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HAUSDORFF OPERATOR IN LEBESGUE SPACES

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Abstract. We study the boundedness of the Hausdorff operator in various types of Lebesgue spaces, e.g. weighted spaces, variable exponent and grand Lebesgue spaces. The results are illustrated by a number of examples.

1. Introduction

Let ϕ be a non-negative fixed kernel defined on $\mathbb{R}_+$, i.e. $\phi \in L_1^{loc}(0,\infty)$, then the Hausdorff operator is defined in the following way

$$H_\phi(f)(x) = \int_0^\infty \frac{\phi(y)}{y} f\left(\frac{x}{y}\right) dy.$$

This integral operator is deeply rooted in the study of one-dimensional Fourier analysis and has become an essential part of modern harmonic analysis. In particular, it is closely related to the summability of the classical Fourier series [13]. Many important operators in analysis are special cases of the Hausdorff operator, by taking suitable choice of ϕ. For example:

1. if $\phi(t) = \frac{\chi_{(1,\infty)}(t)}{t}$, we get the Hardy operator

$$Hf(x) = H_\phi f(x) = \frac{1}{x}\int_0^x f(t)\, dt;$$

2. if $\phi(t) = \chi_{(0,1)}(t)$, we have the adjoint Hardy operator[1]

$$H^* f(x) = H_\phi f(x) = \int_x^\infty \frac{f(t)}{t}\, dt;$$

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[1] This operator is also called Bellman operator.

3. if $\phi(t)=\frac{1}{\max\{1,t\}}$, we get the Hardy-Littlewood-Pólya operator

$$Pf(x)=H_\phi f(x)=\frac{1}{x}\int_0^x f(t)\,dt+\int_x^\infty \frac{f(t)}{t}\,dt;$$

4. if $\phi(t)=\gamma(1-t)^{\gamma-1}\chi_{(0,1)}(t)$ with $\gamma>0$, we obtain the Cesàro operator

$$\mathscr{C}_\gamma f(x)=H_\phi f(x)=\gamma\int_x^\infty \frac{(t-x)^{\gamma-1}}{t^\gamma}f(t)\,dt;$$

5. if $\beta>0$ and $\phi(t)=\frac{1}{\Gamma(\beta)}\frac{\left(1-\frac{1}{t}\right)^{\beta-1}}{t}\chi_{(1,\infty)}(t)$, then the Riemann-Liouville fractional derivative has the following form

$$D_\beta f(x)=x^\beta H_\phi f(x)=\frac{1}{\Gamma(\beta)}\int_0^x (x-t)^{\beta-1}f(t)\,dt.$$

The Hausdorff operator has been extensively studied in recent years, particularly its boundedness on the Lebesgue space as well as on the Hardy space [16, 13, 14], (see also [11, 15, 19] for H_p-estimates, where $0<p<1$). We also refer to [3, 4, 12, 18] for some recent work in this vein.

It is well known that if we consider the Hausdorff operator on Banach space $(E,\|.\|)$ of functions on $\mathbb{R}_+$ with the norm satisfying the norm scaling property[2], then H_ϕ is bounded if

$$\int_0^\infty \frac{\phi(t)}{t^{1-\alpha}}dt<\infty. \tag{1}$$

Of course the classical Lebesgue space L^p with $p>1$ satisfies the norm scaling property with $\alpha=\frac{1}{p}$. Therefore, the Hausdorff operator H_ϕ is bounded in L^p if (1) is satisfied. Moreover, it turns out that condition (1) is also necessary for continuity of H_ϕ on L^p, where $\alpha=\frac{1}{p}$. The main objective of the paper is to study the continuity of the Hausdorff operator in various type of Lebesgue spaces: weighted Lebesgue spaces, grand Lebesgue spaces, variable exponent spaces as well as quasi-Banach spaces.

The remainder of the paper is structured as follows. In Section 2, we introduce various types of function spaces: weighted Lebesgue spaces, variable exponent and grand Lebesgue spaces. Our principal assertions, concerning the continuity of the Hausdorff operator in mentioned spaces are formulated and proven in Section 3. We give necessary and sufficient conditions for the boundedness of the Hausdorff operator. The results are illustrated by a number of examples. It is worth pointing out that in some cases the optimal constants in the inequalities are given.

[2] $(E,\|.\|)$ satisfies the norm scaling property if there exists α such that $\|u(./t)\|=t^\alpha\|u\|$ for all $u\in E, t>0$.

2. Preliminaries

We recall some notation and basic facts about function spaces.

Let ω be a weight function on $\mathbb{R}_+$, i.e. $\omega \in L_1^{loc}(\mathbb{R}_+)$ and $\omega(x) > 0$ almost everywhere. The *weighted Lebesgue space* $L_{p,\omega}(\mathbb{R}_+)$ is the class of all measurable functions f defined on $\mathbb{R}_+$ such that

$$\|f\|_{L_{p,\omega}(\mathbb{R}_+)} = \left(\int_0^\infty |f(x)|^p \, \omega(x) \, dx \right)^{\frac{1}{p}} < \infty.$$

Let U be a measurable subset of $\mathbb{R}$ with positive measure. By a variable exponent we shall mean a measurable function $p : U \to [1,\infty]$, we put

$$p_+(U) = \operatorname*{ess\,sup}_{x\in U} p(x), \quad p_-(U) = \operatorname*{ess\,inf}_{x\in U} p(x).$$

If $U = \mathbb{R}$ or $\mathbb{R}_+$, we shall simply write p_+, p_-. In the paper, we assume that the variable exponent functions are bounded, i.e. $1 \leqslant p_- \leqslant p_+ < \infty$. The *variable exponent Lebesgue space* $L_{p(\cdot)}(U)$ consists of those measurable functions $f : U \to \mathbb{R}$, for which the semimodular

$$\rho_{p(\cdot)}(f) = \int_U |f(x)|^{p(x)} \, dx$$

is finite. This is a Banach space with respect to the Luxemburg-Nakano norm (see e.g. [5, 6])

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \; \rho_{p(\cdot)} \left(\frac{f}{\lambda} \right) \leqslant 1 \right\}.$$

Variable Lebesgue space is a special case of the Musielak-Orlicz spaces. Such kind of spaces find many applications, for example in nonlinear elastic mechanics [21], electrorheological fluids [20] or image restoration [17]. If the variable exponent p is constant, then $L_{p(\cdot)}(U)$ is an ordinary Lebesgue space. It is needed to pass between norm and semimodular very often. In general, there are no functional relationship between norm and modular, but we have the following inequalities

$$\min\left\{ \left(\rho_{p(\cdot)}(f)\right)^{\frac{1}{p_-}}, \left(\rho_{p(\cdot)}(f)\right)^{\frac{1}{p_+}} \right\} \leqslant \|f\|_{L_{p(\cdot)}} \leqslant \max\left\{ \left(\rho_{p(\cdot)}(f)\right)^{\frac{1}{p_-}}, \left(\rho_{p(\cdot)}(f)\right)^{\frac{1}{p_+}} \right\}. \quad (2)$$

By $\mathscr{M}_{0,\infty}(\mathbb{R}_+)$ we denote the set of all measurable functions $p : \mathbb{R}_+ \to \mathbb{R}_+$ such that

i) $1 \leqslant p_- \leqslant p(x) \leqslant p_+ < \infty$, for $x \in \mathbb{R}_+$,

ii) $|p(x) - p(0)| \leqslant \dfrac{C}{\ln \frac{1}{x}}$, for $0 < x \leqslant \frac{1}{2}$,

iii) there exists p_∞ such that $|p(x) - p_\infty| \leqslant \dfrac{C}{\ln x}$, for $x > 2$, where constant C depends on p.

Let $I \subset \mathbb{R}_+$ be a measurable set with finite measure. The *grand Lebesgue space* $L_{p)}(I)$, introduced by Iwaniec and Sbordone in [10], is the class of all measurable functions $f : I \to \mathbb{R}$ such that

$$\|f\|_{L_{p)}(I)} := \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \left(\frac{1}{|I|}\int_I |f(x)|^{p-\varepsilon}dx\right)^{\frac{1}{p-\varepsilon}}$$

is finite. For properties and some applications of grand Lebesgue spaces we refer the reader to the papers [2, 8, 9].

3. Main results

In this section of our paper we state and prove our principal assertions.

3.1. Weighted spaces

Let v, w be weight function, we introduce the following quantities

$$A_{\sup} = \int_0^\infty \frac{\phi(y)}{y} y^{\frac{1}{p}} \left(\sup_{x>0} \frac{w(xy)}{v(x)}\right)^{\frac{1}{p}} dy,$$

$$A_{\inf} = \int_0^\infty \frac{\phi(y)}{y} y^{\frac{1}{p}} \left(\inf_{x>0} \frac{w(xy)}{v(x)}\right)^{\frac{1}{p}} dy.$$

THEOREM 1. *Let $1 < p < \infty$, v, w be weight functions defined on $\mathbb{R}_+$, and $\phi \in L_1^{loc}(\mathbb{R}_+)$. Then*

i) If $A_{\sup} < \infty$, then $H_\phi : L_{p,v}(\mathbb{R}_+) \to L_{p,w}(\mathbb{R}_+)$ is bounded and

$$\|H_\phi f\|_{L_{p,w}(\mathbb{R}_+)} \leqslant A_{\sup} \|f\|_{L_{p,v}(\mathbb{R}_+)};$$

ii) If $H_\phi : L_{p,v}(\mathbb{R}_+) \to L_{p,w}(\mathbb{R}_+)$ is bounded, then

$$\|H_\phi\|_{L_{p,v}(\mathbb{R}_+)\to L_{p,w}(\mathbb{R}_+)} \geqslant A_{\inf}.$$

Proof. $i)$ Applying the generalized Minkowski inequality and changing variables, we have

$$\|H_\phi f\|_{L_{p,w}(\mathbb{R}_+)} = \left(\int_0^\infty w(x) \left(\int_0^\infty \frac{\phi(\frac{x}{y})}{y} f(y)dy\right)^p dx\right)^{\frac{1}{p}}$$

$$= \left(\int_0^\infty \left(\int_0^\infty \frac{\phi(y)}{y} f\left(\frac{x}{y}\right) w^{\frac{1}{p}}(x)dy\right)^p dx\right)^{\frac{1}{p}}$$

$$
\begin{aligned}
&\leqslant \int_0^\infty \frac{\phi(y)}{y}\left(\int_0^\infty f^p\left(\frac{x}{y}\right) w(x)dx\right)^{\frac{1}{p}} dy \\
&= \int_0^\infty \frac{\phi(y)}{y} y^{\frac{1}{p}}\left(\int_0^\infty f^p(x)w(xy)\,dx\right)^{\frac{1}{p}} dy \\
&= \int_0^\infty \frac{\phi(y)}{y} y^{\frac{1}{p}}\left(\int_0^\infty f^p(x)v(x)v^{-1}(x)w(xy)\,dx\right)^{\frac{1}{p}} dy \\
&\leqslant \left(\int_0^\infty \frac{\phi(y)}{y} y^{\frac{1}{p}}\left(\sup_{x>0}\frac{w(xy)}{v(x)}\right)^{\frac{1}{p}} dy\right)\left(\int_0^\infty f^p(x)v(x)\,dx\right)^{\frac{1}{p}} \\
&= A_{\sup}\|f\|_{L_{p,v}(\mathbb{R}_+)}.
\end{aligned}
$$

Thus,

$$
\|H_\phi f\|_{L_{p,w}(\mathbb{R}_+)} \leqslant A_{\sup}\|f\|_{L_{p,v}(\mathbb{R}_+)}.
$$

ii) Let us fix $0<\varepsilon<1$ and define the function

$$
f_\varepsilon(x) = x^{-\frac{1}{p}-\varepsilon} v(x)^{-\frac{1}{p}} \chi_{(1,\infty)}(x).
$$

Then, by straightforward calculations we get

$$
\|f_\varepsilon\|_{L_{p,v}(\mathbb{R}_+)} = \frac{1}{(\varepsilon p)^{\frac{1}{p}}}.
$$

On the other hand, we have

$$
H_\phi f_\varepsilon(x) = x^{-\frac{1}{p}-\varepsilon}\int_0^x \frac{\phi(y)}{y} y^{\frac{1}{p}+\varepsilon} v\left(\frac{x}{y}\right)^{-\frac{1}{p}} dy.
$$

Therefore,

$$
\begin{aligned}
\|H_\phi f_\varepsilon\|_{L_{p,w}(\mathbb{R}_+)} &= \left(\int_0^\infty x^{-1-\varepsilon p}\left(\int_0^x \frac{\phi(y)}{y} y^{\frac{1}{p}+\varepsilon}\left(\frac{w(x)}{v(\frac{x}{y})}\right)^{\frac{1}{p}} dy\right)^p dx\right)^{\frac{1}{p}} \\
&\geqslant \left(\int_{\frac{1}{\varepsilon}}^\infty x^{-1-\varepsilon p}\left(\int_0^{\frac{1}{\varepsilon}} \frac{\phi(y)}{y} y^{\frac{1}{p}+\varepsilon}\left(\frac{w(x)}{v(\frac{x}{y})}\right)^{\frac{1}{p}} dy\right)^p dx\right)^{\frac{1}{p}} \\
&\geqslant B(\varepsilon)\frac{\varepsilon^\varepsilon}{(\varepsilon p)^{\frac{1}{p}}},
\end{aligned}
$$

where

$$B(\varepsilon)=\int_0^{\frac{1}{\varepsilon}}\frac{\phi(y)}{y}y^{\frac{1}{p}+\varepsilon}\left(\inf_{x>0}\frac{w(x)}{v(\frac{x}{y})}\right)^{\frac{1}{p}}dy=\int_0^{\frac{1}{\varepsilon}}\frac{\phi(y)}{y}y^{\frac{1}{p}+\varepsilon}\left(\inf_{x>0}\frac{w(xy)}{v(x)}\right)^{\frac{1}{p}}dy.$$

Hence, we have

$$\|H_\phi\|_{L_{p,w}(\mathbb{R}_+)\to L_{p,v}(\mathbb{R}_+)}\geqslant\varepsilon^{\varepsilon}B(\varepsilon).$$

Finally, by virtue of the Fatou lemma we pass to the limit $\varepsilon\to 0$ and we get

$$\|H_\phi\|_{L_{p,w}(\mathbb{R}_+)\to L_{p,v}(\mathbb{R}_+)}\geqslant\int_0^{\infty}\frac{\phi(y)}{y}y^{\frac{1}{p}}\left(\inf_{x>0}\frac{w(xy)}{v(x)}\right)^{\frac{1}{p}}dy,$$

and this ends the proof of the theorem. □

From Theorem 1 we have that in the case of the Lebesgue space without weights the constant $A=A_{\sup}$ is optimal. Let us give some examples.

EXAMPLE 1. Let $1<p<\infty$, then $\|H\|_{L_p(\mathbb{R}_+)\to L_p(\mathbb{R}_+)}=p'$.

EXAMPLE 2. Let $1<p<\infty$, then $\|H^*\|_{L_p(\mathbb{R}_+)\to L_p(\mathbb{R}_+)}=p$.

EXAMPLE 3. Let $1<p<\infty$, then $\|P\|_{L_p(\mathbb{R}_+)\to L_p(\mathbb{R}_+)}=pp'$.

EXAMPLE 4. Let $1<p<\infty$ and $\gamma>0$, then using the basic properties of the Beta function, we get that $\|\mathscr{C}_\gamma\|_{L_p(\mathbb{R}_+)\to L_p(\mathbb{R}_+)}=\frac{\Gamma\left(\frac{1}{p}\right)\Gamma(\gamma+1)}{\Gamma\left(\gamma+\frac{1}{p}\right)}$.

EXAMPLE 5. Let $1<p<\infty$, $0<\beta<\frac{1}{p'}$, then $\|D_\beta\|_{L_{p,x^{\beta p}}(\mathbb{R}_+)\to L_p(\mathbb{R}_+)}=\frac{\Gamma\left(\frac{1}{p'}-\beta\right)}{\Gamma\left(\frac{1}{p'}\right)}$.

COROLLARY 1. *Let $1<p<\infty$ and v,w be weight functions defined on $\mathbb{R}_+$ such that*

$$\sup_{x>0}\frac{w(xy)}{v(x)}\leqslant C\inf_{x>0}\frac{w(xy)}{v(x)},$$

for some positive constant C. Then $H_\phi: L_{p,v}(\mathbb{R}_+)\to L_{p,w}(\mathbb{R}_+)$ is bounded if and only if $A_{\sup}<\infty$. Furthermore,

$$\frac{1}{C}A_{\sup}\leqslant\|H_\phi\|_{L_{p,v}(\mathbb{R}_+)\to L_{p,w}(\mathbb{R}_+)}\leqslant A_{\sup}.$$

EXAMPLE 6. Let $1<p<\infty$ and $v(x)=\frac{2+x}{1+x}$, $w(x)=1$, then

$$\frac{1}{2^{1/p}}\int_0^{\infty}\frac{\phi(y)}{y}y^{\frac{1}{p}}\,dy\leqslant\left\|H_\phi\right\|_{L_{p,v}(\mathbb{R}_+)\to L_p(\mathbb{R}_+)}\leqslant\int_0^{\infty}\frac{\phi(y)}{y}y^{\frac{1}{p}}\,dy.$$

COROLLARY 2. *Let* $1<p<\infty$, v *and* w *be weight functions on* $\mathbb{R}_+$. *If there exists a function* g *such that* $w(xy)=v(x)g(y)$, *then* $A_{\sup}=A_{\inf}$ *and*

$$\|H_\phi\|_{L_{p,v}(\mathbb{R}_+)\to L_{p,w}(\mathbb{R}_+)}=\int_0^\infty \frac{\phi(y)}{y}(yg(y))^{\frac{1}{p}}\,dy.$$

EXAMPLE 7. Let $v(x)=w(x)=x^\alpha$, then

$$\|H_\phi\|_{L_{p,v}(\mathbb{R}_+)\to L_{p,w}(\mathbb{R}_+)}=\int_0^\infty \frac{\phi(y)}{y}y^{\frac{\alpha+1}{p}}\,dy.$$

We have the generalization of Theorem 1.

THEOREM 2. *Let* $1<q<p<\infty$, v *and* w *be weight functions defined on* $\mathbb{R}_+$ *such that*

$$D=\int_0^\infty \frac{\phi(y)}{y}y^{\frac{1}{q}}\left(\int_0^\infty \frac{[w(xy)]^{\frac{p}{p-q}}}{[v(x)]^{\frac{q}{p-q}}}\,dx\right)^{\frac{p-q}{pq}}dy<\infty,$$

then

$$\|H_\phi f\|_{L_{q,w}(\mathbb{R}_+)}\leqslant D\|f\|_{L_{p,v}(\mathbb{R}_+)}.$$

Proof. By the change of variable formula and the generalized Minkowski inequality we get

$$\begin{aligned}
\|H_\phi f\|_{L_{q,w}(\mathbb{R}_+)} &= \left(\int_0^\infty w(x)\left(\int_0^\infty \frac{\phi(\frac{x}{y})}{y}f(y)dy\right)^q dx\right)^{\frac{1}{q}}\\
&= \left(\int_0^\infty\left(\int_0^\infty \frac{\phi(y)}{y}f\left(\frac{x}{y}\right)w^{\frac{1}{q}}(x)dy\right)^q dx\right)^{\frac{1}{q}}\\
&\leqslant \int_0^\infty \frac{\phi(y)}{y}\left(\int_0^\infty f^q\left(\frac{x}{y}\right)w(x)dx\right)^{\frac{1}{q}}dy\\
&= \int_0^\infty \frac{\phi(y)}{y}y^{\frac{1}{q}}\left(\int_0^\infty f^q(x)w(xy)dx\right)^{\frac{1}{q}}dy\\
&\leqslant \left(\int_0^\infty \frac{\phi(y)}{y}y^{\frac{1}{q}}\left(\int_0^\infty \frac{[w(xy)]^{\frac{p}{p-q}}}{[v(x)]^{\frac{q}{p-q}}}\,dx\right)^{\frac{p-q}{pq}}dy\right)\|f\|_{L_{p,v}(\mathbb{R}_+)}\\
&= D\|f\|_{L_{p,v}(\mathbb{R}_+)},
\end{aligned}$$

and this finishes the proof of the theorem. □

EXAMPLE 8. Let $1 < q < p < \infty$, $w(t) = t^{\alpha}$ and $v(t) = (1+t)^{\beta}$. If $\frac{q-p}{p} < \alpha < q-1$ and $\beta > \frac{(\alpha+1)p}{q} - 1$, then $H : L_{p,v}(\mathbb{R}_+) \mapsto L_{q,w}(\mathbb{R}_+)$ is bounded and

$$\|Hf\|_{L_{q,w}(\mathbb{R}_+)} \leqslant \frac{q}{q-\alpha-1} B^{\frac{p-q}{pq}} \left(\frac{\beta q - \alpha p}{p-q} - 1, \frac{\alpha p}{p-q} + 1 \right) \|f\|_{L_{p,v}(\mathbb{R}_+)}.$$

EXAMPLE 9. Let $1 < q < p < \infty$, $w(t) = t^{\alpha}$ and $v(t) = (1+t)^{\beta}$. If $\alpha > \frac{q-p}{p}$ and $\beta > \frac{(\alpha+1)p}{q} - 1$, then $H^* : L_{p,v}(\mathbb{R}_+) \mapsto L_{q,w}(\mathbb{R}_+)$ is bounded and

$$\|H^* f\|_{L_{q,w}(\mathbb{R}_+)} \leqslant \frac{q}{\alpha+1} B^{\frac{p-q}{pq}} \left(\frac{\beta q - \alpha p}{p-q} - 1, \frac{\alpha p}{p-q} + 1 \right) \|f\|_{L_{p,v}(\mathbb{R}_+)}.$$

EXAMPLE 10. Let $1 < q < p < \infty$, $\gamma > 0$, $w(t) = t^{\alpha}$ and $v(t) = (1+t)^{\beta}$. If $\alpha > \frac{q-p}{p}$ and $\beta > \frac{(\alpha+1)p}{q} - 1$, then $\mathscr{C}_\gamma : L_{p,v}(\mathbb{R}_+) \mapsto L_{q,w}(\mathbb{R}_+)$ is bounded and

$$\left\|\mathscr{C}_\gamma f\right\|_{L_{q,w}(\mathbb{R}_+)} \leqslant \gamma B\left(\gamma, \frac{\alpha+1}{q}\right) B^{\frac{p-q}{pq}} \left(\frac{\beta q - \alpha p}{p-q} - 1, \frac{\alpha p}{p-q} + 1 \right) \|f\|_{L_{p,v}(\mathbb{R}_+)}.$$

In the case of very special weights, we get the following improvement of Theorem 2.

THEOREM 3. *Let $1 < q < p < \infty$ and v be weight function defined on $\mathbb{R}_+$ such that $v(x)^{-\frac{1}{p-q}} \in L^q(\mathbb{R}_+)$ and there exists a function g such that*

$$v\left(\frac{x}{y}\right) \leqslant v(x)g(y) \quad \text{for } x, y \geqslant 0. \tag{3}$$

Then

$$\left(\int_0^\infty v(x)^{-\frac{q}{p-q}} dx\right)^{\frac{p-q}{pq}} \int_0^\infty \frac{\phi(y)}{y} g(y)^{-\frac{1}{p-q}} dy \leqslant \left\|H_\phi\right\|_{L_{p,v}(\mathbb{R}_+)\to L_q(\mathbb{R}_+)}$$
$$\leqslant \left(\int_0^\infty v(x)^{-\frac{q}{p-q}} dx\right)^{\frac{p-q}{pq}} \int_0^\infty \frac{\phi(y)}{y} y^{\frac{1}{q}} dy.$$

In particular, if $\int_0^\infty \frac{\phi(y)}{y} g(y)^{-\frac{1}{p-q}} dy = \int_0^\infty \frac{\phi(y)}{y} y^{\frac{1}{q}} dy$, then

$$\left\|H_\phi f\right\|_{L_{p,v}(\mathbb{R}_+)\to L_q(\mathbb{R}_+)} = \left(\int_0^\infty v(x)^{-\frac{q}{p-q}} dx\right)^{\frac{p-q}{pq}} \int_0^\infty \frac{\phi(y)}{y} y^{\frac{1}{q}} dy.$$

Proof. We define the function $f(x) = v(x)^{-\frac{1}{p-q}}$. It is easy to see that

$$\|f\|_{L_{p,v}(\mathbb{R}_+)} = \left(\int_0^\infty v(x)^{-\frac{q}{p-q}}\,dx\right)^{\frac{1}{p}}.$$

By inequality (3), we obtain

$$\begin{aligned} H_\phi f(x) &= \int_0^\infty \frac{\phi(y)}{y} v\left(\frac{x}{y}\right)^{-\frac{1}{p-q}} dy \\ &\geqslant \int_0^\infty \frac{\phi(y)}{y} g(y)^{-\frac{1}{p-q}} v(x)^{-\frac{1}{p-q}}\,dy \\ &= v(x)^{-\frac{1}{p-q}} \int_0^\infty \frac{\phi(y)}{y} g(y)^{-\frac{1}{p-q}}\,dy. \end{aligned}$$

Therefore, we get

$$\begin{aligned} \left\|H_\phi f\right\|_{L_q(\mathbb{R}_+)} &\geqslant \left(\int_0^\infty v(x)^{-\frac{q}{p-q}}\,dx\right)^{\frac{1}{q}} \int_0^\infty \frac{\phi(y)}{y} g(y)^{-\frac{1}{p-q}}\,dy \\ &= \left(\int_0^\infty v(x)^{-\frac{q}{p-q}}\,dx\right)^{\frac{p-q}{pq}} \left(\int_0^\infty v(x)^{-\frac{q}{p-q}}\,dx\right)^{\frac{1}{p}} \int_0^\infty \frac{\phi(y)}{y} g(y)^{-\frac{1}{p-q}}\,dy. \end{aligned}$$

Thus,

$$\left\|H_\phi\right\|_{L_{p,v}(\mathbb{R}_+)\to L_q(\mathbb{R}_+)} \geqslant \left(\int_0^\infty v(x)^{-\frac{q}{p-q}}\,dx\right)^{\frac{p-q}{pq}} \int_0^\infty \frac{\phi(y)}{y} g(y)^{-\frac{1}{p-q}}\,dy.$$

Finally, by virtue of Theorem 2 we can finish the proof of the theorem. □

EXAMPLE 11. Let $1 < q < p < 2q$ and $v(x) = 1 + x, g(y) = e^{1/y}\chi_{[0,1]}(y) + \chi_{(1,\infty)}(y)$, then condition (3) in Theorem 3 is satisfied. Therefore,

$$\begin{aligned} \left(\frac{p-q}{2q-p}\right)^{\frac{p-q}{pq}} \left(\int_0^1 \frac{\phi(y)}{y} e^{-\frac{1}{(p-q)y}}\,dy + \int_1^\infty \frac{\phi(y)}{y}\,dy\right) &\leqslant \left\|H_\phi\right\|_{L_{p,v}(\mathbb{R}_+)\to L_q(\mathbb{R}_+)} \\ &\leqslant \left(\frac{p-q}{2q-p}\right)^{\frac{p-q}{pq}} \int_0^\infty \frac{\phi(y)}{y} y^{\frac{1}{q}}\,dy. \end{aligned}$$

EXAMPLE 12. Let us assume that assumptions of Theorem 3 are satisfied, then for classical Hardy operator and for the adjoint Hardy operator we have

$$\left(\int_0^\infty v(x)^{\frac{-q}{p-q}}\,dx\right)^{\frac{p-q}{pq}} \int_1^\infty \frac{g(y)^{\frac{-1}{p-q}}}{y^2}\,dy \leqslant \|H\|_{L_{p,v}(\mathbb{R}_+)\to L_q(\mathbb{R}_+)} \leqslant q' \left(\int_0^\infty v(x)^{\frac{-q}{p-q}}\,dx\right)^{\frac{p-q}{pq}},$$

$$\left(\int_0^\infty v(x)^{\frac{-q}{p-q}}\,dx\right)^{\frac{p-q}{pq}} \int_0^1 \frac{g(y)^{\frac{-1}{p-q}}}{y}\,dy \leqslant \|H^*\|_{L_{p,v}(\mathbb{R}_+)\to L_q(\mathbb{R}_+)} \leqslant q \left(\int_0^\infty v(x)^{\frac{-q}{p-q}}\,dx\right)^{\frac{p-q}{pq}}.$$

We close this subsection with a result about boundedness of the Hausdorff operator on finite interval. The below result will be extremely useful in the next section.

THEOREM 4. *Let* $1 < p < \infty$ *and* $\phi \in L_1^{loc}(\mathbb{R}_+)$ *be such that* $supp\phi \subset [1,\infty)$. *Then the following conditions are equivalent*

i) $A(\phi,p) = \int_1^\infty \frac{\phi(y)}{y} y^{1/p} dy < \infty$;

ii) $H_\phi : L_p(0,1) \to L_p(0,1)$ *is bounded.*

Furthermore, if $A(\phi,p) < \infty$ *then* $\|H_\phi\|_{L_p(0,1)\to L_p(0,1)} = A(\phi,p)$.

Proof. Let $A(\phi,p)$ be finite, then the generalized Minkowski inequality and the change of the variables formula give us

$$\begin{aligned}\|H_\phi f\|_{L_p(0,1)} &= \left(\int_0^1 \left(\int_0^\infty \frac{\phi(\frac{x}{y})}{y} f(y)dy\right)^p dx\right)^{\frac{1}{p}} \\ &\leqslant \int_0^\infty \frac{\phi(y)}{y} y^{\frac{1}{p}} \left(\int_0^{\frac{1}{y}} f^p(x)\,dx\right)^{\frac{1}{p}} dy \\ &\leqslant A(\phi,p)\|f\|_{L_p(0,1)}.\end{aligned}$$

Next, let us assume that $H_\phi : L_p(0,1) \to L_p(0,1)$ is continous. We fix $0 < \delta < \frac{1}{p}$ and define

$$g_\delta(x) = x^{\delta-\frac{1}{p}},\ \ x \in (0,1).$$

Then, we get

$$\|g_\delta\|_{L_p(0,1)} = \frac{1}{(\delta p)^{\frac{1}{p}}}.$$

Furthermore, we easily see that g_δ is an eigenfunction of the operator H_ϕ, i.e.

$$H_\phi g_\delta = A\left(\phi, \frac{p}{1-\delta p}\right) g_\delta.$$

Thus,

$$\|H_\phi g_\delta\|_{L_p(0,1)} = \|g_\delta\|_{L_p(0,1)} A\left(\phi, \frac{p}{1-\delta p}\right).$$

Therefore, we get

$$\|H_\phi\|_{L_p(0,1)\to L_p(0,1)} \geqslant A\left(\phi, \frac{p}{1-\delta p}\right),$$

and letting $\delta \to 0$ yields

$$\|H_\phi\|_{L_p(0,1)\to L_p(0,1)} \geqslant \int_1^\infty \frac{\phi(y)}{y} y^{\frac{1}{p}} dy = A(\phi, p).$$

This is the desired conclusion. □

3.2. Grand Lebesgue spaces

Now we turn our attention to the continuity of the Hausdorff operator in the grand Lebesgue spaces.

THEOREM 5. *Let $1 < p < \infty$ and $\phi \in L_1^{loc}(\mathbb{R}_+)$ be such that $supp\phi \subset [1,\infty)$. Then*

i) If $A(\phi,q) = \int_1^\infty \frac{\phi(y)}{y} y^{1/q} dy < \infty$ for some $q \in (0,p)$, then $H_\phi : L_{p)}(0,1) \to L_{p)}(0,1)$ is bounded and

$$\|H_\phi f\|_{L_{p)}(0,1)} \leqslant \inf_{0<\sigma<p-1} (p-1)\sigma^{-\frac{1}{p-\sigma}} A(\phi, p-\sigma) \|f\|_{L_{p)}(0,1)};$$

ii) If $H_\phi : L_{p)}(0,1) \to L_{p)}(0,1)$ is bounded, then

$$\|H_\phi\|_{L_{p)}(0,1)\to L_{p)}(0,1)} \geqslant A(\phi, p).$$

Proof. $i)$ We use the method explored in paper [8], where the boundedness of the Hardy operator in grand Lebesgue spaces has been studied. Let us fix $\sigma \in (0, p-1)$, then

$$\|H_\phi f\|_{L_{p)}(0,1)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_0^1 |H_\phi f(x)|^{p-\varepsilon} dx\right)^{\frac{1}{p-\varepsilon}}$$
$$= \max\left\{ \sup_{0<\varepsilon\leqslant\sigma} \left(\varepsilon \int_0^1 |H_\phi f(x)|^{p-\varepsilon} dx\right)^{\frac{1}{p-\varepsilon}}, \sup_{\sigma<\varepsilon<p-1} \left(\varepsilon \int_0^1 |H_\phi f(x)|^{p-\varepsilon} dx\right)^{\frac{1}{p-\varepsilon}} \right\}.$$

Next, combining the Hölder inequality with Theorem 4 we have

$$\|H_\phi f\|_{L_{p)}(0,1)}$$
$$\leqslant \max\left\{ \sup_{0<\varepsilon\leqslant\sigma} \left(\varepsilon \int_0^1 |H_\phi f|^{p-\varepsilon} dx\right)^{\frac{1}{p-\varepsilon}}, \sup_{\sigma<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \sigma^{\frac{-1}{p-\sigma}} \left(\sigma \int_0^1 |H_\phi f|^{p-\sigma} dx\right)^{\frac{1}{p-\sigma}} \right\}$$
$$\leqslant \max\left\{1, (p-1)\sigma^{-\frac{1}{p-\sigma}}\right\} \sup_{0<\varepsilon<\sigma} \left(\varepsilon \int_0^1 |H_\phi f(x)|^{p-\varepsilon} dx\right)^{\frac{1}{p-\varepsilon}}$$

$$\leqslant \max\left\{1,(p-1)\sigma^{-\frac{1}{p-\sigma}}\right\}\sup_{0<\varepsilon<\sigma} A(\phi,p-\varepsilon)\left(\varepsilon\int_0^1|f(x)|^{p-\varepsilon}dx\right)^{\frac{1}{p-\varepsilon}}$$
$$\leqslant (p-1)\sigma^{-\frac{1}{p-\sigma}}A(\phi,p-\sigma)\|f\|_{L_{p)}(0,1)}.$$

Finally, taking the infimum over $\sigma\in(0,p-1)$ we finish the proof of part $i)$.

$ii)$ Let us assume that $H_\phi : L_{p)}(0,1)\to L_{p)}(0,1)$ is bounded.

We fix $\delta<\min\left(\frac{1}{p},1-\frac{1}{p}\right)$ and we define the test function g_δ in the same way as in the proof of Theorem 4. Then,

$$\|g_\delta\|_{L_{p)}(0,1)} = \sup_{0<\varepsilon<p-1}\left(\frac{\varepsilon}{\left(\delta-\frac{1}{p}\right)(p-\varepsilon)+1}\right)^{\frac{1}{p-\varepsilon}} \leqslant \frac{p-1}{\delta p}.$$

Thus, the family $\{g_\delta\}_{\delta>0}$ is contained in the space $L_{p)}(0,1)$. Furthermore, since g_δ is an eigenfunction of H_ϕ, we have

$$\|H_\phi g_\delta\|_{L_{p)}(0,1)} = \|g_\delta\|_{L_{p)}(0,1)}A\left(\phi,\frac{p}{1-\delta p}\right).$$

Hence,

$$\|H_\phi\|_{L_{p)}(0,1)\to L_{p)}(0,1)} \geqslant A\left(\phi,\frac{p}{1-\delta p}\right),$$

and letting $\delta\to 0$ we get

$$\|H_\phi\|_{L_{p)}(0,1)\to L_{p)}(0,1)} \geqslant \int_1^\infty \frac{\phi(y)}{y}y^{\frac{1}{p}}dy = A(\phi,p). \quad \Box$$

EXAMPLE 13. Let $1<p<\infty$ then $H : L_{p)}(0,1)\mapsto L_{p)}(0,1)$ is bounded and

$$\|Hf\|_{L_{p)}(0,1)} \leqslant \inf_{0<\sigma<p-1}(p-1)\sigma^{-\frac{1}{p-\sigma}}\frac{p-\sigma}{p-\sigma-1}\|f\|_{L_{p)}(0,1)}.$$

3.3. Quasi-Banach spaces

In this subsection we shall study the boundedness of the Hausdorff operator in quasi-Banach spaces, i.e. $0<p<1$. We need to recall the following lemma.

LEMMA 1. [1] *Let* $0<s<1$, $-\infty<a<b\leqslant\infty$ *and* f *be a non-negative and non-increasing function defined on* (a,b)*, then*

$$\left(\int_a^b f(t)\,dt\right)^s \leqslant s\int_a^b f^s(t)\,(t-a)^{s-1}\,dt.$$

We denote by $\mathscr{M}_\phi$ the class of measurable functions $f:\mathbb{R}_+\to\mathbb{R}_+$ such that $y\mapsto \dfrac{\phi(x/y)}{y}f(y)$ is non-increasing.

EXAMPLE 14. If $\phi(t)=\dfrac{\chi_{(1,\infty)}(t)}{t}$ or $\phi(t)=\min\{1,1/t\}$, then any non-increasing function f belongs to the class $\mathscr{M}_\phi$.

Let us introduce the following quantity

$$B=\left(\int_0^\infty \phi^p(y)\,y^\alpha\,dy\right)^{\frac{1}{p}}.$$

THEOREM 6. *Let $0<p<1$ and let ϕ be a non-negative function defined on $\mathbb{R}_+$. If $B<\infty$, then for any $f\in\mathscr{M}_\phi\cap L_{p,x^\alpha}(\mathbb{R}_+)$ the following inequality*

$$\|H_\phi f\|_{L_{p,x^\alpha}}\leqslant p^{\frac{1}{p}}B\|f\|_{L_{p,x^\alpha}} \tag{4}$$

holds.

Proof. By Lemma 1 with $a=0,\ b=\infty,\ s=p$ and by the Fubini theorem, we have

$$\begin{aligned}
\left(\int_0^\infty\left(\int_0^\infty \frac{\phi\left(\frac{x}{y}\right)}{y}f(y)dy\right)^p x^\alpha dx\right)^{\frac{1}{p}} &\leqslant p^{\frac{1}{p}}\left(\int_0^\infty\left(\int_0^\infty \frac{\phi^p\left(\frac{x}{y}\right)}{y^p}f^p(y)\,y^{p-1}dy\right)x^\alpha dx\right)^{\frac{1}{p}}\\
&= p^{\frac{1}{p}}\left(\int_0^\infty\left(\int_0^\infty \frac{\phi^p(y)}{y}f^p\left(\frac{x}{y}\right)dy\right)x^\alpha dx\right)^{\frac{1}{p}}\\
&= p^{\frac{1}{p}}\left(\int_0^\infty \frac{\phi^p(y)}{y}\left(\int_0^\infty f^p\left(\frac{x}{y}\right)x^\alpha dx\right)dy\right)^{\frac{1}{p}}\\
&= p^{\frac{1}{p}}\left(\int_0^\infty \frac{\phi^p(y)}{y}\left(\int_0^\infty f^p(x)\,x^\alpha\,y^{\alpha+1}dx\right)dy\right)^{\frac{1}{p}}\\
&= p^{\frac{1}{p}}\left(\int_0^\infty \phi^p(y)\,y^\alpha\,dy\right)^{\frac{1}{p}}\left(\int_0^\infty f^p(x)\,x^\alpha\,dx\right)^{\frac{1}{p}},
\end{aligned}$$

and this ends the proof of inequality (4). □

THEOREM 7. *Let $0 < p < 1$ and let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be a measurable map. If $H_\phi : L_{p,x^\alpha}(\mathbb{R}_+) \to L_{p,x^\alpha}(\mathbb{R}_+)$ is bounded, then*

$$\|H_\phi\|_{L_{p,x^\alpha}(\mathbb{R}_+)\to L_{p,x^\alpha}(\mathbb{R}_+)} \geqslant p^{\frac{1}{p}} B.$$

Proof. By assumptions we have

$$\left(\int_0^\infty \left(\int_0^\infty \frac{\phi\left(\frac{x}{y}\right)}{y} f(y)dy\right)^p x^\alpha dx\right)^{\frac{1}{p}} \leqslant C\left(\int_0^\infty f^p(x)x^\alpha dx\right)^{\frac{1}{p}}, \tag{5}$$

where $C = \|H_\phi\|_{L_{p,x^\alpha}(\mathbb{R}_+)\to L_{p,x^\alpha}(\mathbb{R}_+)}$. Let us define the following test function

$$f_0(x) = x^{-\frac{\alpha+1}{p}-1}\chi_{(1,\infty)}(x).$$

Then, we get

$$\|f_0\|_{L_{p,x^\alpha}(\mathbb{R}_+)} = \frac{1}{p^{\frac{1}{p}}}.$$

Furthermore, by the reverse Minkowski inequality we have

$$\begin{aligned}
\left(\int_0^\infty \left|H_\phi f_0(x)\right|^p x^\alpha dx\right)^{\frac{1}{p}} &= \left(\int_0^\infty \left[\int_1^\infty \frac{\phi\left(\frac{x}{y}\right)}{y} y^{-\frac{\alpha+1}{p}-1} x^{\frac{\alpha}{p}} dy\right]^p dx\right)^{\frac{1}{p}} \\
&\geqslant \int_1^\infty \left(\int_0^\infty \frac{\phi^p\left(\frac{x}{y}\right)}{y^p} y^{p\left(-\frac{\alpha+1}{p}-1\right)} x^\alpha dx\right)^{\frac{1}{p}} dy \\
&= \left(\int_1^\infty \frac{dy}{y^2}\right)\left(\int_0^\infty \phi^p(x)x^\alpha dx\right)^{\frac{1}{p}} \\
&= \left(\int_0^\infty \phi^p(x)x^\alpha dx\right)^{\frac{1}{p}} = B.
\end{aligned}$$

Therefore, from inequality (5) we obtain $p^{\frac{1}{p}} B \leqslant C$ and the proof is completed. □

COROLLARY 3. *Let $0 < p < 1$ and ϕ be a non-negative measurable function defined on $\mathbb{R}_+$ such that $\dfrac{\phi(x/y)}{y}$ is non-increasing function with respect to y for any x. Assume that $-1 < \alpha < p-1$ and let $B = \|\phi\|_{L_{p,x^\alpha}}$. Then the inequality*

$$\|H_\phi f\|_{L_{p,x^\alpha}} \leqslant C\|f\|_{L_{p,x^\alpha}} \tag{6}$$

holds for all and non-increasing functions f with $C \leqslant p^{1/p}B$. Conversely, if inequality (6) holds for all non-increasing functions, then $p(\alpha+1)^{\frac{1}{p}-1}B \leqslant C$.

Proof. It is obvious that for any x the function $y \mapsto \dfrac{\phi(x/y)}{y} f(y)$ is non-increasing, i.e. $f \in \mathscr{M}_\phi$. Therefore by Theorem 6 we get that inequality (6) holds. Now, we assume that estimate (6) is satisfied. If we take

$$f_1(x) = \chi_{(0,1)}(x),$$

then

$$\|f_1\|_{L_{p,x^\alpha}(\mathbb{R}_+)} = \frac{1}{(\alpha+1)^{\frac{1}{p}}}.$$

Applying the reverse Minkowski inequality, we get

$$\begin{aligned}
\left(\int_0^\infty |H_\phi f_1(x)|^p \, x^\alpha \, dx\right)^{\frac{1}{p}} &= \left(\int_0^\infty \left[\int_0^1 \frac{\phi\left(\frac{x}{y}\right)}{y} x^{\frac{\alpha}{p}} \, dy\right]^p dx\right)^{\frac{1}{p}} \\
&\geqslant \int_0^1 \left(\int_0^\infty \frac{\phi^p\left(\frac{x}{y}\right)}{y^p} x^\alpha \, dx\right)^{\frac{1}{p}} dy \\
&= \int_0^1 y^{\frac{\alpha+1}{p}-1} \left(\int_0^\infty \phi^p(x) \, x^\alpha \, dx\right)^{\frac{1}{p}} dy \\
&= \frac{p}{\alpha+1} B.
\end{aligned}$$

Thus $p(\alpha+1)^{\frac{1}{p}-1} B \leqslant C$ and the proof of Corollary 3 is completed. □

EXAMPLE 15. Let $0 < p < 1$ and $\alpha < p-1$, then by Corollary 3, for all non-negative and non-increasing functions f the following inequality holds[3]

$$\|Hf\|_{L_{p,x^\alpha}(\mathbb{R}_+)} \leqslant \left(\frac{p}{p-\alpha-1}\right)^{\frac{1}{p}} \left(\int_0^\infty f^p(x) x^\alpha \, dx\right)^{\frac{1}{p}}.$$

On the other hand, let f be a non-negative and non-increasing, then

$$Hf(x) = \frac{1}{x}\int_0^x f(t)dt \geqslant f(x)\,\frac{1}{x}\int_0^x dt = f(x) \ \text{ for all } x > 0.$$

Thus,

$$\|Hf\|_{L_{p,x^\alpha}(\mathbb{R}_+)} \geqslant \left(\int_0^\infty f^p(x) x^\alpha dx\right)^{\frac{1}{p}}.$$

[3] The assumption $-1 < \alpha$ in Corollary 3 was needed only in the proof of the estimate $p(\alpha+1)^{\frac{1}{p}-1} B \leqslant C$, where C is the constant from inequality (6).

EXAMPLE 16. Let $0<p<1$ and $\alpha<p-1$, then for all non-negative and non-increasing functions f the following inequality holds

$$\left(\int_0^\infty |Pf(x)|^p x^\alpha dx\right)^{\frac{1}{p}} \leqslant \left(\frac{p}{\alpha+1}+\frac{p}{p-\alpha-1}\right)^{\frac{1}{p}} \left(\int_0^\infty f^p(x)x^\alpha\, dx\right)^{\frac{1}{p}}.$$

3.4. Variable exponent Lebesgue spaces

For $p:\mathbb{R}_+\to\mathbb{R}_+$, we define $p_\star:\mathbb{R}\to\mathbb{R}$ in the following manner $p_\star(t)=p(e^{-t})$.

THEOREM 8. *Let $p\in\mathscr{M}_{0,\infty}(\mathbb{R}_+)$ be such that $p(0)=p_\infty$. Moreover, let $\phi:\mathbb{R}_+\to\mathbb{R}_+$ be such that $\int_{\mathbb{R}_+} y^{1/p(0)}\phi(y)\frac{dy}{y}$ and $\int_{\mathbb{R}_+} y^{s_0/p(0)}\phi^{s_0}(y)\frac{dy}{y}$ are finite, where $\frac{1}{s_0}=1-\frac{1}{p_\star^-}+\frac{1}{p_\star^+}$. Then there exists C such that for all $f\in L_{p(\cdot)}(\mathbb{R}_+)$ the inequality holds*

$$\|H_\phi f\|_{L_{p(\cdot)}(\mathbb{R}_+)}\leqslant C\|f\|_{L_{p(\cdot)}(\mathbb{R}_+)}.$$

REMARK 1. If we define $h(t)=e^{-\frac{t}{p(0)}}\phi(e^{-t})$, then

$$\int_{\mathbb{R}_+} y^{1/p(0)}\phi(y)\frac{dy}{y}=\|h\|_{L_1(\mathbb{R})},\qquad \int_{\mathbb{R}_+} y^{s_0/p(0)}\phi^{s_0}(y)\frac{dy}{y}=\|h\|^{s_0}_{L_{s_0}(\mathbb{R})}.$$

Proof. Let us introduce the operator

$$\left(W_{p(\cdot)}f\right)(t)=e^{-\frac{t}{p(0)}}f\left(e^{-t}\right),\quad t\in\mathbb{R}.$$

By Lemma 5.1 from [7] we have that the operator

$$W_{p(\cdot)}:L_{p(\cdot)}(\mathbb{R}_+)\to L_{p_\star(\cdot)}(\mathbb{R})$$

is an isomorphism and the inverse operator has the form

$$\left(W^{-1}_{p(\cdot)}g\right)(x)=x^{-\frac{1}{p(0)}}g\left(-\ln x\right),\quad x\in\mathbb{R}_+.$$

Furthermore, let us define the following kernel

$$h(t)=e^{-\frac{t}{p(0)}}\phi(e^{-t}),\quad t\in\mathbb{R}$$

and the integral operator

$$(K_\phi\varphi)(t)=\int_{\mathbb{R}} h(t-\tau)\varphi(\tau)\,d\tau=(h\star\varphi)(t).$$

It is easy to see that

$$W_{p(\cdot)}H_\phi=K_\phi W_{p(\cdot)},$$

in other words we get

$$H_\phi = W_{p(\cdot)}^{-1} K_\phi W_{p(\cdot)}. \tag{7}$$

By our assumptions we have that $h \in L_1 \cap L_{s_0}$, therefore since K_ϕ is a convolution operator, by Theorem 4.6 from [7], we get

$$K_\phi : L_{p_\star(\cdot)}(\mathbb{R}) \to L_{p_\star(\cdot)}(\mathbb{R}).$$

Finally, since $W_{p(\cdot)} : L_{p(\cdot)}(\mathbb{R}_+) \to L_{p_\star(\cdot)}(\mathbb{R})$ and $W_{p(\cdot)}^{-1} : L_{p_\star(\cdot)}(\mathbb{R}) \to L_{p(\cdot)}(\mathbb{R}_+)$ we get, by (7), that H_ϕ is indeed a bounded operator from $L_{p(\cdot)}(\mathbb{R}_+)$ to $L_{p(\cdot)}(\mathbb{R}_+)$. □

PROPOSITION 1. Let p satisfies condition i) and condition iii) of $\mathscr{M}_{0,\infty}(\mathbb{R}_+)$, $p_\infty = p_-$ and $\phi \in L_1^{loc}(\mathbb{R}_+)$. If $H_\phi : L_{p(\cdot)}(\mathbb{R}_+) \to L_{p(\cdot)}(\mathbb{R}_+)$ is bounded, then

$$\int_0^\infty \frac{\phi(y)}{y} y^{\frac{1}{p_-}} dy < \infty.$$

Proof. Let us fix $0 < \varepsilon < 1/p_-$. We take the following test function

$$f_\varepsilon(x) = x^{-\varepsilon - \frac{1}{p_-}} \chi_{(2,\infty)}(x).$$

By inequality (2), we get the string of inequalities

$$\begin{aligned}
&\|f_\varepsilon\|_{L_{p(\cdot)}(\mathbb{R}_+)} \\
&\leqslant \max\left\{ \left(\int_2^\infty x^{-\left(\varepsilon + \frac{1}{p_-}\right) p(x)} dx \right)^{\frac{1}{p_-}}, \left(\int_2^\infty x^{-\left(\varepsilon + \frac{1}{p_-}\right) p(x)} dx \right)^{\frac{1}{p_+}} \right\} \\
&\leqslant \max\left\{ e^{\frac{C}{p_-}\left(\varepsilon + \frac{1}{p_-}\right)} \left(\int_1^\infty x^{-\left(\varepsilon + \frac{1}{p_-}\right) p_-} dx \right)^{\frac{1}{p_-}}, e^{\frac{A_2}{p_+}\left(\varepsilon + \frac{1}{p_-}\right)} \left(\int_1^\infty x^{-\left(\varepsilon + \frac{1}{p_-}\right) p_-} dx \right)^{\frac{1}{p_+}} \right\} \\
&= \max\left\{ e^{\frac{C}{p_-}\left(\varepsilon + \frac{1}{p_-}\right)} \frac{1}{(\varepsilon p_-)^{\frac{1}{p_-}}}, e^{\frac{A_2}{p_+}\left(\varepsilon + \frac{1}{p_-}\right)} \frac{1}{(\varepsilon p_-)^{\frac{1}{p_+}}} \right\} \\
&\leqslant e^{\frac{C}{p_-}\left(\varepsilon + \frac{1}{p_-}\right)} \frac{1}{(\varepsilon p_-)^{\frac{1}{p_-}}}.
\end{aligned}$$

Moreover, we have

$$H_\phi f_\varepsilon(x) = x^{-\frac{1}{p_-} - \varepsilon} \int_0^{x/2} \frac{\phi(y)}{y} y^{\frac{1}{p_-} + \varepsilon} dy.$$

On the other hand, by assumptions we have that $1 \in L_{s(\cdot)}(\mathbb{R})$, where $\frac{1}{s(x)} = \frac{1}{p(x)} - \frac{1}{p_\infty}$. Therefore, by Lemma 3.3.5 from [6], we have $L_{p(\cdot)}(\mathbb{R}_+) \hookrightarrow L_{p_-}(\mathbb{R}_+)$. Hence, we obtain

$$
\begin{aligned}
\left\|H_\phi f\right\|_{L_{p(\cdot)}(\mathbb{R}_+)} &\geqslant \tilde{C}\left\|H_\phi f\right\|_{L_{p_-}(\mathbb{R}_+)} \geqslant \tilde{C}\left\|H_\phi f\right\|_{L_{p_-}(\frac{1}{\varepsilon},\infty)} \\
&= \tilde{C}\left(\int\limits_{1/\varepsilon}^{\infty} x^{-1-\varepsilon p_-}\left(\int\limits_0^{x/2} \frac{\phi(y)}{y} y^{\frac{1}{p_-}+\varepsilon} dy\right)^{p_-} dx\right)^{\frac{1}{p_-}} \\
&\geqslant \tilde{C}\left(\int\limits_{\frac{1}{\varepsilon}}^{\infty} x^{-1-\varepsilon p_-}\left(\int\limits_0^{\frac{1}{2\varepsilon}} \frac{\phi(y)}{y} y^{\frac{1}{p_-}+\varepsilon} dy\right)^{p_-} dx\right)^{\frac{1}{p_-}} \\
&= \tilde{C}\frac{\varepsilon^\varepsilon}{(\varepsilon p_-)^{\frac{1}{p_-}}}\left(\int\limits_0^{\frac{1}{2\varepsilon}} \frac{\phi(y)}{y} y^{\frac{1}{p_-}+\varepsilon} dy\right).
\end{aligned}
$$

Thus,

$$
\|H_\phi\|_{L_{p(\cdot)}(\mathbb{R}_+)\to L_{p(\cdot)}(\mathbb{R}_+)} \geqslant \tilde{C}\varepsilon^\varepsilon e^{-\frac{C}{p_-}\left(\varepsilon+\frac{1}{p_-}\right)}\left(\int\limits_0^{\frac{1}{2\varepsilon}} \frac{\phi(y)}{y} y^{\frac{1}{p_-}+\varepsilon} dy\right).
$$

Therefore, we can pass to the limit $\varepsilon \to 0$ and we obtain

$$
\|H_\phi\|_{L_{p(\cdot)}(\mathbb{R}_+)\to L_{p(\cdot)}(\mathbb{R}_+)} \geqslant \tilde{C}e^{-\frac{C}{(p_-)^2}}\left(\int\limits_0^{\infty} \frac{\phi(y)}{y} y^{\frac{1}{p_-}} dy\right),
$$

and this ends the proof of the Proposition. □

From Theorem 3.7 and Proposition 3.8 we have the following corollary.

COROLLARY 4. *Let $p \in \mathscr{M}_{0,\infty}(\mathbb{R}_+)$ and $p(0) = p_\infty = p_-$. Suppose $h \in L_{s_0}(\mathbb{R})$, where $h(t) = e^{-\frac{t}{p(0)}}\phi(e^{-t})$. Then the Hausdorff operator is bounded in $L_{p(\cdot)}(\mathbb{R}_+)$ if and only if*

$$
\int\limits_0^{\infty} \frac{\phi(y)}{y} y^{\frac{1}{p(0)}} dy < \infty.
$$

EXAMPLE 17. Let $p \in \mathscr{M}_{0,\infty}(\mathbb{R}_+)$ and $p(0) = p_\infty$. Then

i) if $p(0) > 1$, then the Hardy operator H is bounded in $L_{p(\cdot)}(\mathbb{R}_+)$;

ii) the adjoint Hardy operator H^* is bounded in $L_{p(\cdot)}(\mathbb{R}_+)$;

iii) if $p(0) > 1$, then the Hardy-Littlewood-Pólya operator P is bounded in $L_{p(\cdot)}(\mathbb{R}_+)$;

iv) if $\gamma > \frac{1}{p_-} - \frac{1}{p_+}$, then the Cesàro operator $\mathscr{C}_\gamma$ is bounded in $L_{p(\cdot)}(\mathbb{R}_+)$.

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