorem and substitution $t_is_i=r_i$, we see that

$$\begin{aligned}
&\int_0^\infty\int_0^\infty\int_0^1\int_0^1\prod_{i=1}^{2}t_i{}^{n_i-1}s_i{}^{n_i\alpha_i+n_i-1}W(t_1x_1',t_2x_2')g(t_1s_1z_1',t_2s_2z_2')ds_2ds_1dt_2dt_1\\
&=\int_0^\infty\int_0^\infty\int_0^1\int_0^1\prod_{i=1}^{2}r_i{}^{n_i-1}s_i{}^{n_i\alpha_i-1}g(r_1z_1',r_2z_2')W\left(\frac{r_1}{s_1}x_1',\frac{r_2}{s_2}x_2'\right)ds_2ds_1dr_2dr_1
\end{aligned}$$

$$(\text{substituting } \frac{r_i}{s_i}=\tau_i,\ ds_i=-\frac{r_i}{\tau_i^2}d\tau_i)$$

$$=\int_0^\infty\int_0^\infty\int_{r_1}^\infty\int_{r_2}^\infty\prod_{i=1}^2 r_i^{n_i\alpha_i+n_i-1}\tau_i^{-n_i\alpha_i-1}g(r_1z_1',r_2z_2')W(\tau_1x_1',\tau_2x_2')d\tau_2d\tau_1dr_2dr_1.$$

Substituting into (15) shows that

$$\begin{aligned}&\int_{\mathbb{R}^{n_1+n_2}}W(x)\exp\left(\frac{1}{\prod_{k=1}^2|B(|x_k|)|}\int_{|y_1|\leqslant|x_1|}\int_{|y_2|\leqslant|x_2|}\log g(y_1,y_2)dy_2dy_1\right)dx\\
&\leqslant\frac{e^{(\alpha_1+\alpha_2)}}{\nu_{n_1}\nu_{n_2}}\int_{\Sigma_{n_1-1}}\int_{\Sigma_{n_2-1}}\int_{\Sigma_{n_1-1}}\int_{\Sigma_{n_2-1}}\int_0^\infty\int_0^\infty r_1^{n_1-1}r_2^{n_2-1}g(r_1z_1',r_2z_2')r_1^{n_1\alpha_1}r_2^{n_2\alpha_2}\\
&\quad\times\int_{r_1}^\infty\int_{r_2}^\infty\frac{\tau_1^{n_1-1}\tau_2^{n_2-1}W(\tau_1x_1',\tau_2x_2')}{\tau_1^{n_1(\alpha_1+1)}\tau_2^{n_2(\alpha_2+1)}}d\tau_2d\tau_1dr_2dr_1d\sigma(x_2')d\sigma(x_1')d\sigma(z_2')d\sigma(z_1')\\
&=\frac{e^{(\alpha_1+\alpha_2)}}{\nu_{n_1}\nu_{n_2}}\int_{\mathbb{R}^{n_1+n_2}}g(x)\\
&\quad\times\left(|x_1|^{n_1\alpha_1}|x_2|^{n_2\alpha_2}\int_{|y_1|\geqslant|x_1|}\int_{|y_2|\geqslant|x_2|}\frac{W(y_1,y_2)}{|y_1|^{n_1(\alpha_1+1)}|y_2|^{n_2(\alpha_2+1)}}dy_2dy_1\right)dx\\
&\leqslant\frac{e^{(\alpha_1+\alpha_2)}}{\nu_{n_1}\nu_{n_2}}A_1\int_{\mathbb{R}^{n_1+n_2}}g(x)dx,\end{aligned}$$

where in the last inequality we have used the condition (11).

Conversely, suppose that (14) holds, we shall show that (11) holds for the case $m=2$. Fixing $t_1,t_2>0$, for $x=(x_1,x_2)\in\mathbb{R}^{n_1+n_2}$, we take the piecewise function

$$g(x)=\begin{cases}t_1^{-n_1}t_2^{-n_2} & |x_1|\leqslant t_1,|x_2|\leqslant t_2,\\ t_1^{-n_1}e^{-n_2(1+\alpha_2)}|x_2|^{-n_2(1+\alpha_2)}t_2^{n_2\alpha_2} & |x_1|\leqslant t_1,|x_2|>t_2,\\ t_2^{-n_2}e^{-n_1(1+\alpha_1)}|x_1|^{-n_1(1+\alpha_1)}t_1^{n_1\alpha_1} & |x_1|>t_1,|x_2|\leqslant t_2,\\ e^{-n_1(1+\alpha_1)-n_2(1+\alpha_2)}|x_1|^{-n_1(1+\alpha_1)}|x_2|^{-n_2(1+\alpha_2)}t_1^{n_1\alpha_1}t_2^{n_2\alpha_2} & |x_1|>t_1,|x_2|>t_2.\end{cases}$$

Obviously, g is radial and the right hand side of (14) is

$$\begin{aligned}&c_1\int_{\Sigma_{n_1-1}}\int_{\Sigma_{n_2-1}}\int_0^\infty\int_0^\infty s_1^{n_1-1}s_2^{n_2-1}g(s_1,s_2)ds_2ds_1d\sigma(x_2')d\sigma(x_1')\\
&=c_1\omega_{n_1-1}\omega_{n_2-1}\left(\int_0^{t_1}\int_0^{t_2}+\int_0^{t_1}\int_{t_2}^\infty+\int_{t_1}^\infty\int_0^{t_2}+\int_{t_1}^\infty\int_{t_2}^\infty\right)\prod_{i=1}^2 s_i^{n_i-1}g(s_1,s_2)ds_2ds_1\\
&=c_1\nu_{n_1}\nu_{n_2}\left(1+\frac{e^{-n_2(1+\alpha_2)}}{\alpha_2}+\frac{e^{-n_1(1+\alpha_1)}}{\alpha_1}+\frac{e^{-n_1(1+\alpha_1)-n_2(1+\alpha_2)}}{\alpha_1\alpha_2}\right)\\
&=:C_1(n_1,n_2,\alpha_1,\alpha_2).\end{aligned}$$

Hence, the left hand side of (14) satisfies

$$\begin{aligned}&C_1(n_1,n_2,\alpha_1,\alpha_2)\\
&\geqslant\int_{\mathbb{R}^{n_1+n_2}}W(x)\exp\left(\frac{1}{\prod_{k=1}^2|B(|x_k|)|}\int_{|y_1|\leqslant|x_1|}\int_{|y_2|\leqslant|x_2|}\log g(y_1,y_2)dy_2dy_1\right)dx\end{aligned}$$

$$
\begin{aligned}
&= \int_{\Sigma_{n_1-1}} \int_{\Sigma_{n_2-1}} \int_0^\infty \int_0^\infty s_1^{n_1-1} s_2^{n_2-1} W(s_1 x_1', s_2 x_2') \\
&\quad \times \exp\left(\frac{n_1 n_2}{s_1{}^{n_1} s_2{}^{n_2}} \int_0^{s_1} \int_0^{s_2} r_1^{n_1-1} r_2^{n_2-1} \log g(r_1, r_2) dr_2 dr_1 \right) ds_2 ds_1 d\sigma(x_2') d\sigma(x_1') \\
&= \int_{\Sigma_{n_1-1}} \int_{\Sigma_{n_2-1}} \sum_{j=1}^4 I_j(x_1', x_2') d\sigma(x_2') d\sigma(x_1'),
\end{aligned}
$$

where

$$
\begin{aligned}
I_1(x_1', x_2') &= \int_0^{t_1} \int_0^{t_2} \prod_{j=1}^2 s_j{}^{n_j-1} W(s_1 x_1', s_2 x_2') E(s_1, s_2) ds_2 ds_1, \\
I_2(x_1', x_2') &= \int_0^{t_1} \int_{t_2}^\infty \prod_{j=1}^2 s_j{}^{n_j-1} W(s_1 x_1', s_2 x_2') E(s_1, s_2) ds_2 ds_1, \\
I_3(x_1', x_2') &= \int_{t_1}^\infty \int_0^{t_2} \prod_{j=1}^2 s_j{}^{n_j-1} W(s_1 x_1', s_2 x_2') E(s_1, s_2) ds_2 ds_1, \\
I_4(x_1', x_2') &= \int_{t_1}^\infty \int_{t_2}^\infty \prod_{j=1}^2 s_j{}^{n_j-1} W(s_1 x_1', s_2 x_2') E(s_1, s_2) ds_2 ds_1,
\end{aligned}
$$

and

$$
E(s_1, s_2) = \exp\left(\frac{n_1 n_2}{s_1{}^{n_1} s_2{}^{n_2}} \int_0^{s_1} \int_0^{s_2} \prod_{i=1}^2 r_i{}^{n_i-1} \log g(r_1, r_2) dr_2 dr_1 \right).
$$

Since each integral $\int_{\Sigma_{n_1-1}} \int_{\Sigma_{n_2-1}} I_j(x_1', x_2') d\sigma(x_2') d\sigma(x_1')$ is positive, we have

$$
C_1(n_1, n_2, \alpha_1, \alpha_2) \geqslant \int_{\Sigma_{n_1-1}} \int_{\Sigma_{n_2-1}} I_4(x_1', x_2') d\sigma(x_2') d\sigma(x_1').
$$

It is enough to estimate the term

$$
\int_{\Sigma_{n_1-1}} \int_{\Sigma_{n_2-1}} I_4(x_1', x_2') d\sigma(x_2') d\sigma(x_1').
$$

Split the integral $\int_0^{s_1} \int_0^{s_2}$ into four parts,

$$
\int_0^{s_1} \int_0^{s_2} = \int_0^{t_1} \int_0^{t_2} + \int_0^{t_1} \int_{t_2}^{s_2} + \int_{t_1}^{s_1} \int_0^{t_2} + \int_{t_1}^{s_1} \int_{t_2}^{s_2},
$$

and write

$$
\begin{aligned}
E_1(s_1, s_2) &= \exp\left(\prod_{k=1}^2 \frac{n_k}{s_k^{n_k}} \int_0^{t_1} \int_0^{t_2} \prod_{i=1}^2 r_i^{n_i-1} \log(t_1{}^{-n_1} t_2{}^{-n_2}) dr_2 dr_1 \right), \\
E_2(s_1, s_2) &= \exp\left(\prod_{k=1}^2 \frac{n_k}{s_k^{n_k}} \int_0^{t_1} \int_{t_2}^{s_2} \prod_{i=1}^2 r_i^{n_i-1} \log(t_1{}^{-n_1} e^{-n_2(1+\alpha_2)} r_2{}^{-n_2(1+\alpha_2)} t_2{}^{n_2 \alpha_2}) dr_2 dr_1 \right),
\end{aligned}
$$

$$E_3(s_1,s_2)=\exp\left(\prod_{k=1}^{2}\frac{n_k}{s_k^{n_k}}\int_{t_1}^{s_1}\int_0^{t_2}\prod_{i=1}^{2}r_i^{n_i-1}\log(t_2{}^{-n_2}e^{-n_1(1+\alpha_1)}r_1{}^{-n_1(1+\alpha_1)}t_1{}^{n_1\alpha_1})dr_2dr_1\right),$$

$$E_4(s_1,s_2)=\exp\left(\prod_{k=1}^{2}\frac{n_k}{s_k^{n_k}}\int_{t_1}^{s_1}\int_{t_2}^{s_2}\prod_{i=1}^{2}r_i^{n_i-1}\log\left(\prod_{i=1}^{2}e^{-n_i(1+\alpha_i)}r_i{}^{-n_i(1+\alpha_i)}t_i{}^{n_i\alpha_i}\right)dr_2dr_1\right).$$

We have

$$I_4(x_1',x_2')=\int_{t_1}^{\infty}\int_{t_2}^{\infty}s_1^{n_1-1}s_2^{n_2-1}W(s_1x_1',s_2x_2')\prod_{k=1}^{4}E_k(s_1,s_2)ds_2ds_1.$$

It is easy to get

$$E_1(s_1,s_2)=\exp\left(\frac{t_1^{n_1}t_2^{n_2}}{s_1^{n_1}s_2^{n_2}}(-n_1\log t_1-n_2\log t_2)\right).$$

An elementary calculation shows that

$$\begin{aligned}E_2(s_1,s_2)=\exp\Bigg(&\frac{t_1^{n_1}t_2^{n_2}}{s_1^{n_1}s_2^{n_2}}\big(n_1\log t_1+n_2\log t_2+(n_2-1)(1+\alpha_2)\big)\\&+\left(\frac{t_1}{s_1}\right)^{n_1}\big(-n_1\log t_1+n_2\alpha_2\log t_2-n_2(\alpha_2+1)\log s_2+(1-n_2)(1+\alpha_2)\big)\Bigg),\\E_3(s_1,s_2)=\exp\Bigg(&\frac{t_1^{n_1}t_2^{n_2}}{s_1^{n_1}s_2^{n_2}}\big(n_2\log t_2+n_1\log t_1+(n_1-1)(1+\alpha_1)\big)\\&+\left(\frac{t_2}{s_2}\right)^{n_2}\big(-n_2\log t_2+n_1\alpha_1\log t_1-n_1(\alpha_1+1)\log s_1+(1-n_1)(1+\alpha_1)\big)\Bigg),\\E_4(s_1,s_2)=\exp\Bigg(&\sum_{i=1}^{2}\big(n_i\alpha_i\log t_i-n_i(\alpha_i+1)\log s_i+(1-n_i)(1+\alpha_i)\big)\\&+\frac{t_1^{n_1}t_2^{n_2}}{s_1^{n_1}s_2^{n_2}}\big(\sum_{i=1}^{2}(1-n_i)(1+\alpha_i)-n_i\log t_i\big)\\&+\left(\frac{t_1}{s_1}\right)^{n_1}\big(\sum_{i=1}^{2}(n_i-1)(1+\alpha_i)+n_1\log t_1+n_2(\alpha_2+1)\log s_2-n_2\alpha_2\log t_2\big)\\&+\left(\frac{t_2}{s_2}\right)^{n_2}\big(\sum_{i=1}^{2}(n_i-1)(1+\alpha_i)+n_2\log t_2+n_1(\alpha_1+1)\log s_1-n_1\alpha_1\log t_1\big)\Bigg).\end{aligned}$$

From these estimates we obtain that

$$\begin{aligned}\prod_{k=1}^{4}E_k(s_1,s_2)=&\frac{t_1^{n_1\alpha_1}t_2^{n_2\alpha_2}}{s_1{}^{n_1(\alpha_1+1)}s_2{}^{n_2(\alpha_2+1)}}e^{\sum_{i=1}^{2}(1-n_i)(1+\alpha_i)}\\&\times\exp\left(\left(\frac{t_1}{s_1}\right)^{n_1}(n_1-1)(1+\alpha_1)+\left(\frac{t_2}{s_2}\right)^{n_2}(n_2-1)(1+\alpha_2)\right).\end{aligned}$$

Clearly we have

$$\begin{aligned}&\int_{\Sigma_{n_1-1}}\int_{\Sigma_{n_2-1}} I_4(x_1',x_2')d\sigma(x_2')d\sigma(x_1')\\ &\geqslant e^{\sum_{i=1}^{2}(1-n_i)(1+\alpha_i)}t_1^{n_1\alpha_1}t_2^{n_2\alpha_2}\int_{|x_1|\geqslant t_1}\int_{|x_2|\geqslant t_2}\frac{W(x_1,x_2)}{|x_1|^{n_1(\alpha_1+1)}|x_2|^{n_2(\alpha_2+1)}}dx_2dx_1\end{aligned}$$

and this implies (11) by taking the supremum over all $t_1,t_2>0$. □

3. Proof of Theorem 2

Proof. Theorem 2 can be proved in a similar way as Theorem 1. Hence, we only point out some differences of the corresponding relations. It is easy to conclude that (12) is equivalent to

$$\int_{\mathbb{R}^{n_1+n_2}} W^*(x_1,x_2)\exp\big(\mathscr{H}_2^*(\log g)\big)(x_1,x_2)dx\leqslant c_3\int_{\mathbb{R}^{n_1+n_2}} g(x_1,x_2)dx. \tag{16}$$

Assume that (13) holds. For any $\alpha_1,\alpha_2>0$,

$$\frac{\alpha_1\alpha_2}{\omega_{n_1-1}\omega_{n_2-1}}\int_{|z_1|\geqslant 1}\int_{|z_2|\geqslant 1}\frac{\log(|z_1|^{n_1}|z_2|^{n_2})}{|z_1|^{n_1+\alpha_1}|z_2|^{n_2+\alpha_2}}dz_2dz_1=\frac{n_1}{\alpha_1}+\frac{n_2}{\alpha_2},$$

then by using Jensen's inequality, the integral becomes

$$\begin{aligned}&\int_{\mathbb{R}^{n_1+n_2}} W^*(x_1,x_2)\exp\big(\mathscr{H}_2^*(\log g)\big)(x_1,x_2)dx\\ &=e^{-\frac{n_1}{\alpha_1}-\frac{n_2}{\alpha_2}}\int_{\mathbb{R}^{n_1+n_2}} W^*(x_1,x_2)\exp\left(\frac{\int_{|z_1|\geqslant 1}\int_{|z_2|\geqslant 1}\frac{\log(|z_1|^{n_1}|z_2|^{n_2}g(|x_1|z_1,|x_2|z_2))}{|z_1|^{n_1+\alpha_1}|z_2|^{n_2+\alpha_2}}dz_2dz_1}{\frac{\omega_{n_1-1}\omega_{n_2-1}}{\alpha_1\alpha_2}}\right)dx\\ &\leqslant e^{-\frac{n_1}{\alpha_1}-\frac{n_2}{\alpha_2}}\frac{\alpha_1\alpha_2}{\omega_{n_1-1}\omega_{n_2-1}}\int_{\mathbb{R}^{n_1+n_2}} W^*(x_1,x_2)\int_{|z_1|\geqslant 1}\int_{|z_2|\geqslant 1}\frac{g(|x_1|z_1,|x_2|z_2)}{|z_1|^{\alpha_1}|z_2|^{\alpha_2}}dz_2dz_1dx.\end{aligned}$$

By a similar argument as in Theorem 1, the last integral above is expressed by the polar coordinates form:

$$\begin{aligned}&\int_{\Sigma_{n_1-1}}\int_{\Sigma_{n_2-1}}\int_{\Sigma_{n_1-1}}\int_{\Sigma_{n_2-1}}\int_0^\infty\int_0^\infty t_1{}^{n_1-1}t_2{}^{n_2-1}W^*(t_1x_1',t_2x_2') \\ &\times\int_1^\infty\int_1^\infty\prod_{i=1}^{2}s_i{}^{n_i-1-\alpha_i}g(t_1s_1z_1',t_2s_2z_2')ds_2ds_1dt_2dt_1d\sigma(z_2')d\sigma(z_1')d\sigma(x_2')d\sigma(x_1').\end{aligned}\tag{17}$$

Interchanging the order of integration and changing variables $t_is_i=r_i$, one has

$$\begin{aligned}&\int_0^\infty\int_0^\infty\int_1^\infty\int_1^\infty\prod_{i=1}^{2}t_i{}^{n_i-1}s_i{}^{n_i-1-\alpha_i}W^*(t_1x_1',t_2x_2')g(t_1s_1z_1',t_2s_2z_2')ds_2ds_1dt_2dt_1\\ &=\int_0^\infty\int_0^\infty\int_1^\infty\int_1^\infty\prod_{i=1}^{2}r_i{}^{n_i-1}s_i{}^{-1-\alpha_i}g(r_1z_1',r_2z_2')W^*\left(\frac{r_1}{s_1}x_1',\frac{r_2}{s_2}x_2'\right)ds_2ds_1dr_2dr_1\end{aligned}$$

$$(\text{substituting } \frac{r_i}{s_i} = \tau_i,\ ds_i = -\frac{r_i}{\tau_i{}^2} d\tau_i)$$

$$= \int_0^\infty \int_0^\infty \int_0^{r_1} \int_0^{r_2} \prod_{i=1}^2 r_i^{n_i-1-\alpha_i} \tau_i^{\alpha_i-1} g(r_1 z_1', r_2 z_2') W^*(\tau_1 x_1', \tau_2 x_2') d\tau_2 d\tau_1 dr_2 dr_1.$$

Substituting into (17) implies that

$$\int_{\mathbb{R}^{n_1+n_2}} W^*(x_1,x_2) \exp\big(\mathscr{H}_2^*(\log g)\big)(x_1,x_2) dx$$
$$\leqslant e^{-\frac{n_1}{\alpha_1}-\frac{n_2}{\alpha_2}} \frac{\alpha_1\alpha_2}{\omega_{n_1-1}\omega_{n_2-1}} \int_{\Sigma_{n_1-1}} \int_{\Sigma_{n_2-1}} \int_{\Sigma_{n_1-1}} \int_{\Sigma_{n_2-1}} \int_0^\infty \int_0^\infty r_1{}^{n_1-1} r_2{}^{n_2-1} g(r_1 z_1', r_2 z_2')$$
$$\times \prod_{k=1}^2 r_k{}^{-\alpha_k} \int_0^{r_1} \int_0^{r_2} \frac{\prod_{i=1}^2 \tau_i^{n_i-1} W^*(\tau_1 x_1', \tau_2 x_2')}{\tau_1^{n_1-\alpha_1} \tau_2^{n_2-\alpha_2}} d\tau_2 d\tau_1 dr_2 dr_1 d\sigma(z_2') d\sigma(z_1') d\sigma(x_2') d\sigma(x_1')$$
$$= e^{-\frac{n_1}{\alpha_1}-\frac{n_2}{\alpha_2}} \frac{\alpha_1\alpha_2}{\omega_{n_1-1}\omega_{n_2-1}}$$
$$\times \int_{\mathbb{R}^{n_1+n_2}} g(x) \left(|x_1|^{-\alpha_1} |x_2|^{-\alpha_2} \int_{|y_1|\leqslant|x_1|} \int_{|y_2|\leqslant|x_2|} \frac{W^*(y_1,y_2)}{|y_1|^{n_1-\alpha_1} |y_2|^{n_2-\alpha_2}} dy_2 dy_1 \right) dx$$
$$\leqslant e^{-\frac{n_1}{\alpha_1}-\frac{n_2}{\alpha_2}} \frac{\alpha_1\alpha_2}{\omega_{n_1-1}\omega_{n_2-1}} A_2 \int_{\mathbb{R}^{n_1+n_2}} g(x) dx,$$

which conclude that (16) holds under the assumption of $A_2 < \infty$.

On the other hand, we take the radial piecewise function

$$g(x) = \begin{cases} e^{-2} t_1{}^{-\alpha_1} t_2{}^{-\alpha_2} |x_1|^{\alpha_1-n_1} |x_2|^{\alpha_2-n_2} & |x_1| \leqslant t_1, |x_2| \leqslant t_2, \\ e^{-1} t_1{}^{\alpha_1} t_2{}^{-\alpha_2} |x_1|^{-\alpha_1-n_1} |x_2|^{\alpha_2-n_2} & |x_1| > t_1, |x_2| \leqslant t_2, \\ e^{-1} t_1{}^{-\alpha_1} t_2{}^{\alpha_2} |x_1|^{\alpha_1-n_1} |x_2|^{-\alpha_2-n_2} & |x_1| \leqslant t_1, |x_2| > t_2, \\ t_1{}^{\alpha_1} t_2{}^{\alpha_2} |x_1|^{-\alpha_1-n_1} |x_2|^{-\alpha_2-n_2} & |x_1| > t_1, |x_2| > t_2 \end{cases}$$

with $t_1, t_2 > 0$ and $x = (x_1, x_2) \in \mathbb{R}^{n_1+n_2}$.

Obviously, the right hand side of (16) is

$$c_2 \int_{\Sigma_{n_1-1}} \int_{\Sigma_{n_2-1}} \int_0^\infty \int_0^\infty s_1^{n_1-1} s_2^{n_2-1} g(s_1,s_2) ds_2 ds_1 d\sigma(x_2') d\sigma(x_1')$$
$$= \frac{c_2\ \omega_{n_1-1}\omega_{n_2-1}}{\alpha_1\alpha_2} (e^{-2} + 2e^{-1} + 1)$$
$$=: C_2(n_1,n_2,\alpha_1,\alpha_2).$$

Similar arguments as that of proof of Theorem 1, we have

$$C_2(n_1,n_2,\alpha_1,\alpha_2) \geqslant \int_{\Sigma_{n_1-1}} \int_{\Sigma_{n_2-1}} \mathbf{J}_1(s_1,s_2) d\sigma(x_2') d\sigma(x_1'),$$

where $\mathbf{J}_1(s_1,s_2)$ is

$$\mathbf{J}_1(s_1,s_2) = \int_0^{t_1} \int_0^{t_2} s_1{}^{n_1-1} s_2{}^{n_2-1} W^*(s_1 x_1', s_2 x_2') \exp\big(\alpha_1\alpha_2 s_1{}^{\alpha_1} s_2^{\alpha_2}$$
$$\times \left(\int_{s_1}^{t_1} \int_{s_2}^{t_2} + \int_{t_1}^{\infty} \int_{s_2}^{t_2} + \int_{s_1}^{t_1} \int_{t_2}^{\infty} + \int_{t_1}^{\infty} \int_{t_2}^{\infty} \right) \frac{\log g(r_1,r_2)}{r_1{}^{1+\alpha_1} r_2{}^{1+\alpha_2}} dr_2 dr_1 \Big) ds_2 ds_1$$

$$= \int_0^{t_1} \int_0^{t_2} {s_1}^{n_1-1} {s_2}^{n_2-1} W^*(s_1x_1', s_2x_2') \prod_{k=1}^{4} F_k(s_1,s_2) ds_2 ds_1.$$

By the tedious calculations without any technique, we obtain that

$$\prod_{k=1}^{4} F_k(s_1,s_2) = e^{-2-\frac{n_1}{\alpha_1}-\frac{n_2}{\alpha_2}} \frac{t_1^{-\alpha_1} t_2^{-\alpha_2}}{{s_1}^{n_1-\alpha_1} {s_2}^{n_2-\alpha_2}} \exp\left(2 - \left(\frac{t_1}{s_1}\right)^{\alpha_1} - \left(\frac{t_2}{s_2}\right)^{\alpha_2}\right),$$

holds and then we have

$$\begin{aligned} C_2(n_1,n_2,\alpha_1,\alpha_2) &\geqslant \int_{\Sigma_{n_1-1}} \int_{\Sigma_{n_2-1}} \mathbf{J}_1(s_1,s_2) d\sigma(x_2') d\sigma(x_1') \\ &\geqslant e^{-2-\frac{n_1}{\alpha_1}-\frac{n_2}{\alpha_2}} t_1^{-\alpha_1} t_2^{-\alpha_2} \int_{|x_1|\leqslant t_1} \int_{|x_2|\leqslant t_2} \frac{W^*(x_1,x_2)}{|x_1|^{n_1-\alpha_1} |x_2|^{n_2-\alpha_2}} dx_2 dx_1. \end{aligned}$$

Thus, we finish the proof by taking the supremum over all $t_1, t_2 > 0$. □

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