Mathematical
Inequalities
& Applications
Volume 22, Number 2 (2019), 631–655 doi:10.7153/mia-2019-22-44

# COMMUTATORS OF FRACTIONAL INTEGRALS ON MARTINGALE MORREY SPACES

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(*Communicated by L. Pick*)

*Abstract.* On martingale Morrey spaces we give necessary and sufficient conditions for the boundedness and compactness of the commutator generated by the fractional integral and a function in the martingale Campanato space. We also give the conditions for the boundedness and compactness from martingale Morrey spaces to martingale Triebel-Lizorkin-Morrey spaces.

## 1. Introduction

It is well known as the Hardy-Littlewood-Sobolev theorem that the fractional integral operators $I_\alpha$ on the Euclidean space $\mathbb{R}^n$ is bounded from $L_p$ to $L_q$ for $1<p<q<\infty$, $0<\alpha<n$ and $-n/p+\alpha=-n/q$. For any BMO function $b$, Chanillo [2] proved the same boundedness of the commutator $[b,I_\alpha]$. This boundedness was extended to Morrey spaces by Di Fazio and Ragusa [6]. See also Ragusa and Scapellato [15]. Paluszyński [14] proved that, for any $\beta$-Lipschitz function $b$, $0<\beta<1$, the commutator $[b,I_\alpha]$ is bounded from $L_p$ to $L_q$ for $-n/p+\alpha+\beta=-n/q$ and from $L_p$ to the Triebel-Lizorkin space $\dot{F}^{\beta}_{p,\infty}$. Further, the compactness of the commutators on Morrey spaces was investigated by Chen, Ding and Wang [5].

In martingale theory, based on the result by Watari [18, Theorem 1.1], Chao and Ombe [3] proved the boundedness of the fractional integrals for $H_p$, $L_p$, BMO and Lipschitz spaces of the dyadic martingales. These fractional integrals were defined for more general martingales in [16]. See also Hao and Jiao [7]. On the other hand, martingale Morrey spaces and their generalization were introduced by [11] and [13], respectively, and the boundedness of fractional integrals as martingale transforms were established. Moreover, necessary and sufficient conditions for the boundedness of fractional integrals on the martingale Morrey spaces were given in [12]. On the other hand the compactness of the commutators of martingale transforms was investigated by Janson [8] and Chao and Peng [4], etc.

In this paper, we investigate the boundedness of the commutator $[b,I_\alpha]$ with a function $b$ in the martingale Campanato spaces introduced in [11]. We give necessary and sufficient conditions for the boundedness of $[b,I_\alpha]$ from a martingale Morrey space to another martingale Morrey space or to a martingale Triebel-Lizorkin-Morrey space

*Mathematics subject classification* (2010): Primary 60G46, Secondary 42B30, 46E30, 42B35.
*Keywords and phrases*: Martingale, Morrey space, fractional integral, commutator.

(Theorems 1.1 and 1.2). As a corollary we get the martingale version of Paluszyński's result (Corollary 1.3). Further, we give the conditions for the compactness of the commutators (Theorems 1.4 and 1.5 and Corollary 1.6).

Let $(\Omega,\mathscr{F},P)$ be a probability space and let $\{\mathscr{F}_n\}_{n\geqslant 0}$ be a nondecreasing sequence of sub-$\sigma$-algebras of $\mathscr{F}$ such that $\mathscr{F}=\sigma(\bigcup_n \mathscr{F}_n)$. We suppose that every $\sigma$-algebra $\mathscr{F}_n$ is generated by countable atoms, where $B\in\mathscr{F}_n$ is called an atom (more precisely a $(\mathscr{F}_n,P)$-atom), if any $A\subset B$ with $A\in\mathscr{F}_n$ satisfies $P(A)=P(B)$ or $P(A)=0$. Denote by $A(\mathscr{F}_n)$ the set of all atoms in $\mathscr{F}_n$. We also suppose that $(\Omega,\mathscr{F},P)$ is non-atomic.

The expectation operator is denoted by $E$. For a measurable set $G\in\mathscr{F}$, its characteristic function is denoted by $\chi_G$. Let $L_{p,\mathrm{loc}}$ be the set of all measurable functions such that $|f|^p\chi_B$ is integrable for all $B\in A(\mathscr{F}_0)$. If $\mathscr{F}_0=\{\Omega,\emptyset\}$, then $L_{p,\mathrm{loc}}=L_p$. An $\mathscr{F}_n$-measurable function $g\in L_{1,\mathrm{loc}}$ is called the conditional expectation of $f\in L_{1,\mathrm{loc}}$ relative to $\mathscr{F}_n$ if

$$E[g\chi_B\chi_G]=E[f\chi_B\chi_G]\quad \text{for all}\quad B\in A(\mathscr{F}_0)\quad \text{and}\quad G\in\mathscr{F}_n.$$

We denote by $E_nf$ the conditional expectation of $f$ relative to $\mathscr{F}_n$. We say a sequence $(f_n)_{n\geqslant 0}$ in $L_{1,\mathrm{loc}}$ is a martingale relative to $\{\mathscr{F}_n\}_{n\geqslant 0}$ if it is adapted to $\{\mathscr{F}_n\}_{n\geqslant 0}$ and satisfies $E_n[f_m]=f_n$ for every $n\leqslant m$.

We first recall the definition of fractional integrals. Let

$$\beta_n=\sum_{B\in A(\mathscr{F}_n)} P(B)\chi_B,\quad n=0,1,2,\cdots. \tag{1.1}$$

For $\alpha>0$ and a martingale $f=(f_n)_{n\geqslant 0}$ relative to $\{\mathscr{F}_n\}_{n\geqslant 0}$, we define the fractional integral $I_\alpha f=((I_\alpha f)_n)_{n\geqslant 0}$ of $f$ by

$$(I_\alpha f)_n=\sum_{k=0}^{n}(\beta_{k-1})^\alpha(f_k-f_{k-1}) \tag{1.2}$$

with the conventions $\beta_{-1}=\beta_0$ and $f_{-1}=0$. In what follows we always use these conventions and $E_{-1}f=0$. As is shown in [11, Remark 5.3], the series $\chi_B\sum_{k=0}^{\infty}(\beta_{k-1})^\alpha(f_k-f_{k-1})$ converges in $L_1$ for every $B\in A(\mathscr{F}_0)$. By this reason, for a function $f\in L_{1,\mathrm{loc}}$ with its corresponding martingale $f=(E_nf)_{n\geqslant 0}$, we define

$$I_\alpha f=\sum_{k=0}^{\infty}(\beta_{k-1})^\alpha(E_kf-E_{k-1}f), \tag{1.3}$$

which is in $L_{1,\mathrm{loc}}$. By this definition the commutator

$$[b,I_\alpha]f=bI_\alpha f-I_\alpha(bf)$$

is well-defined for $f\in L_{p,\mathrm{loc}}$ and $b\in L_{p',\mathrm{loc}}$, where $p,p'\in[1,\infty]$ and $1/p+1/p'=1$.

Next, we recall the definition of martingale Morrey and Campanato spaces.

DEFINITION 1.1. Let $p \in [1,\infty)$ and $\lambda \in (-\infty,\infty)$. For $f \in L_{1,\mathrm{loc}}$, let

$$\|f\|_{L_{p,\lambda}} = \sup_{n\geqslant 0} \sup_{B\in A(\mathscr{F}_n)} \frac{1}{P(B)^\lambda} \left( \frac{1}{P(B)} \int_B |f|^p \, dP \right)^{1/p},$$

$$\|f\|_{\mathscr{L}_{p,\lambda}} = \sup_{n\geqslant 0} \sup_{B\in A(\mathscr{F}_n)} \frac{1}{P(B)^\lambda} \left( \frac{1}{P(B)} \int_B |f - E_n f|^p \, dP \right)^{1/p},$$

and define

$$L_{p,\lambda} = \{f \in L_{p,\mathrm{loc}} : \|f\|_{L_{p,\lambda}} < \infty\}, \quad \mathscr{L}_{p,\lambda} = \{f \in L_{p,\mathrm{loc}} : \|f\|_{\mathscr{L}_{p,\lambda}} < \infty\}.$$

If $p = 1$ and $\lambda = 0$, then $\mathscr{L}_{1,0}$ is the martingale BMO space and $\|f\|_{\mathrm{BMO}} = \|f\|_{\mathscr{L}_{1,0}}$.

The stochastic basis $\{\mathscr{F}_n\}_{n\geqslant 0}$ is said to be regular, if there exists a constant $R \geqslant 2$ such that

$$f_n \leqslant R f_{n-1} \tag{1.4}$$

holds for all $n \geqslant 1$ and all nonnegative martingales $(f_n)_{n\geqslant 0}$. It was shown in [10] that, if $\{\mathscr{F}_n\}_{n\geqslant 0}$ is regular, then $\|f\|_{\mathscr{L}_{p,\lambda}}$ is equivalent to

$$\|f\|_{\mathscr{L}^-_{p,\lambda}} = \sup_{n\geqslant 0} \sup_{B\in A(\mathscr{F}_n)} \frac{1}{P(B)^\lambda} \left( \frac{1}{P(B)} \int_B |f - E_{n-1} f|^p \, dP \right)^{1/p},$$

if $E_0 f = 0$.

We next recall the definition of sharp functions. Let $\delta \geqslant 0$. For $f \in L_{1,\mathrm{loc}}$, let

$$M^\sharp_\delta f = \sup_{n\geqslant 0} (\beta_n)^{-\delta} E_n |f - E_{n-1} f|, \tag{1.5}$$

with the convention $E_{-1} f = 0$. If $\delta = 0$ we denote $M^\sharp_0$ by $M^\sharp$, that is,

$$M^\sharp f = \sup_{n\geqslant 0} E_n |f - E_{n-1} f|. \tag{1.6}$$

DEFINITION 1.2. Let $p \in [1,\infty)$ and $\delta \in [0,\infty)$. For $f \in L_{1,\mathrm{loc}}$, let

$$\|f\|_{F^\delta_{L_{p,\lambda}}} = \|M^\sharp_\delta f\|_{L_{p,\lambda}},$$

and define

$$F^\delta_{L_{p,\lambda}} = \{f \in L_{p,\mathrm{loc}} : \|M^\sharp_\delta f\|_{L_{p,\lambda}} < \infty\}.$$

If $\lambda = -1/p$ and the number of the elements in $A(\mathscr{F}_0)$ is finite, then we use the symbol $F^\delta_{p,\infty}$ instead of $F^\delta_{L_{p,\lambda}}$ because it coincides with the martingale Triebel-Lizorkin space introduced in [17].

Our main results are the following:

THEOREM 1.1. *Let* $\alpha > 0$, $1 < p < q < \infty$, $-1/p \leqslant \lambda < 0$, $\delta \geqslant 0$, *and*

$$\lambda + \alpha + \delta < 0, \quad p\lambda = q(\lambda + \alpha + \delta).$$

*Assume that* $\{\mathscr{F}_n\}_{n \geqslant 0}$ *is regular and that* $b$ *is in* $L_{1,\mathrm{loc}}$ *and satisfies*

$$\|b\|_{\mathscr{L}_{1,\delta}(\mathscr{F}_0)} := \sup_{B \in A(\mathscr{F}_0)} \frac{1}{P(B)^{1+\delta}} \int_B |b - E_0 b| \, dP < \infty. \tag{1.7}$$

(i) *If* $b \in \mathscr{L}_{1,\delta}$, *then the commutator* $[b, I_\alpha]$ *is bounded from* $L_{p,\lambda}$ *to* $L_{q,\lambda+\alpha+\delta}$ *and*

$$\|[b, I_\alpha] f\|_{L_{q,\lambda+\alpha+\delta}} \leqslant C \|b\|_{\mathscr{L}_{1,\delta}} \|f\|_{L_{p,\lambda}}, \quad f \in L_{p,\lambda},$$

*where the constant* $C$ *is independent of* $b$ *and* $f$.

(ii) *Conversely, if* $[b, I_\alpha]$ *is bounded from* $L_{p,\lambda}$ *to* $L_{q,\lambda+\alpha+\delta}$ *with the operator norm* $\|[b, I_\alpha]\|_{L_{p,\lambda} \to L_{q,\lambda+\alpha+\delta}}$, *then* $b \in \mathscr{L}_{1,\delta}$ *and*

$$\|b\|_{\mathscr{L}_{1,\delta}} \leqslant C\left(\|[b, I_\alpha]\|_{L_{p,\lambda} \to L_{q,\lambda+\alpha+\delta}} + \|b\|_{\mathscr{L}_{1,\delta}(\mathscr{F}_0)}\right),$$

*where the constant* $C$ *is independent of* $b$.

REMARK 1.1. (i) If $b \in \mathscr{L}_{1,\delta}$, then $b$ satisfies (1.7) with $\|b\|_{\mathscr{L}_{1,\delta}(\mathscr{F}_0)} \leqslant \|b\|_{\mathscr{L}_{1,\delta}}$.

(ii) If the number of the elements in $A(\mathscr{F}_0)$ is finite, then $L_{1,\mathrm{loc}} = L_1$ and every $b \in L_1$ satisfies (1.7) with $\|b\|_{\mathscr{L}_{1,\delta}(\mathscr{F}_0)} \leqslant 2(\min_{B \in A(\mathscr{F}_0)} P(B))^{-1-\delta} \|b\|_{L_1}$.

THEOREM 1.2. *Let* $\alpha > 0$, $1 < p < q < \infty$, $-1/p \leqslant \lambda < 0$, $\delta \geqslant 0$, *and*

$$\lambda + \alpha < 0, \quad p\lambda = q(\lambda + \alpha).$$

*Assume that* $\{\mathscr{F}_n\}_{n \geqslant 0}$ *is regular and that* $b$ *is in* $L_{1,\mathrm{loc}}$ *and satisfies* (1.7).

(i) *If* $b \in \mathscr{L}_{1,\delta}$, *then the commutator* $[b, I_\alpha]$ *is bounded from* $L_{p,\lambda}$ *to* $F^{\delta}_{L_{q,\lambda+\alpha}}$ *and*

$$\|[b, I_\alpha] f\|_{F^{\delta}_{L_{q,\lambda+\alpha}}} \leqslant C \|b\|_{\mathscr{L}_{1,\delta}} \|f\|_{L_{p,\lambda}}, \quad f \in L_{p,\lambda},$$

*where the constant* $C$ *is independent of* $b$ *and* $f$.

(ii) *Conversely, if* $[b, I_\alpha]$ *is bounded from* $L_{p,\lambda}$ *to* $F^{\delta}_{L_{q,\lambda+\alpha}}$ *with the operator norm* $\|[b, I_\alpha]\|_{L_{p,\lambda} \to F^{\delta}_{L_{q,\lambda+\alpha}}}$, *then* $b \in \mathscr{L}_{1,\delta}$ *and*

$$\|b\|_{\mathscr{L}_{1,\delta}} \leqslant C\left(\|[b, I_\alpha]\|_{L_{p,\lambda} \to F^{\delta}_{L_{q,\lambda+\alpha}}} + \|b\|_{\mathscr{L}_{1,\delta}(\mathscr{F}_0)}\right),$$

*where the constant* $C$ *is independent of* $b$.

COROLLARY 1.3. *Let* $\alpha > 0$, $1 < p < q < \infty$ *and* $\delta \geqslant 0$. *Assume that* $\{\mathscr{F}_n\}_{n \geqslant 0}$ *is regular and* $b \in L_1$. *If the number of the elements in* $A(\mathscr{F}_0)$ *is finite, then, the following conditions are equivalent:*

(i) $b \in \mathscr{L}_{1,\delta}$.

(ii) $[b, I_\alpha]$ *is bounded from* $L_p$ *to* $F^{\delta}_{q,\infty}$, *if* $-1/p + \alpha = -1/q$.

(iii) $[b, I_\alpha]$ *is bounded from* $L_p$ *to* $L_q$, *if* $-1/p + \alpha + \delta = -1/q$.

Next we state the compactness of the commutators. Let $L_0$ be the set of all $\mathscr{F}$-measurable functions. Let

$$L = \{f \in L_0 : f \text{ is } \mathscr{F}_n\text{-measurable for some } n \geqslant 0\},$$

and define

$$\mathscr{W}_{1,\delta} = \overline{\mathscr{L}_{1,\delta} \cap L},$$

where $\overline{\mathscr{L}_{1,\delta} \cap L}$ stands for the closure of $\mathscr{L}_{1,\delta} \cap L$ in $\mathscr{L}_{1,\delta}$. Further, let $A(\mathscr{F}_0) = \{A_n\}_{n=1}^{\infty}$ and let $D_n = \bigcup_{k=1}^{n} A_k$. Let

$$L_C = \{f \in L_0 : f = \chi_{D_n} f \text{ for some } n \geqslant 0\},$$

and define

$$\mathscr{C}_{1,\delta} = \overline{\mathscr{L}_{1,\delta} \cap L \cap L_C},$$

where $\overline{\mathscr{L}_{1,\delta} \cap L \cap L_C}$ stands for the closure of $\mathscr{L}_{1,\delta} \cap L \cap L_C$ in $\mathscr{L}_{1,\delta}$. If the number of the elements in $A(\mathscr{F}_0)$ is finite, then $L_C = L_0$ and $\mathscr{W}_{1,\delta} = \mathscr{C}_{1,\delta}$.

THEOREM 1.4. *Let* $\alpha > 0$, $1 < p < q < \infty$, $-1/p \leqslant \lambda < 0$, $\delta \geqslant 0$, *and*

$$\lambda + \alpha + \delta < 0, \quad p\lambda = q(\lambda + \alpha + \delta).$$

*Assume that* $\{\mathscr{F}_n\}_{n \geqslant 0}$ *is regular and that* $b$ *is in* $\mathscr{L}_{1,\delta}$ *and satisfies*

$$\lim_{n \to \infty} \|b - b\chi_{D_n}\|_{\mathscr{L}_{1,\delta}(\mathscr{F}_0)} = 0. \tag{1.8}$$

*Then,* $b \in \mathscr{C}_{1,\delta}$ *if and only if the commutator* $[b, I_\alpha]$ *is compact from* $L_{p,\lambda}$ *to* $L_{q,\lambda+\alpha+\delta}$.

REMARK 1.2. (i) If $b \in \mathscr{C}_{1,\delta}$, then $b$ satisfies (1.8).

(ii) If the number of the elements in $A(\mathscr{F}_0)$ is finite, then $L_{1,\mathrm{loc}} = L_1$ and every $b \in L_1$ satisfies (1.8).

THEOREM 1.5. *Let* $\alpha > 0$, $1 < p < q < \infty$, $-1/p \leqslant \lambda < 0$, $\delta \geqslant 0$, *and*

$$\lambda + \alpha < 0, \quad p\lambda = q(\lambda + \alpha).$$

*Assume that* $\{\mathscr{F}_n\}_{n \geqslant 0}$ *is regular and that* $b$ *is in* $\mathscr{L}_{1,\delta}$ *and satisfies* (1.8). *Then,* $b \in \mathscr{C}_{1,\delta}$ *if and only if the commutator* $[b, I_\alpha]$ *is compact from* $L_{p,\lambda}$ *to* $F^{\delta}_{L_{q,\lambda+\alpha}}$.

COROLLARY 1.6. *Let* $\alpha > 0$, $1 < p < q < \infty$ *and* $\delta \geqslant 0$. *Assume that* $\{\mathscr{F}_n\}_{n \geqslant 0}$ *is regular and* $b \in L_1$. *If the number of the elements in* $A(\mathscr{F}_0)$ *is finite, then, the following conditions are equivalent:*

(i) $b \in \mathscr{W}_{1,\delta}$.

(ii) $[b, I_\alpha]$ *is compact from* $L_p$ *to* $F^{\delta}_{q,\infty}$, *if* $-1/p+\alpha = -1/q$.

(iii) $[b, I_\alpha]$ *is compact from* $L_p$ *to* $L_q$, *if* $-1/p+\alpha+\delta = -1/q$.

In the second section we recall the boundedness of the fractional integrals and prove the boundedness of the fractional maximal functions. In Sections 3 and 4 we show the Morrey norm estimate of the sharp maximal function and the pointwise estimate for the sharp maximal function of the commutator, respectively. Then, using these results, we prove the main results in Sections 5 and 6.

At the end of this section, we make some conventions. Throughout this paper, we always use $C$ to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as $C_p$, is dependent on the subscripts. If $f \leqslant Cg$, we then write $f \lesssim g$ or $g \gtrsim f$; and if $f \lesssim g \lesssim f$, we then write $f \sim g$.

## 2. Fractional integrals and fractional maximal functions

First we recall the boundedness of the maximal operator $M$ and the fractional integral $I_\alpha$ on Morrey spaces $L_{p,\lambda}$.

For a martingale $f = (f_n)_{n\geqslant 0}$ relative to $\{\mathscr{F}_n\}_{n\geqslant 0}$, the maximal functions are defined by

$$Mf = \sup_{n\geqslant 0} |f_n|, \quad M_n f = \sup_{0\leqslant m\leqslant n} |f_m|, \quad M^{(n)} f = \sup_{m\geqslant n} |f_m|.$$

For a function $f \in L_{p,\mathrm{loc}}$ with $p \in [1,\infty)$, let $f_n = E_n f$, $n \geqslant 0$. Then $(f_n)_{n\geqslant 0}$ is a martingale and $\lim_{n\to\infty} f_n = f$ in $L_p(B)$ for each $B \in A(\mathscr{F}_0)$. For this reason a function $f \in L_{1,\mathrm{loc}}$ and its corresponding martingale $(f_n)_{n\geqslant 0}$ with $f_n = E_n f$ will be denoted by the same symbol $f$. In this case, for $f \in L_{1,\mathrm{loc}}$,

$$Mf = \sup_{n\geqslant 0} |E_n f|, \quad M_n f = \sup_{0\leqslant m\leqslant n} |E_m f|, \quad M^{(n)} f = \sup_{m\geqslant n} |E_m f|.$$

It is known as Doob's inequality that (see for example [19, pages 20–21])

$$\|Mf\|_{L_p} \leqslant \frac{p}{p-1} \|f\|_{L_p}, \quad f \in L_p \ (p>1), \tag{2.1}$$

$$\|Mf\|_{\mathrm{w}L_1} \leqslant \|f\|_{L_1}, \quad f \in L_1. \tag{2.2}$$

Since $M^{(n)}(f\chi_B) = (M^{(n)} f)\chi_B$ for $B \in A(\mathscr{F}_n)$, we deduce

$$E[(M^{(n)} f)^p \chi_B]^{1/p} \leqslant \frac{p}{p-1} E[|f|^p \chi_B]^{1/p}, \quad f \in L_{p,\mathrm{loc}} \ (p>1), \tag{2.3}$$

$$\sup_{t>0} t E[\chi_{\{M^{(n)} f > t\}} \chi_B] \leqslant E[|f|\chi_B], \quad f \in L_{1,\mathrm{loc}}, \tag{2.4}$$

from (2.1) and (2.2), respectively. Furthermore, since $P(B)E_n[g]\chi_B = E[g\chi_B]\chi_B$ for any $g \in L_{1,\mathrm{loc}}$ and $B \in A(\mathscr{F}_n)$, we have

$$E_n[(M^{(n)}f)^p\chi_B]^{1/p}\chi_B \leqslant \frac{p}{p-1}E_n[|f|^p]^{1/p}\chi_B, \quad f \in L_{p,\mathrm{loc}}\ (p>1), \tag{2.5}$$

$$\sup_{t\geqslant 0} tP(B\cap\{M^{(n)}f>t\})\chi_B \leqslant P(B)E_n[|f|]\chi_B, \quad f \in L_{1,\mathrm{loc}} \tag{2.6}$$

from (2.3) and (2.4) respectively.

For the boundedness of $M$ on Morrey spaces, we have the following theorem:

THEOREM 2.1. ([11, 12]) *Let $1<p<\infty$ and $\lambda<0$. Then $M$ is bounded from $L_{p,\lambda}$ to itself.*

For the boundedness of $I_\alpha$ we have the following theorem:

THEOREM 2.2. ([11, 12]) *Let $1<p<q<\infty$, $-1/p\leqslant\lambda<\lambda+\alpha<0$ and $p\lambda = q(\lambda+\alpha)$. Assume that $\{\mathscr{F}_n\}_{n\geqslant 0}$ is regular. Then $I_\alpha$ is bounded from $L_{p,\lambda}$ to $L_{q,\lambda+\alpha}$.*

In the above theorem, if $\lambda=-1/p$ and $\mathscr{F}_0=\{\emptyset,\Omega\}$, then $-1/p+\alpha=-1/q$, $L_{p,\lambda}=L_p$ and $L_{q,\lambda+\alpha}=L_q$. Then we have the following corollary:

COROLLARY 2.3. ([11]) *Let $\mathscr{F}_0=\{\emptyset,\Omega\}$, $1<p<q<\infty$ and $-1/p+\alpha=-1/q$. Then $I_\alpha$ is bounded from $L_p$ to $L_q$.*

This boundedness proved by [3] in the case of dyadic martingale.

For $\alpha>0$ and for $f\in L_{1,\mathrm{loc}}$, let

$$I_\alpha^{(n)}f = I_\alpha f - E_n[I_\alpha f] = \sum_{k>n}(\beta_{k-1})^\alpha(E_kf-E_{k-1}f). \tag{2.7}$$

Then we have the following corollary:

COROLLARY 2.4. *Let $1<p<q<\infty$ and $-1/p+\alpha=-1/q$. Let $R$ be the constant in (1.4). Then there exists a positive constant $C_{p,q,R}$ such that, for all $n\geqslant 0$ and $B\in A(\mathscr{F}_n)$,*

$$\left(E_n\left[|I_\alpha^{(n)}f|^q\right]\right)^{1/q}\chi_B \leqslant C_{p,q,R}P(B)^\alpha\left(E_n[|f|^p]\right)^{1/p}\chi_B. \tag{2.8}$$

*Proof.* For $B\in A(\mathscr{F}_n)$, we denote $\{A\cap B : A\in\mathscr{F}\}$ and $\{A\cap B : A\in\mathscr{F}_k\}$ by $\mathscr{F}\cap B$ and $\mathscr{F}_k\cap B$ respectively. Note that $\mathscr{F}_n\cap B=\{\emptyset,B\}$. Then, on the probability space $(B,\mathscr{F}\cap B,P/P(B))$ with filtration $\{\mathscr{F}_k\cap B\}_{k\geqslant n}$, the fractional integral is defined by

$$I_\alpha^B f = (\beta_n/P(B))^\alpha(E_nf-0)+\sum_{k>n}(\beta_{k-1}/P(B))^\alpha(E_kf-E_{k-1}f) \quad \text{on } B.$$

By Corollary 2.3 with the fact $I_\alpha^{(n)} f = P(B)^\alpha I_\alpha^B(f - E_n f)$ on $B$, we have

$$\left(E_n\left[|I_\alpha^{(n)} f|^q\right]\right)^{1/q} = \|I_\alpha^{(n)} f\|_{L_q(B,P/P(B))} = P(B)^\alpha \|I_\alpha^B(f - E_n f)\|_{L_q(B,P/P(B))}$$
$$\leqslant CP(B)^\alpha \|f - E_n f\|_{L_p(B,P/P(B))} \leqslant 2CP(B)^\alpha \left(E_n[|f|^p]\right)^{1/p} \quad \text{on } B,$$

where $C$ is a constant depending only on $p$, $q$ and $R$. $\square$

We also note that, if $g$ is $\mathscr{F}_n$-measurable, then

$$I_\alpha^{(n)}(fg) = (I_\alpha^{(n)} f)g. \tag{2.9}$$

For $f \in L_{1,\mathrm{loc}}$, its fractional maximal function $M_\alpha f$ is defined by

$$M_\alpha f = \sup_{n \geqslant 0} (\beta_n)^\alpha |E_n f|. \tag{2.10}$$

As a corollary of Theorems 2.1 and 2.2 we have the boundedness of $M_\alpha$.

COROLLARY 2.5. *Assume that $\{\mathscr{F}_n\}_{n\geqslant 0}$ is regular. Let $\alpha > 0$, $1 < p < q < \infty$, $-1/p \leqslant \lambda < 0$. If $\lambda + \alpha < 0$ and $p\lambda = q(\lambda + \alpha)$, then $M_\alpha$ is bounded from $L_{p,\lambda}$ to $L_{q,\lambda+\alpha}$. If $\lambda + \alpha = 0$, then $M_\alpha$ is bounded from $L_{p,\lambda}$ to $L_\infty$.*

*Proof.* Let $\lambda + \alpha = 0$. Then, for any $B \in A(\mathscr{F}_n)$,

$$(\beta_n)^\alpha |E_n f| \chi_B \leqslant (\beta_n)^\alpha \left(\frac{1}{P(B)} \int_B |f|^p\right)^{1/p} \leqslant \|f\|_{L_{p,\lambda}}.$$

This shows that

$$\|M_\alpha f\|_{L_\infty} \leqslant \|f\|_{L_{p,\lambda}}.$$

Let $\lambda + \alpha < 0$ and $p\lambda = q(\lambda + \alpha)$. By a simple calculation, we have

$$\begin{aligned}
E_0(I_\alpha|f|) &= (\beta_0)^\alpha E_0|f| \geqslant (\beta_0)^\alpha |E_0 f|, \\
E_1(I_\alpha|f|) &= (\beta_0)^\alpha E_1|f| \geqslant (\beta_1)^\alpha |E_1 f|, \\
E_n(I_\alpha|f|) &= (\beta_0)^\alpha E_0|f| + \sum_{k=1}^n (\beta_{k-1})^\alpha (E_k|f| - E_{k-1}|f|) \\
&= \sum_{k=1}^n ((\beta_{k-1})^\alpha - (\beta_k)^\alpha) E_k|f| + (\beta_{n-1})^\alpha E_n|f| \\
&\geqslant (\beta_{n-1})^\alpha |E_n f| \geqslant (\beta_n)^\alpha |E_n f|, \quad \text{if } n \geqslant 2.
\end{aligned}$$

That is,

$$M_\alpha f \leqslant M(I_\alpha|f|).$$

Therefore, by Theorems 2.1 and 2.2, we have

$$\|M_\alpha f\|_{L_{q,\lambda+\alpha}} \leqslant C_{p,q} \|f\|_{L_{p,\lambda}}. \quad \square$$

## 3. Sharp maximal functions

Recall that the sharp maximal functions $M_\delta^\sharp f$ and $M^\sharp f$ are defined by (1.5) and (1.6), respectively.

For $f \in L_{1,\mathrm{loc}}$ and $B \in \bigcup_{n=0}^{\infty} A(\mathscr{F}_n)$, let

$$f_B = \frac{1}{P(B)} \int_B f\, dP. \tag{3.1}$$

First we show the following good $\lambda$ inequality.

PROPOSITION 3.1. *Let $f \in L_{1,\mathrm{loc}}$ and $B \in A(\mathscr{F}_n)$. Then, for any $\lambda \in [|f_B|, \infty)$ and $\theta \in (0,\infty)$,*

$$P(B \cap \{M^{(n)} f > 2\lambda, M^\sharp f \leqslant \theta\lambda\}) \leqslant \theta P(B \cap \{M^{(n)} f > \lambda\}). \tag{3.2}$$

*Proof.* Let $\tau_\lambda^{(n)}(f) = \inf\{m \geqslant n : |E_m f| > \lambda\}$, and let $\Omega_{\lambda,m}^B = B \cap \{\tau_\lambda^{(n)}(f) = m\}$. Then

$$B \cap \{M^{(n)} f > 2\lambda\} \subset B \cap \{M^{(n)} f > \lambda\} = \bigcup_{m \geqslant n} \Omega_{\lambda,m}^B.$$

Hence, it is enough to prove that, for each $m \geqslant n$,

$$P(\Omega_{\lambda,m}^B \cap \{M^{(n)} f > 2\lambda, M^\sharp f \leqslant \theta\lambda\}) \leqslant \theta P(\Omega_{\lambda,m}^B). \tag{3.3}$$

Note that $\Omega_{\lambda,n}^B = \emptyset$, since $|E_n f(\omega)| = |f_B| \leqslant \lambda$ if $\omega \in B$. Let $\Omega_{\lambda,m}^B \neq \emptyset$. Then there exist atoms $B_\nu \in A(\mathscr{F}_m)$, $\nu = 1, 2, \cdots$, such that

$$\Omega_{\lambda,m}^B = \bigcup_\nu B_\nu.$$

Hence, for (3.3) it is enough to prove that, for each $\nu$,

$$P(B_\nu \cap \{M^{(n)} f > 2\lambda, M^\sharp f \leqslant \theta\lambda\}) \leqslant \theta P(B_\nu). \tag{3.4}$$

Now, if $k \geqslant m$, then

$$\begin{aligned} |E_k f| \chi_{B_\nu} &= |E_k[f \chi_{B_\nu}]| \\ &\leqslant E_k[|(f - E_{m-1} f)\chi_{B_\nu}|] + |E_{m-1} f| \chi_{B_\nu} \\ &\leqslant M^{(m)}[|(f - E_{m-1} f)\chi_{B_\nu}|]\chi_{B_\nu} + \lambda \chi_{B_\nu}. \end{aligned}$$

Then

$$(M^{(m)} f)\chi_{B_\nu} \leqslant M^{(m)}[|(f - E_{m-1} f)\chi_{B_\nu}|]\chi_{B_\nu} + \lambda \chi_{B_\nu}. \tag{3.5}$$

If $\omega \in B_\nu$ and $M^{(n)} f(\omega) > 2\lambda$, then $M^{(m)} f(\omega) > 2\lambda$, and then

$$M^{(m)}[|(f - E_{m-1} f)\chi_{B_\nu}|](\omega) \geqslant M^{(m)} f(\omega) - \lambda > \lambda$$

by (3.5). Hence, we have

$$B_\nu \cap \{M^{(n)}f > 2\lambda\} \subset B_\nu \cap \{M^{(m)}[|(f - E_{m-1}f)\chi_{B_\nu}|] > \lambda\}.$$

By Doob's inequality (2.6) for $M^{(m)}$ on $B_\nu$, we have

$$\begin{aligned} P(B_\nu \cap \{M^{(n)}f > 2\lambda\})\chi_{B_\nu} &\leqslant P(B_\nu \cap \{M^{(m)}[|(f - E_{m-1}f)\chi_{B_\nu}|] > \lambda\})\chi_{B_\nu} \\ &\leqslant \frac{P(B_\nu)}{\lambda} E_m[|(f - E_{m-1}f)|]\chi_{B_\nu} \\ &\leqslant \frac{P(B_\nu)}{\lambda} (M^\sharp f)\chi_{B_\nu}. \end{aligned}$$

If $P(B_\nu \cap \{M^\sharp f(\omega) \leqslant \theta\lambda\}) = 0$, then the left hand side of (3.4) is zero. If $P(B_\nu \cap \{M^\sharp f(\omega) \leqslant \theta\lambda\}) > 0$, then we have $\operatorname{ess\,inf}_{\omega\in B_\nu} M^\sharp f(\omega) \leqslant \theta\lambda$ and

$$P(B_\nu \cap \{M^{(n)}f > 2\lambda, M^\sharp f \leqslant \theta\lambda\}) \leqslant \frac{P(B_\nu)}{\lambda} \operatorname*{ess\,inf}_{\omega\in B_\nu} M^\sharp f(\omega) \leqslant \theta P(B_\nu).$$

Therefore, we have (3.4) and the conclusion. □

Next, by using the good $\lambda$ inequality and ideas in [1] and [9], we show the following proposition.

PROPOSITION 3.2. *Assume that $\{\mathscr{F}_n\}_{n\geqslant 0}$ is regular. Let $f \in L_{p,\mathrm{loc}}$. Let $1 \leqslant p < \infty$ and $\lambda < 0$. If $M^\sharp f \in L_{p,\lambda}$, then $f \in L_{p,\lambda}$ and*

$$\|f\|_{L_{p,\lambda}} \leqslant C\|M^\sharp f\|_{L_{p,\lambda}}, \tag{3.6}$$

*where the constant $C$ is independent of $f$.*

To show the proposition we use the following two lemmas.

LEMMA 3.3. *Let $1 \leqslant p < \infty$, $\lambda < 0$ and $n \geqslant 0$. Let $f \in L_{p,\mathrm{loc}}$ and $B \in A(\mathscr{F}_n)$. If $M^\sharp f \in L_{p,\lambda}$, then*

$$\left(\int_B |f - f_B|^p \, dP\right)^{1/p} \leqslant C\left(\left(\int_B (M^\sharp f)^p \, dP\right)^{1/p} + P(B)^{1/p}|f_B|\right), \tag{3.7}$$

*where the constant $C$ is independent of $f$ and $B$.*

*Proof.* First we show that, for all $B \in A(\mathscr{F}_n)$,

$$\left(\int_B (M^{(n)}f)^p \, dP\right)^{1/p} \leqslant C\left(\left(\int_B (M^\sharp f)^p \, dP\right)^{1/p} + P(B)^{1/p}|f_B|\right). \tag{3.8}$$

For any $L > 2|f_B|$,

$$\begin{aligned}
&\int_0^L p\lambda^{p-1} P(B\cap\{M^{(n)}f>\lambda\})\,d\lambda \\
&= \int_0^{2|f_B|} p\lambda^{p-1} P(B\cap\{M^{(n)}f>\lambda\})\,d\lambda \\
&\quad + \int_{2|f_B|}^{L} p\lambda^{p-1} P(B\cap\{M^{(n)}f>\lambda\})\,d\lambda \\
&\leqslant (2|f_B|)^p P(B) + 2^p \int_{|f_B|}^{L/2} p\lambda^{p-1} P(B\cap\{M^{(n)}f>2\lambda\})\,d\lambda.
\end{aligned}$$

By the good $\lambda$ inequality (3.2) we have

$$\begin{aligned}
&2^p \int_{|f_B|}^{L/2} p\lambda^{p-1} P(B\cap\{M^{(n)}f>2\lambda\})\,d\lambda \\
&\leqslant 2^p\theta \int_{|f_B|}^{L/2} p\lambda^{p-1} P(B\cap\{M^{(n)}f>\lambda\})\,d\lambda \\
&\quad + 2^p \int_{|f_B|}^{L/2} p\lambda^{p-1} P(B\cap\{M^{\sharp}f>\theta\lambda\})\,d\lambda \\
&\leqslant 2^p\theta \int_0^{L} p\lambda^{p-1} P(B\cap\{M^{(n)}f>\lambda\})\,d\lambda \\
&\quad + 2^p\theta^{-p} \int_0^{\infty} p\lambda^{p-1} P(B\cap\{M^{\sharp}f>\theta\lambda\})\,d\lambda.
\end{aligned}$$

Then, for small $\theta > 0$,

$$\begin{aligned}
&(1-2^p\theta)\int_0^L p\lambda^{p-1} P(B\cap\{M^{(n)}f>\lambda\})\,d\lambda \\
&\leqslant (2|f_B|)^p P(B) + 2^p\theta^{-p} \int_0^{\infty} p\lambda^{p-1} P(B\cap\{M^{\sharp}f>\lambda\})\,d\lambda.
\end{aligned}$$

Letting $L\to\infty$, we have (3.8).

On the other hand, noting that $\lim_{n\to\infty} E_n f = f$ in $L_p(B)$, we have

$$\begin{aligned}
\left(\int_B |f-f_B|^p\,dP\right)^{1/p} &\leqslant \left(\int_B |f|^p\,dP\right)^{1/p} + P(B)^{1/p}|f_B| \\
&\leqslant \left(\int_B (M^{(n)}f)^p\,dP\right)^{1/p} + P(B)^{1/p}|f_B|.
\end{aligned}$$

Combining this with (3.8), we have the conclusion. □

LEMMA 3.4. ([11], Lemma 3.3) *Let $\{\mathscr{F}_n\}_{n\geqslant 0}$ be regular. Then every sequence*

$$B_0 \supset B_1 \supset \cdots \supset B_n \supset \cdots, \quad B_n \in A(\mathscr{F}_n),$$

*has the following property: For each* $n \geqslant 1$,

$$B_n = B_{n-1} \text{ or } (1+1/R)P(B_n) \leqslant P(B_{n-1}) \leqslant RP(B_n),$$

*where* $R$ *is the constant in* (1.4).

*Proof of Proposition* 3.2. Let $R$ be the constant in (1.4). First we show that there exists a positive constant $C_{\lambda,R}$ such that, for any atom $B$,

$$|f_B| \leqslant C_{\lambda,R} P(B)^{\lambda} \|M^{\sharp} f\|_{L_{p,\lambda}}. \tag{3.9}$$

Note that

$$\begin{aligned}
\operatorname*{ess\,inf}_{\omega \in B} M^{\sharp} f(\omega) &\leqslant \frac{1}{P(B)} \int_B M^{\sharp} f \, dP \\
&\leqslant \left( \frac{1}{P(B)} \int_B (M^{\sharp} f)^p \, dP \right)^{1/p} \\
&\leqslant P(B)^{\lambda} \|M^{\sharp} f\|_{L_{p,\lambda}}.
\end{aligned}$$

If $B \in A(\mathscr{F}_0)$, then

$$\begin{aligned}
|f_B| &\leqslant \frac{1}{P(B)} \int_B |f - E_{-1} f| \, dP \leqslant \operatorname*{ess\,inf}_{\omega \in B} M^{\sharp} f(\omega) \\
&\leqslant P(B)^{\lambda} \|M^{\sharp} f\|_{L_{p,\lambda}},
\end{aligned}$$

since $E_{-1} f = 0$. Let $B \in A(\mathscr{F}_n)$ with $n \geqslant 1$. Choose atoms $B_k \in A(\mathscr{F}_k)$, $k = 0, 1, \cdots, n$, such that $B = B_n \subset B_{n-1} \subset \cdots \subset B_0$. We may assume that $P(B_k) \neq P(B_{k-1})$, $k = 1, 2, \cdots, n$. Then, by Lemma 3.4,

$$\left(1 + \frac{1}{R}\right) P(B_k) \leqslant P(B_{k-1}) \leqslant RP(B_k).$$

Using the inequalities

$$\begin{aligned}
|f_{B_k} - f_{B_{k-1}}| &\leqslant \frac{1}{P(B_k)} \int_{B_k} |f - E_{k-1} f| \, dP \leqslant \operatorname*{ess\,inf}_{\omega \in B_k} M^{\sharp} f(\omega) \\
&\leqslant P(B_k)^{\lambda} \|M^{\sharp} f\|_{L_{p,\lambda}}, \quad k = 1, 2, \cdots, n,
\end{aligned}$$

and

$$\begin{aligned}
P(B_k)^{\lambda} &= \frac{1}{\log\left(P(B_k)/P(B_{k+1})\right)} \int_{P(B_{k+1})}^{P(B_k)} \frac{P(B_k)^{\lambda}}{t} \, dt \\
&\leqslant \frac{C_{\lambda}}{\log(1+1/R)} \int_{P(B_{k+1})}^{P(B_k)} \frac{t^{\lambda}}{t} \, dt, \quad k = 0, 1, \cdots n-1,
\end{aligned}$$

we have

$$\begin{aligned}
|f_B| &\leqslant |f_{B_0}| + \sum_{k=1}^{n} |f_{B_k} - f_{B_{k-1}}| \\
&\leqslant \sum_{k=0}^{n} P(B_k)^{\lambda} \|M^{\sharp} f\|_{L_{p,\lambda}} \\
&\leqslant \left( \frac{C_\lambda}{\log(1+1/R)} \int_{P(B_n)}^{P(B_0)} \frac{t^{\lambda}}{t}\, dt + P(B_n)^{\lambda} \right) \|M^{\sharp} f\|_{L_{p,\lambda}} \\
&\leqslant C_{\lambda,R} P(B_n)^{\lambda} \|M^{\sharp} f\|_{L_{p,\lambda}}.
\end{aligned}$$

This shows (3.9).

Then, combining (3.9) and Lemma 3.3, we have

$$\begin{aligned}
\left( \frac{1}{P(B)} \int_B |f|^p\, dP \right)^{1/p} &\leqslant \left( \frac{1}{P(B)} \int_B |f - f_B|^p\, dP \right)^{1/p} + |f_B| \\
&\lesssim P(B)^{\lambda} \|M^{\sharp} f\|_{L_{p,\lambda}},
\end{aligned}$$

which shows the conclusion. □

## 4. Pointwise estimate for the sharp maximal function

In this section we show the following proposition.

PROPOSITION 4.1. *Assume that $\{\mathscr{F}_n\}_{n\geqslant 0}$ is regular. Let $\delta > 0$, $1 < p \leqslant \infty$ and $1 < v < p$. Then there exists a positive constant $C$ such that, for all $b \in \mathscr{L}_{1,\delta}$ and $f \in L_{p,\mathrm{loc}}$,*

$$M^{\sharp}([b, I_\alpha] f) \leqslant C \|b\|_{\mathscr{L}_{1,\delta}} \Big( M_\delta(I_\alpha f) + (M_{(\alpha+\delta)v}(|f|^v))^{1/v} \Big), \tag{4.1}$$

*and*

$$M_\delta^{\sharp}([b, I_\alpha] f) \leqslant C \|b\|_{\mathscr{L}_{1,\delta}} \Big( M(I_\alpha f) + (M_{\alpha v}(|f|^v))^{1/v} \Big). \tag{4.2}$$

To prove the proposition we state two lemmas.

LEMMA 4.2. ([10, Theorem 2.9]) *Assume that $\{\mathscr{F}_n\}_{n\geqslant 0}$ is regular. Let $1 \leqslant p < \infty$ and $\delta \geqslant 0$. Then $\mathscr{L}_{p,\delta} = \mathscr{L}_{1,\delta}$ with equivalent norms.*

LEMMA 4.3. *Let $\alpha > 0$, and let $I_\alpha^{(n)}$ be as in (2.7), $n \geqslant 0$. Let $f \in L_{p,\mathrm{loc}}$ and $b \in L_{p',\mathrm{loc}}$ with $p, p' \in [1, \infty]$ and $1/p + 1/p' = 1$. Let $b_n = E_n b$. Then,*

$$\begin{aligned}
&E_n \big| [b, I_\alpha] f - E_n([b, I_\alpha] f) \big| \\
&\qquad \leqslant 2E_n \big| (b - b_n) I_\alpha^{(n)} f \big| + 2E_n \big| I_\alpha^{(n)}((b - b_n) f) \big| + |(I_\alpha f)_n| E_n |b - b_n|.
\end{aligned}$$

*Proof.* Noting that $b_n I_\alpha^{(n)} f = I_\alpha^{(n)}(b_n f)$, we have

$$\begin{aligned}
[b, I_\alpha]f &= [b, I_\alpha^{(n)}]f + [b, E_n I_\alpha]f \\
&= [b - b_n, I_\alpha^{(n)}]f + b(I_\alpha f)_n - (I_\alpha(bf))_n.
\end{aligned}$$

Hence, we obtain

$$\left|[b, I_\alpha]f - E_n([b, I_\alpha]f)\right| \leqslant \left|[b - b_n, I_\alpha^{(n)}]f - E_n([b - b_n, I_\alpha^{(n)}]f)\right| + |(b - b_n)(I_\alpha f)_n|.$$

Therefore,

$$E_n\left|[b, I_\alpha]f - E_n[[b, I_\alpha]f]\right| \leqslant 2E_n\left|[b - b_n, I_\alpha^{(n)}]f\right| + |(I_\alpha f)_n| E_n |b - b_n|.$$

Since $[b - b_n, I_\alpha^{(n)}]f = (b - b_n) I_\alpha^{(n)} f - I_\alpha^{(n)}((b - b_n)f)$, we have the conclusion. $\square$

*Proof of Proposition* 4.1. Let $g = [b, I_\alpha]f$. Then

$$E_n\left|g - E_{n-1}[g]\right| \leqslant R E_{n-1}\left|g - E_{n-1}[g]\right|$$

by the regularity assumption on $\{\mathscr{F}_n\}_{n \geqslant 0}$. Hence,

$$\begin{aligned}
M_\delta^\sharp g &\leqslant (\beta_0)^{-\delta} E_0|g| + R^{1+\delta} \sup_{n \geqslant 0} (\beta_n)^{-\delta} E_n\left|g - E_n[g]\right|, \\
M^\sharp g &\leqslant E_0|g| + R \sup_{n \geqslant 0} E_n\left|g - E_n[g]\right|.
\end{aligned}$$

Let $b_n = E_n b$. By Lemma 4.3, it is enough to show that, for all $n \geqslant 0$ and all $B \in A(\mathscr{F}_n)$,

$$\begin{aligned}
&\left(2E_n\left|(b - b_n) I_\alpha^{(n)} f\right| + 2E_n\left|I_\alpha^{(n)}((b - b_n)f)\right| + |(I_\alpha f)_n| E_n |b - b_n|\right)\chi_B \\
&\leqslant \begin{cases} CP(B)^\delta \|b\|_{\mathscr{L}_{1,\delta}} \left(M(I_\alpha f) + (M_{\alpha v}(|f|^v))^{1/v}\right)\chi_B, \\ C\|b\|_{\mathscr{L}_{1,\delta}} \left(M_\delta(I_\alpha f) + (M_{(\alpha+\delta)v}(|f|^v))^{1/v}\right)\chi_B, \end{cases}
\end{aligned}$$

and that, for all $B \in A(\mathscr{F}_0)$,

$$E_0|g|\chi_B \leqslant \begin{cases} CP(B)^\delta \|b\|_{\mathscr{L}_{1,\delta}} \left(M(I_\alpha f) + (M_{\alpha v}(|f|^v))^{1/v}\right)\chi_B, \\ C\|b\|_{\mathscr{L}_{1,\delta}} \left(M_\delta(I_\alpha f) + (M_{(\alpha+\delta)v}(|f|^v))^{1/v}\right)\chi_B. \end{cases}$$

Let $n \geqslant 0$ and $B \in A(\mathscr{F}_n)$. Choose $p_1$ and $v$ such that $1 < p_1 < v < p$, and let $-1/p_1 + \alpha = -1/q_1$ and $1/p_1 = 1/u + 1/v$. Then $1 < p_1 < q_1 < \infty$ and $1 < u, v < \infty$. From Hölder's inequality and the boundedness (2.8) of $I_\alpha^{(n)}$ it follows that, for any

$B \in A(\mathscr{F}_n)$,

$$
\begin{aligned}
E_n\big|(b-b_n)I_\alpha^{(n)} f\big|\chi_B &\leqslant \Big(E_n\big[|(b-b_n)|^{q_1'}\big]\Big)^{1/q_1'} \Big(E_n\big[|I_\alpha^{(n)} f|^{q_1}\big]\Big)^{1/q_1} \chi_B \\
&\leqslant CP(B)^\delta \|b\|_{\mathscr{L}_{q_1',\delta}} P(B)^\alpha \left(E_n\big[|f|^{p_1}\big]\right)^{1/p_1} \chi_B \\
&\leqslant CP(B)^\delta \|b\|_{\mathscr{L}_{1,\delta}} P(B)^\alpha \left(E_n[|f|^v]\right)^{1/v} \chi_B \\
&= \begin{cases} CP(B)^\delta \|b\|_{\mathscr{L}_{1,\delta}} \left(P(B)^{\alpha v} E_n[|f|^v]\right)^{1/v} \chi_B, \\ C\|b\|_{\mathscr{L}_{1,\delta}} \Big(P(B)^{(\alpha+\delta)v} E_n[|f|^v]\Big)^{1/v} \chi_B. \end{cases}
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
E_n\big|I_\alpha^{(n)}((b-b_n)f)\big|\chi_B &\leqslant \Big(E_n\big[|I_\alpha^{(n)}((b-b_n)f)|^{q_1}\big]\Big)^{1/q_1} \chi_B \\
&\leqslant CP(B)^\alpha \left(E_n\big[|(b-b_n)f)|^{p_1}\big]\right)^{1/p_1} \chi_B \\
&\leqslant CP(B)^\alpha \left(E_n\big[|b-b_n|^u\big]\right)^{1/u} \left(E_n\big[|f|^v\big]\right)^{1/v} \chi_B \\
&\leqslant CP(B)^{\alpha+\delta} \|b\|_{\mathscr{L}_{u,\delta}} (E_n[|f|^v])^{1/v} \chi_B \\
&= \begin{cases} CP(B)^\delta \|b\|_{\mathscr{L}_{1,\delta}} \left(P(B)^{\alpha v} E_n[|f|^v]\right)^{1/v} \chi_B, \\ C\|b\|_{\mathscr{L}_{1,\delta}} \Big(P(B)^{(\alpha+\delta)v} E_n[|f|^v]\Big)^{1/v} \chi_B, \end{cases}
\end{aligned}
$$

and

$$
|(I_\alpha f)_n| E_n|b-b_n|\chi_B \leqslant \|b\|_{\mathscr{L}_{1,\delta}} P(B)^\delta |E_n[I_\alpha f]|\chi_B \leqslant \begin{cases} P(B)^\delta \|b\|_{\mathscr{L}_{1,\delta}} M(I_\alpha f)\chi_B, \\ \|b\|_{\mathscr{L}_{1,\delta}} M_\delta(I_\alpha f)\chi_B. \end{cases}
$$

Next, we note that, for $B \in A(\mathscr{F}_0)$,

$$
E_0|g|\chi_B = E_0|[b-b_0, I_\alpha]f|\chi_B \leqslant \left(E_0\big|(b-b_0)I_\alpha f\big| + E_0\big|I_\alpha((b-b_0)f)\big|\right)\chi_B.
$$

By the same way as above, we obtain

$$
E_0|g|\chi_B \leqslant \begin{cases} CP(B)^\delta \|b\|_{\mathscr{L}_{1,\delta}} \left(P(B)^{\alpha v} E_0[|f|^v]\right)^{1/v} \chi_B, \\ C\|b\|_{\mathscr{L}_{1,\delta}} \Big(P(B)^{(\alpha+\delta)v} E_0[|f|^v]\Big)^{1/v} \chi_B. \end{cases}
$$

These show the conclusion. $\square$

## 5. Proofs of Theorems 1.1 and 1.2

*Proof of Theorem* 1.1. (i) Let $b \in \mathscr{L}_{1,\delta}$ and $f \in L_{p,\lambda}$. Then by Proposition 4.1 we have

$$
M^\sharp([b, I_\alpha]f) \leqslant C\|b\|_{\mathscr{L}_{1,\delta}} \Big(M_\delta(I_\alpha f) + (M_{(\alpha+\delta)v}(|f|^v))^{1/v}\Big).
$$

Take $p_0$ such that $p\lambda = p_0(\lambda+\alpha) = q(\lambda+\alpha+\delta)$. Then $I_\alpha$ is bounded from $L_{p,\lambda}$ to $L_{p_0,\lambda+\alpha}$ and $M_\delta$ is bounded from $L_{p_0,\lambda+\alpha}$ to $L_{q,\lambda+\alpha+\delta}$. That is,

$$\|M_\delta(I_\alpha f)\|_{L_{q,\lambda+\alpha+\delta}} \lesssim \|f\|_{L_{p,\lambda}}.$$

On the other hand, from $p\lambda = q(\lambda+\alpha+\delta)$ it follows that $(p/v)\lambda v = (q/v)(\lambda v + (\alpha+\delta)v)$, that is, $M_{(\alpha+\delta)v}$ is bounded from $L_{p/v,\lambda v}$ to $L_{q/v,(\lambda+\alpha+\delta)v}$. Then

$$\begin{aligned}\|(M_{(\alpha+\delta)v}(|f|^v))^{1/v}\|_{L_{q,\lambda+\alpha+\delta}} &= \Big(\|M_{(\alpha+\delta)v}(|f|^v)\|_{L_{q/v,(\lambda+\alpha+\delta)v}}\Big)^{1/v} \\ &\lesssim \Big(\||f|^v\|_{L_{p/v,\lambda v}}\Big)^{1/v} = \|f\|_{L_{p,\lambda}}.\end{aligned}$$

Combining these and Proposition 3.2, we have

$$\|[b,I_\alpha]f\|_{L_{q,\lambda+\alpha+\delta}} \lesssim \|M^\sharp([b,I_\alpha]f)\|_{L_{q,\lambda+\alpha+\delta}} \lesssim \|b\|_{\mathscr{L}_{1,\delta}}\|f\|_{L_{p,\lambda}}.$$

This is the conclusion. $\square$

*Proof of Theorem* 1.2. (i) Let $b \in \mathscr{L}_{1,\delta}$ and $f \in L_{p,\lambda}$. Then by Proposition 4.1 we have

$$M^\sharp_\delta([b,I_\alpha]f) \leqslant C\|b\|_{\mathscr{L}_{1,\delta}}\Big(M(I_\alpha f) + (M_{\alpha v}(|f|^v))^{1/v}\Big).$$

Since $I_\alpha$ is bounded from $L_{p,\lambda}$ to $L_{q,\lambda+\alpha}$ and $M$ is bounded from $L_{q,\lambda+\alpha}$ to itself, we have

$$\|M(I_\alpha f)\|_{L_{q,\lambda+\alpha}} \lesssim \|f\|_{L_{p,\lambda}}.$$

On the other hand, from $p\lambda = q(\lambda+\alpha)$ it follows that $(p/v)\lambda v = (q/v)(\lambda v + \alpha v)$, that is, $M_{\alpha v}$ is bounded from $L_{p/v,\lambda v}$ to $L_{q/v,(\lambda+\alpha)v}$. Then

$$\begin{aligned}\|(M_{\alpha v}(|f|^v))^{1/v}\|_{L_{q,\lambda+\alpha}} &= \Big(\|M_{\alpha v}(|f|^v)\|_{L_{q/v,(\lambda+\alpha)v}}\Big)^{1/v} \\ &\lesssim \Big(\||f|^v\|_{L_{p/v,\lambda v}}\Big)^{1/v} = \|f\|_{L_{p,\lambda}}.\end{aligned}$$

Therefore, we have

$$\|[b,I_\alpha]f\|_{F^\delta_{L_{q,\lambda+\alpha}}} = \|M^\sharp_\delta([b,I_\alpha]f)\|_{L_{q,\lambda+\alpha}} \lesssim \|b\|_{\mathscr{L}_{1,\delta}}\|f\|_{L_{p,\lambda}}.$$

This is the conclusion. $\square$

Next, to prove Theorem 1.1 (ii) and Theorem 1.2 (ii) we show the following two lemmas.

LEMMA 5.1. *Let $b \in L_{1,\mathrm{loc}}$. Let $B \in A(\mathscr{F}_n)$, $n \geqslant 0$, and let $f \in L_\infty$ with $f\chi_B = 0$. Then,*

$$\chi_B E_n\big|[b,I_\alpha]f - E_n[[b,I_\alpha]f]\big| = \chi_B|I_\alpha f|E_n|b - E_n b|. \tag{5.1}$$

*Proof.* From the assumption, it follows that

$$I_\alpha f = E_n[I_\alpha f],\ I_\alpha(fb) = E_n[I_\alpha(fb)] \quad \text{on } B.$$

Hence, we have

$$[b,I_\alpha]f = bE_n[I_\alpha f] - E_n[I_\alpha(bf)] \quad \text{on } B.$$

Therefore,

$$[b,I_\alpha]f - E_n([b,I_\alpha]f) = (b-E_nb)E_n[I_\alpha f] = (b-E_nb)I_\alpha f \quad \text{on } B,$$

and

$$E_n\big|[b,I_\alpha]f - E_n[[b,I_\alpha]f]\big| = |I_\alpha f|E_n|b-E_nb| \quad \text{on } B,$$

which shows the conclusion. □

LEMMA 5.2. *Assume that $\{\mathscr{F}_n\}_{n\geqslant 0}$ is regular. Let $B\in A(\mathscr{F}_n)$, $n\geqslant 2$, and let*

$$B = B_n \subset B_{n-1} \subset \ldots \subset B_0, \quad B_k \in A(\mathscr{F}_k).$$

*Assume that $0\leqslant \ell < m < n$, where*

$$m = \max\{k : P(B_n) < P(B_k)\}, \quad \ell = \max\{k : P(B_m) < P(B_k)\}.$$

*Let $h = \chi_{B_m\setminus B}$ and let $b\in L_{1,\mathrm{loc}}$. Then*

$$C_R\chi_B P(B)^\alpha E_n|b-E_nb| \leqslant \chi_B E_n\big|[b,I_\alpha]h - E_n([b,I_\alpha]h)\big|, \tag{5.2}$$

*where $C_R$ is a positive constant depending only on the constant $R$ in (1.4).*

*Proof.* Since $h\chi_B=0$, we have $I_\alpha h = E_n[I_\alpha h]$ on $B$. Then, observing Lemma 3.4, we have

$$\begin{aligned}
\chi_B I_\alpha h &= \chi_B\sum_{k=0}^{n}(\beta_{k-1})^\alpha(E_kh - E_{k-1}h)\\
&= \chi_B\sum_{k=1}^{n-1}\left((\beta_{k-1})^\alpha - (\beta_k)^\alpha\right)E_kh + \chi_B(\beta_{n-1})^\alpha E_nh\\
&\geqslant \chi_B\left((\beta_\ell)^\alpha - (\beta_m)^\alpha\right)E_mh\\
&= \chi_B\big(P(B_\ell)^\alpha - P(B_m)^\alpha\big)\left(1-\frac{P(B)}{P(B_m)}\right)\\
&\geqslant C_R\chi_B P(B)^\alpha,
\end{aligned} \tag{5.3}$$

where $C_R$ is a positive constant depending only on $R$. Hence, by Lemma 5.1, we have

$$C_R\chi_B P(B)^\alpha E_n|b-E_nb| \leqslant \chi_B I_\alpha h E_n|b-E_nb| = \chi_B E_n\big|[b,I_\alpha]h - E_n([b,I_\alpha]h)\big|.$$

This is the conclusion. □

*Proof of Theorem* 1.1. (ii) Let $n \geqslant 0$ and $B \in A(\mathscr{F}_n)$. Let $B_k \in A(\mathscr{F}_k)$ $(0 \leqslant k \leqslant n)$ such that

$$B = B_n \subset B_{n-1} \subset \ldots \subset B_0.$$

Let $N = N_B$ be the number of elements in the set $\{k : 1 \leqslant k \leqslant n, B_k \neq B_{k-1}\}$.
*Case* 1: $N = 0$.

In this case, $B = B_0 \in A(\mathscr{F}_0)$. Therefore,

$$\chi_B E_n|b - E_n b| = \chi_B E_0|b - E_0 b| \leqslant P(B)^{\delta} \|b\|_{\mathscr{L}_{1,\delta}(\mathscr{F}_0)}.$$

*Case* 2: $N = 1$.

In this case, there exists an integer $k$ such that $1 \leqslant k \leqslant n$ and that $B = B_k$, $B_0 = B_{k-1}$. Then, by Lemma 3.4, we have $P(B_0) \leqslant RP(B)$. Using the regularity assumption on $\{\mathscr{F}_n\}_{n \geqslant 0}$ again, we obtain

$$\begin{aligned}
\chi_B E_n|b - E_n b| &= \chi_B E_k|b - E_k b| \\
&\leqslant \chi_B(E_k|b - E_{k-1} b| + |E_{k-1} b - E_k b|) \\
&\leqslant 2\chi_B E_k|b - E_{k-1} b| \\
&\leqslant 2R\chi_B E_{k-1}|b - E_{k-1} b| \\
&\leqslant 2R\chi_{B_0} E_0|b - E_0 b| \\
&\leqslant 2RP(B_0)^{\delta} \|b\|_{\mathscr{L}_{1,\delta}(\mathscr{F}_0)} \\
&\leqslant 2R^{1+\delta} P(B)^{\delta} \|b\|_{\mathscr{L}_{1,\delta}(\mathscr{F}_0)}.
\end{aligned}$$

*Case* 3: $N \geqslant 2$.

In this case, we can apply Lemma 5.2. Let $h$ be as in Lemma 5.2. Then, using the boundedness of $[b, I_\alpha]$ from $L_{p,\lambda}$ to $L_{q,\lambda+\alpha+\delta}$ and $\|h\|_{L_{p,\lambda}} \lesssim P(B)^{-\lambda}$, we have

$$\begin{aligned}
\chi_B E_n\big|[b, I_\alpha]h - E_n([b, I_\alpha]h)\big| &\leqslant 2\chi_B E_n(|[b, I_\alpha]h|^q)^{1/q} \\
&\leqslant 2P(B)^{\lambda+\alpha+\delta} \|[b, I_\alpha]h\|_{L_{q,\lambda+\alpha+\delta}} \\
&\lesssim P(B)^{\lambda+\alpha+\delta} \|[b, I_\alpha]\|_{L_{p,\lambda} \to L_{q,\lambda+\alpha+\delta}} \|h\|_{L_{p,\lambda}} \\
&\lesssim P(B)^{\delta+\alpha} \|[b, I_\alpha]\|_{L_{p,\lambda} \to L_{q,\lambda+\alpha+\delta}}.
\end{aligned}$$

Hence, by (5.2) we have

$$\chi_B E_n|b - E_n b| \lesssim P(B)^{\delta} \|[b, I_\alpha]\|_{L_{p,\lambda} \to L_{q,\lambda+\alpha+\delta}}.$$

Thus, we have the conclusion. □

*Proof of Theorem* 1.2. (ii) Let $n \geqslant 0$ and $B \in A(\mathscr{F}_n)$. Define $N = N_B$ be the same as in the proof of Theorem 1.1 (ii).

If $N = 0$ or $N = 1$, then by the same as in the proof of Theorem 1.1 (ii), we have

$$\chi_B E_n|b - E_n b| \lesssim P(B)^{\delta} \|b\|_{\mathscr{L}_{1,\delta}(\mathscr{F}_0)}.$$

We now consider the case $N \geqslant 2$. Let $h$ be as in Lemma 5.2. Then, using the boundedness of $[b,I_\alpha]$ from $L_{p,\lambda}$ to $F^{\delta}_{q,L_{q,\lambda+\alpha}}$ and $\|h\|_{L_{p,\lambda}} \lesssim P(B)^{-\lambda}$, we have

$$\begin{aligned}
\chi_B E_n\big|[b,I_\alpha]h - E_n([b,I_\alpha]h)\big| &\lesssim P(B)^{\delta} \operatorname*{ess\,inf}_{\omega\in B} M^{\sharp}_{\delta}([b,I_\alpha]h) \\
&\leqslant \frac{P(B)^{\delta}}{P(B)} \left(\int_B \left[M^{\sharp}_{\delta}([b,I_\alpha]h)\right]^q\right)^{1/q} \\
&\leqslant P(B)^{\lambda+\alpha+\delta} \|M^{\sharp}_{\delta}([b,I_\alpha]h)\|_{L_{q,\lambda+\alpha}} \\
&\lesssim P(B)^{\lambda+\alpha+\delta} \|[b,I_\alpha]\|_{L_{p,\lambda}\to F^{\delta}_{L_{q,\lambda+\alpha}}} \|h\|_{L^{p,\lambda}} \\
&\lesssim P(B)^{\alpha+\delta} \|[b,I_\alpha]\|_{L_{p,\lambda}\to F^{\delta}_{L_{q,\lambda+\alpha}}}.
\end{aligned}$$

Hence, by (5.2) we have

$$\chi_B E_n|b - E_n b| \lesssim P(B)^{\delta} \|[b,I_\alpha]\|_{L_{p,\lambda}\to F^{\delta}_{L_{q,\lambda+\alpha}}},$$

which shows the conclusion. □

## 6. Proofs of Theorems 1.4 and 1.5

Recall that $A(\mathscr{F}_0) = \{A_n\}_{n=1}^{\infty}$ and that $D_n = \bigcup_{k=1}^{n} A_k$. First we show several lemmas to prove Theorems 1.4 and 1.5.

LEMMA 6.1. *Let $\delta \geqslant 0$ and let $f \in \mathscr{L}_{1,\delta}$. Then, $f$ belongs to $\mathscr{W}_{1,\delta}$ if and only if*

$$\lim_{n\to\infty} \|f - E_n f\|_{\mathscr{L}_{1,\delta}} = 0. \tag{6.1}$$

*Proof.* It is easy to see that

$$\|E_n f\|_{\mathscr{L}_{1,\delta}} \leqslant \|f\|_{\mathscr{L}_{1,\delta}} \tag{6.2}$$

for all $n \geqslant 0$. Hence, we have $E_n f \in \mathscr{L}_{1,\delta} \cap L$. Therefore, (6.1) implies $f \in \mathscr{W}_{1,\delta}$.

For the converse, let $f \in \mathscr{W}_{1,\delta}$. Then, for any $\varepsilon > 0$, there exists $g \in \mathscr{L}_{1,\delta}$ such that $g$ is $\mathscr{F}_{n_0}$-measurable for some $n_0 \geqslant 0$ and that $\|f-g\|_{\mathscr{L}_{1,\delta}} < \varepsilon$. In this case, if $n \geqslant n_0$, then

$$\|f - E_n f\|_{\mathscr{L}_{1,\delta}} \leqslant \|f-g\|_{\mathscr{L}_{1,\delta}} + \|E_n(g-f)\|_{\mathscr{L}_{1,\delta}} \leqslant 2\|f-g\|_{\mathscr{L}_{1,\delta}} \leqslant 2\varepsilon.$$

This shows the conclusion. □

Since $\mathscr{C}_{1,\delta} = \overline{\mathscr{L}_{1,\delta} \cap L \cap L_C} = \overline{\mathscr{W}_{1,\delta} \cap L_C}$, we have the following corollary:

COROLLARY 6.2. *Let $\delta \geqslant 0$ and let $f \in \mathscr{L}_{1,\delta}$. Then, $f$ belongs to $\mathscr{C}_{1,\delta}$ if and only if*

$$\lim_{n\to\infty} \|f - \chi_{D_n} E_n f\|_{\mathscr{L}_{1,\delta}} = 0. \tag{6.3}$$

LEMMA 6.3. *Let $\delta \geqslant 0$ and let $f \in \mathscr{L}_{1,\delta}(\mathscr{F}_0)$. Assume that $\{\mathscr{F}_n\}_{n\geqslant 0}$ is regular. Then*

$$\|E_n f\|_{\mathscr{L}_{1,\delta}} \leqslant (2R^{1+\delta})^{n-1}\|f\|_{\mathscr{L}_{1,\delta}(\mathscr{F}_0)}. \tag{6.4}$$

*Proof.* Let $g = E_n f$. Then $g - E_k g = 0$ for $k \geqslant n$. For each $B \in A(\mathscr{F}_k)$, $1 \leqslant k \leqslant n-1$,

$$\begin{aligned}\chi_B E_k|g - E_k g| &\leqslant \chi_B(E_k|g - E_{k-1}g| + |E_{k-1}g - E_k g|)\\ &\leqslant 2\chi_B E_k|g - E_{k-1}g| \leqslant 2R\chi_B E_{k-1}|g - E_{k-1}g|.\end{aligned}$$

Then, taking $B_0 \in A(\mathscr{F}_0)$ such that $B \subset B_0$, we have

$$\chi_B E_k|g - E_k g| \leqslant (2R)^k P(B_0)^\delta \|g\|_{\mathscr{L}_{1,\delta}(\mathscr{F}_0)} \leqslant (2R^{1+\delta})^k P(B)^\delta \|g\|_{\mathscr{L}_{1,\delta}(\mathscr{F}_0)},$$

which shows

$$\begin{aligned}\|g\|_{\mathscr{L}_{1,\delta}} &= \sup_{0\leqslant k\leqslant n-1}\ \sup_{B\in A(\mathscr{F}_k)} \frac{1}{P(B)^{1+\delta}}\int_B |g - E_k g|\,dP\\ &\leqslant (2R^{1+\delta})^{n-1}\|g\|_{\mathscr{L}_{1,\delta}(\mathscr{F}_0)} \leqslant (2R^{1+\delta})^{n-1}\|f\|_{\mathscr{L}_{1,\delta}(\mathscr{F}_0)}.\end{aligned}$$

This is the conclusion. □

LEMMA 6.4. *Let $\delta \geqslant 0$ and let $f \in \mathscr{L}_{1,\delta}$. Assume that $\{\mathscr{F}_n\}_{n\geqslant 0}$ is regular and that $f$ satisfies* (1.8). *Then, $f$ belongs to $\mathscr{C}_{1,\delta}$ if and only if $f$ belongs to $\mathscr{W}_{1,\delta}$.*

*Proof.* If $f \in \mathscr{C}_{1,\delta}$, then $f \in \mathscr{W}_{1,\delta}$ by the definition. Conversely, let $f \in \mathscr{W}_{1,\delta}$. Then, by Lemma 6.1, for any $\varepsilon > 0$, there exists $n \geqslant 0$ such that $\|f - E_n f\|_{\mathscr{L}_{1,\delta}} < \varepsilon$. By the assumption (1.8), there exists $k_0 \geqslant 0$ such that, if $k \geqslant k_0$, then $\|f - \chi_{D_k} f\|_{\mathscr{L}_{1,\delta}(\mathscr{F}_0)} < \varepsilon/(2R^{1+\delta})^{n-1}$. That is, $\|E_n(f - \chi_{D_k} f)\|_{\mathscr{L}_{1,\delta}} < \varepsilon$ by (6.4). In this case,

$$\begin{aligned}\|f - \chi_{D_k} f\|_{\mathscr{L}_{1,\delta}} &\leqslant \|f - E_n f\|_{\mathscr{L}_{1,\delta}} + \|E_n f - \chi_{D_k} E_n f\|_{\mathscr{L}_{1,\delta}} + \|\chi_{D_k}(E_n f - f)\|_{\mathscr{L}_{1,\delta}}\\ &\leqslant \|f - E_n f\|_{\mathscr{L}_{1,\delta}} + \|E_n(f - \chi_{D_k} f)\|_{\mathscr{L}_{1,\delta}} + \|f - E_n f\|_{\mathscr{L}_{1,\delta}}\\ &< 3\varepsilon.\end{aligned}$$

This shows that $f \in \mathscr{C}_{1,\delta}$. Therefore, we have the desired conclusion. □

Next, we recall the notation used in the previous section. For $B \in A(\mathscr{F}_n)$, let $B_j \in A(\mathscr{F}_j)$, $j = 0, 1, \ldots, n$, such that

$$B = B_n \subset B_{n-1} \subset \cdots \subset B_0.$$

Denote by $N_B$ the number of the set $\{j : 1 \leqslant j \leqslant n, P(B_j) \neq P(B_{j-1})\}$.

LEMMA 6.5. *Assume that* $\{\mathscr{F}_n\}_{n\geqslant 0}$ *is regular. Let* $b$ *be in* $\mathscr{L}_{1,\delta}$ *and satisfy* (1.8) *with* $\|b\|_{\mathscr{L}_{1,\delta}}=1$. *For a positive constant* $c$, *choose an integer* $J$ *such that*

$$\|b-\chi_{D_J}b\|_{\mathscr{L}_{1,\delta}(\mathscr{F}_0)}\leqslant c/(8R^{1+\delta}).$$

*If* $B\in A(\mathscr{F}_n)$ *satisfies*

$$P(B)\leqslant R^{-2}\min\{P(A_j):A_j\in A(\mathscr{F}_0),1\leqslant j\leqslant J\} \tag{6.5}$$

*and*

$$\int_B|b-E_nb|\,dP\geqslant cP(B)^{1+\delta}/2, \tag{6.6}$$

*then* $N_B\geqslant 2$.

*Proof.* If $B\subset D_J$, then we see that $N_B\geqslant 2$ by (6.5) and the regularity of $\{\mathscr{F}_n\}_{n\geqslant 0}$. If $B\subset\Omega\setminus D_J$, we also conclude that $N_B\geqslant 2$. Indeed, if $N_B\leqslant 1$, then we have

$$\begin{aligned}\int_B|b-E_nb|\,dP&=\int_B\chi_{\Omega\setminus D_J}|b-E_nb|\,dP\\&\leqslant 2R^{1+\delta}P(B)^{1+\delta}\|b-\chi_{D_J}b\|_{\mathscr{L}_{1,\delta}(\mathscr{F}_0)}\leqslant cP(B)^{1+\delta}/4\end{aligned}$$

by the same way as in the proof of Theorem 1.1 (ii), which contradict (6.6). □

Further, we recall the following fact.

THEOREM 6.6. ([11, Theorem 5.8]) *Assume that* $\{\mathscr{F}_n\}_{n\geqslant 0}$ *is regular. Let* $\alpha>0$ *and* $\delta\geqslant 0$. *Then,*

$$I_\alpha\in B(\mathscr{L}_{1,\delta},\mathscr{L}_{1,\delta+\alpha}). \tag{6.7}$$

We now prove Theorem 1.4.

*Proof of Theorem* 1.4. Let $b$ be in $\mathscr{L}_{1,\delta}$ and satisfy (1.8).
*Part* 1. We first show that, if $b\in\mathscr{C}_{1,\delta}$, then $[b,I_\alpha]$ is compact from $L_{p,\lambda}$ to $L_{q,\lambda+\alpha+\delta}$. Note that $b\in\mathscr{C}_{1,\delta}$ implies that $b$ satisfies (1.8). From

$$[\chi_{D_n}E_nb,I_\alpha]f=\chi_{D_n}E_n([E_n(b),I_\alpha]f),$$

we obtain that the range of the commutator $[\chi_{D_n}E_nb,I_\alpha]$ is finite dimensional, since the number of the elements in $\{B\in A(\mathscr{F}_n):B\subset D_n\}$ is finite for each $n\geqslant 0$ by the regularity on $\{\mathscr{F}_n\}_{n\geqslant 0}$. Besides, by Theorem 1.1,

$$\|[b,I_\alpha]-[\chi_{D_n}E_nb,I_\alpha]\|_{L_{p,\lambda}\to L_{q,\lambda+\alpha+\delta}}\leqslant C\|b-\chi_{D_n}E_nb\|_{\mathscr{L}_{1,\delta}}.$$

Combining this with Corollary 6.2, we obtain

$$\lim_{n\to\infty}\|[b,I_\alpha]-[\chi_{D_n}E_nb,I_\alpha]\|_{L_{p,\lambda}\to L_{q,\lambda+\alpha+\delta}}=0.$$

Thus, we have the compactness of $[b,I_\alpha]$ from $L_{p,\lambda}$ to $L_{q,\lambda+\alpha+\delta}$.
*Part* 2. For the converse, we show that, if $b \notin \mathscr{C}_{1,\delta}$, then $[b,I_\alpha]$ is not compact from $L_{p,\lambda}$ to $L_{q,\lambda+\alpha+\delta}$. Let $\|b\|_{\mathscr{L}_{1,\delta}} = 1$. By Lemma 6.4, $b \notin \mathscr{C}_{1,\delta}$ implies $b \notin \mathscr{W}_{1,\delta}$. Then, $c := \limsup_{n\to\infty} \|b - E_n b\|_{\mathscr{L}_{1,\delta}} > 0$ by Lemma 6.1. Hence, the set

$$N = \{n \geqslant 0 : \int_B |b - E_n b|\,dP \geqslant cP(B)^{1+\delta}/2 \text{ for some } B \in A(\mathscr{F}_n)\}$$

is infinite. By the fact $\lim_{n\to\infty} \max\{P(B) : B \in A(\mathscr{F}_n)\} = 0$ and Lemma 6.5, we can choose an increasing sequence $\{n_j\}_{j\geqslant 1}$ in $N$ and a sequence $\{B_{n_j}\}_{j\geqslant 1}$ of $\mathscr{F}_{n_j}$-atoms, inductively in the following manner. Let $n_1 = \min N$ and let $B_{n_1} \in A(\mathscr{F}_{n_1})$ satisfy

$$\int_{B_{n_1}} |b - E_{n_1} b|\,dP \geqslant cP(B_{n_1})^{1+\delta}/2.$$

Next, we choose $n_2 < n_3 < \dots$ and $B_{n_j} \in A(\mathscr{F}_{n_j})$, $j = 2,3,\dots$, satisfying

$$\int_{B_{n_j}} |b - E_{n_j} b|\,dP \geqslant cP(B_{n_j})^{1+\delta}/2, \tag{6.8}$$

$$P(B_{n_j}) < P(B_{n_{j-1}}), \tag{6.9}$$

$$P(B_{n_j})^{\lambda+\alpha} > \frac{2}{R^\lambda C_R} \|I_\alpha\|_{L_{p,\lambda}\to L_{q_1,\lambda+\alpha}} P(B_{n_{j-1}})^{\lambda+\alpha}, \tag{6.10}$$

$$P(B_{n_j})^{\lambda} > \frac{8}{cC_R} \|I_\alpha\|_{\mathscr{L}_{1,\delta}\to \mathscr{L}_{1,\alpha+\delta}} P(B_{n_{j-1}})^{\lambda}, \tag{6.11}$$

$$N_{B_{n_j}} \geqslant 2, \tag{6.12}$$

where $C_R$ is the constant in Lemma 5.2, $q_1$ is the number defined by $p\lambda = q_1(\lambda + \alpha)$, and, $\|I_\alpha\|_{L_{p,\lambda}\to L_{q_1,\lambda+\alpha}}$ and $\|I_\alpha\|_{\mathscr{L}_{1,\delta}\to \mathscr{L}_{1,\alpha+\delta}}$ are the operator norms determined by Theorems 2.2 and 6.6, respectively. Note that we do not assume $B_{n_j} \subset B_{n_{j-1}}$.

For each $n_j$ above, we define an integer $m_j$ by

$$m_j = \max\{m : m < n_j, B_{n_j} \subset B, P(B_{n_j}) \neq P(B), B \in A(\mathscr{F}_m)\}.$$

and let

$$h_j = P(B_{n_j})^\lambda (\chi_{B_{m_j}} - \chi_{B_{n_j}}).$$

By the regularity on $\{\mathscr{F}_n\}_{n\geqslant 0}$, it is easy to see that

$$\|h_j\|_{L_{p,\lambda}} \leqslant R^{-\lambda} \quad \text{for all} \quad j \geqslant 1. \tag{6.13}$$

Therefore, to prove that $[b,I_\alpha]$ is not compact, we only have to show

$$\inf_{k\neq j} \|[b,I_\alpha](h_j - h_k)\|_{L_{q,\lambda+\alpha+\delta}} > 0. \tag{6.14}$$

In the following we show (6.14). Let $k < j$. Then $h_j - h_k$ and $I_\alpha(h_j - h_k)$ are $\mathscr{F}_{n_j}$-measurable. Hence

$$\begin{aligned}
&\chi_{B_{n_j}} E_{n_j} \left| [b, I_\alpha](h_j - h_k) - E_{n_j}[[b, I_\alpha](h_j - h_k)] \right| \\
&= \chi_{B_{n_j}} E_{n_j} \Big[ |b I_\alpha(h_j - h_k) - E_{n_j}[b] I_\alpha(h_j - h_k) \\
&\qquad\qquad - \{ I_\alpha[b(h_j - h_k)] - E_{n_j}(I_\alpha[b(h_j - h_k)]) \}| \Big] \\
&\geqslant \chi_{B_{n_j}} |I_\alpha(h_j - h_k)| E_{n_j} |b - E_{n_j} b| - \chi_{B_{n_j}} |h_j - h_k| E_{n_j} \left| I_\alpha(b) - E_{n_j}(I_\alpha(b)) \right|. \qquad (6.15)
\end{aligned}$$

In the above we use the property (2.9). By the same way as (5.3) in the proof of Lemma 5.2, we have

$$\chi_{B_{n_j}} |I_\alpha h_j| \geqslant \chi_{B_{n_j}} C_R P(B_{n_j})^{\lambda+\alpha}. \qquad (6.16)$$

Further, by (6.13) and (6.10),

$$\begin{aligned}
\chi_{B_{n_j}} |I_\alpha h_k| &\leqslant \chi_{B_{n_j}} R^{-\lambda} \|I_\alpha\|_{L_{p,\lambda} \to L_{q_1,\lambda+\alpha}} P(B_{n_k})^{\lambda+\alpha} \qquad (6.17) \\
&\leqslant \chi_{B_{n_j}} R^{-\lambda} \|I_\alpha\|_{L_{p,\lambda} \to L_{q_1,\lambda+\alpha}} P(B_{n_{j-1}})^{\lambda+\alpha}. \\
&\leqslant \chi_{B_{n_j}} \frac{C_R}{2} P(B_{n_j})^{\lambda+\alpha}.
\end{aligned}$$

Combining (6.8), (6.16) and (6.17), we have

$$\chi_{B_{n_j}} |I_\alpha(h_j - h_k)| E_{n_j} |b - E_{n_j} b| \geqslant \chi_{B_{n_j}} \frac{cC_R}{4} P(B_{n_j})^{\lambda+\alpha+\delta}. \qquad (6.18)$$

Besides, using Theorem 6.6, we have

$$\begin{aligned}
\chi_{B_{n_j}} |h_j - h_k| E_{n_j} |I_\alpha(b) - E_{n_j}(I_\alpha(b))| &= \chi_{B_{n_j}} |h_k| E_{n_j} |I_\alpha(b) - E_{n_j}(I_\alpha(b))| \qquad (6.19) \\
&\leqslant \chi_{B_{n_j}} P(B_{n_k})^\lambda P(B_{n_j})^{\alpha+\delta} \|I_\alpha(b)\|_{\mathscr{L}_{1,\alpha+\delta}} \\
&\leqslant \chi_{B_{n_j}} P(B_{n_k})^\lambda P(B_{n_j})^{\alpha+\delta} \|I_\alpha\|_{\mathscr{L}_{1,\delta} \to \mathscr{L}_{1,\alpha+\delta}} \\
&\leqslant \chi_{B_{n_j}} \frac{cC_R}{8} P(B_{n_j})^{\lambda+\alpha+\delta}.
\end{aligned}$$

Thus, combining (6.15), (6.18) and (6.19), we have

$$\chi_{B_{n_j}} E_{n_j} \left| [b, I_\alpha](h_j - h_k) - E_{n_j}[[b, I_\alpha](h_j - h_k)] \right| \geqslant \chi_{B_{n_j}} \frac{cC_R}{8} P(B_{n_j})^{\lambda+\alpha+\delta}, \qquad (6.20)$$

that is,

$$\|[b, I_\alpha](h_j - h_k)\|_{L_{q,\lambda+\alpha+\delta}} \gtrsim \frac{cC_R}{8}.$$

Therefore, we have (6.14). The proof is complete. $\square$

*Proof of Theorem* 1.5. The necessity part is obtained by the same way as in Theorem 1.4

For the converse part, let $b$ be in $\mathscr{L}_{1,\delta}$ and satisfy (1.8) with $\|b\|_{\mathscr{L}_{1,\delta}} = 1$. We show that, if $b \notin \mathscr{W}_{1,\delta}$, then $[b, I_\alpha]$ is not compact from $L_{p,\lambda}$ to $F^{\delta}_{L_{q,\lambda+\alpha}}$. We take a sequence $(h_j)_{j\geqslant 1}$ by a similar way to the proof of Theorem 1.4. Then, we obtain that, for $k < j$,

$$\begin{aligned}
&\chi_{B_{n_j}} P(B_{n_j})^{\lambda+\alpha+\delta} \|M^{\sharp}_{\delta}([b,I_\alpha](h_j - h_k))\|_{L_{q,\lambda+\alpha}} \\
&\geqslant \chi_{B_{n_j}} P(B_{n_j})^{\delta} \left( \frac{1}{P(B_{n_j})} \int_{B_{n_j}} \left[ M^{\sharp}_{\delta}([b,I_\alpha](h_j - h_k)) \right]^q dP \right)^{1/q} \\
&\geqslant \chi_{B_{n_j}} P(B_{n_j})^{\delta} \operatorname*{ess\,inf}_{\omega\in B} M^{\sharp}_{\delta}([b,I_\alpha](h_j - h_k)) \\
&\gtrsim \chi_{B_{n_j}} E_{n_j} \left| [b,I_\alpha](h_j - h_k) - E_{n_j}[[b,I_\alpha](h_j - h_k)] \right| \\
&\geqslant \chi_{B_{n_j}} \frac{cC_R}{8} P(B_{n_j})^{\lambda+\alpha+\delta}.
\end{aligned}$$

In the above we use (6.20). This shows the conclusion. $\square$

*Acknowledgement.* The authors would like to thank the referee for her/his careful reading and useful comments. The first author was supported by Grant-in-Aid for Scientific Research (B), No. 15H03621, Japan Society for the Promotion of Science. The second author was supported by Grant-in-Aid for Scientific Research (C), No. 16K05203, Japan Society for the Promotion of Science.

REFERENCES

[1] R. ARAI AND E. NAKAI, *Commutators of Calderón-Zygmund and generalized fractional integral operators on generalized Morrey spaces*, Rev. Mat. Complut. **31** (2018), Vol. 2, 287–331, `https://doi.org/10.1007/s13163-017-0251-4`.

[2] S. CHANILLO, *A note on commutators*, Indiana Univ. Math. J. **31** (1982), no. 1, 7–16.

[3] J.-A. CHAO AND H. OMBE, *Commutators on Dyadic Martingales*, Proc. Japan Acad. **61**, Ser. A (1985), 35–38.

[4] J.-A. CHAO AND L. PENG, *Schatten classes and commutators on simple martingales*, Colloq. Math. **71** (1996), no. 1, 7–21.

[5] Y. CHEN, Y. DING AND X. WANG, *Compactness of commutators of Riesz potential on Morrey spaces*, Potential Anal. **30** (2009), no. 4, 301–313.

[6] G. DI FAZIO AND M. A. RAGUSA, *Commutators and Morrey spaces*, Boll. Un. Mat. Ital. A (7) **5** (1991), no. 3, 323–332.

[7] Z. HAO AND Y. JIAO, *Fractional integral on martingale Hardy spaces with variable exponents*, Fract. Calc. Appl. Anal. **18** (2015), no. 5, 1128–1145.

[8] S. JANSON, *BMO and commutators of martingale transforms*, Ann. Inst. Fourier (Grenoble) **31** (1981), no. 1, viii, 265–270.

[9] Y. KOMORI-FURUYA, *Local good-$\lambda$ estimate for the sharp maximal function and weighted Morrey space*, J. Funct. Spaces **2015**, Art. ID 651825, 4 pp.

[10] T. MIYAMOTO, E. NAKAI AND G. SADASUE, *Martingale Orlicz-Hardy spaces*, Math. Nachr. **285** (2012), no. 5–6, 670–686.

[11] E. NAKAI AND G. SADASUE, *Martingale Morrey-Campanato spaces and fractional integrals*, J. Funct. Spaces Appl. **2012** (2012), Article ID 673929, 29 pages, `doi:10.1155/2012/673929`.

[12] E. NAKAI AND G. SADASUE, *Characterizations of boundedness for generalized fractional integrals on martingale Morrey spaces*, Math. Inequalities Appl. **20** (2017), no. 4, 929–947, doi:10.7153/mia-2017-20-58.

[13] E. NAKAI, G. SADASUE AND Y. SAWANO, *Martingale Morrey-Hardy and Campanato-Hardy Spaces*, J. Funct. Spaces Appl. **2013** (2013), Article ID 690258, 14 pages, doi:10.1155/2013/690258.

[14] M. PALUSZYŃSKI, *Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss*, Indiana Univ. Math. J. **44** (1995), no. 1, 1–17.

[15] M. A. RAGUSA AND A. SCAPELLATO, *Mixed Morrey spaces and their applications to partial differential equations*, Nonlinear Anal. **151** (2017), 51–65.

[16] G. SADASUE, *Fractional integrals on martingale Hardy spaces for* $0 < p \leqslant 1$, Mem. Osaka Kyoiku Univ. Ser. III Nat. Sci. Appl. Sci. 60 (2011), no. 1 1–7.

[17] G. SADASUE, *Martingale Besov spaces and martingale Triebel-Lizorkin spaces*, to appear in Sci. Math. Jpn.

[18] C. WATARI, *Multipliers for Walsh Fourier series*, Tohoku Math. J. **16** (1964), 239–251.

[19] F. WEISZ, *Martingale Hardy spaces and their applications in Fourier analysis*, Lecture Notes in Mathematics, 1568, Springer-Verlag, Berlin, 1994.

(Received March 12, 2018)

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