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BILINEAR FOURIER MULTIPLIER OPERATORS ON VARIABLE TRIEBEL SPACES

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Abstract. In this paper, we prove the boundedness of bilinear Fourier multiplier operators on variable exponent Triebel-Lizorkin spaces.

1. Introduction

The Triebel-Lizorkin space $F^s_{p,q}(\mathbb{R}^n)$ and the Besov space $B^s_{p,q}(\mathbb{R}^n)$ were introduced and studied accompanying with the development of the theory of function spaces between 1960's and 1970's, see [43]. These spaces form a very general unifying scale of many well-known classical concrete function spaces such as Lebesgue spaces, Bessel-potential spaces, Sobolev spaces, Hölder-Zygmund spaces, Hardy spaces and BMO, which have their own history. A comprehensive treatment of these function spaces and their history can be founded in Triebel's monographs, see [43], etc..

On the other hand, function spaces with variable exponents have received more and more attention in recent years, and have been extensively studied in harmonic analysis, fluid dynamics, image processing, partial differential equations and variational calculus, see [1], [2], [4], [5], [6], [15], [16], [17], [20], [29], [33], [37], [40],[42], [45], [49], etc., and the references therein. Variable exponent Lebesgue spaces are a generalization of the classical $L^p(\mathbb{R}^n)$ spaces, via replacing the constant exponent p with an exponent function $p(\cdot)\colon\ \mathbb{R}^n \to (0,\infty)$, that is, they consist of all measurable functions f such that

$$\int_{\mathbb{R}^n} |f(x)|^{p(x)}dx < \infty.$$

These spaces were introduced by Birnbaum-Orlicz [11] (see also Luxemburg [34] and Nakano [38], [39]) and then systematically developed in [15], [16].

In [30] and [31], when Leopold and Schrohe studied pseudo-differential operators, they introduced related Besov spaces with variable smoothness, $B^{s(\cdot)}_{p,p}(\mathbb{R}^n)$, which were further generalized to the case that $q \neq p$, including $F^{s(\cdot)}_{p,q}(\mathbb{R}^n)$ and $B^{s(\cdot)}_{p,q}(\mathbb{R}^n)$, by Besov,

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see [9] and [10]. Along a different line of study, Xu studied Triebel-Lizorkin spaces $F^{s}_{p(\cdot),q}(\mathbb{R}^n)$ and Besov spaces $B^{s}_{p(\cdot),q}(\mathbb{R}^n)$ with variable exponent $p(\cdot)$ but fixed q and s in [46] and [47]. In 2009, Diening, Hästö and Roudenko [17] defined and investigated Triebel-Lizorkin spaces with variable smoothness and integrability $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ with $s(\cdot)\geqslant 0$. Later, Almeida and Hästö introduced and studied the Besov spaces $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ in [5]. Additional results, including the Sobolev embedding, have subsequently been studied by Vybíral and Kempka [25], [26], [27], [28], [44], and others [18], [19], [21], [24], etc., and the references therein. Recently, Noi [41] also gave a research about Triebel-Lizorkin spaces and Besov spaces with variable exponents.

Within the framework of Calderón-Zygmund theory, the study of multilinear operators is not just motivated by a quest to generalize the theory of linear operators but rather by their natural appearance in analysis. The study of such operators using Littlewood-Paley theory and related decomposition techniques, which is originated in the works of Coifman and Meyer [13], [14] and has been extensively researched since then with applications to harmonic analysis and partial differential equations [3], [7], [8], [12], [35], [48], etc., and the references therein. In [23], Grafakos and Torres obtained some conclusions about the bilinear operators for Hardy spaces, Sobolev spaces, and other Triebel-Lizorkin spaces. In [36], on the scales of inhomogeneous Triebel-Lizorkin and Besov spaces of positive smoothness, Naibo investigated the boundedness of pseudodifferential operators with symbols belonging to certain bilinear Hörmander classes. In 2017, Zhao et al. [32] studied the boundedness of bilinear Fourier multiplier operators on Triebel-Lizorkin and Besov spaces. In this paper, we consider the boundedness of bilinear Fourier multiplier operators on variable Triebel-Lizorkin spaces $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

For the purpose of this article, the Fourier transform of $f\in\mathscr{S}'(\mathbb{R}^n)$ will be denoted by $\mathscr{F}(f)$ or $\widehat{f}$; in particular, we use the formula

$$\widehat{f}(\xi):=\int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\,dx \text{ if } f\in\mathscr{S}(\mathbb{R}^n).$$

The inverse Fourier transform will be denoted by $\mathscr{F}^{-1}$ or $\check{f}$. Given a real number $r\geqslant 0$, the homogeneous derivative of order r, D^r, acts as

$$\widehat{D^r f}(\xi):=|\xi|^r\widehat{f}(\xi),\qquad \xi\in\mathbb{R}^n.$$

For a function h, we denote $h(D)$ as the multiplier operator given by $\widehat{h(D)f}=h\widehat{f}$ for $f\in\mathscr{S}'(\mathbb{R}^n)$.

If $m\in L^\infty(\mathbb{R}^{2n})$, the bilinear Fourier multiplier operator T_m is defined by

$$T_m(f,g)(x)=\int_{\mathbb{R}^{2n}} m(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\,d\xi\, d\eta,$$

for $f,g\in\mathscr{S}(\mathbb{R}^n)$.

For a measurable function $p(\cdot):\mathbb{R}^n\to(0,\infty)$ and a measurable set E of $\mathbb{R}^n$, let

$$p_+(E)\equiv \operatorname*{ess\,sup}_{x\in E} p(x),\quad p_-(E)\equiv \operatorname*{ess\,inf}_{x\in E} p(x).$$

When $E = \mathbb{R}^n$, we write simply $p_+ := p_+(\mathbb{R}^n)$ and $p_- := p_-(\mathbb{R}^n)$. Denote by $\mathscr{P}(\mathbb{R}^n)$ the collection of all measurable functions $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ satisfying $0 < p_- \leqslant p_+ < \infty$.

For a measurable function f, let

$$\|f\|_{L^{p(\cdot)}} \equiv \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n}\left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \leqslant 1\right\}.$$

For a measurable function p, if there exists a positive constant $C_{\log}(p)$ such that, for all $x, y \in \mathbb{R}^n$,

$$|p(x) - p(y)| \lesssim \frac{C_{\log}(p)}{\log(e + \frac{1}{|x-y|})},$$

we call $p(\cdot)$ satisfies the locally log-Hölder continuous condition, denoted by $p(\cdot) \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$.

If $p(\cdot) \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$ and there exist a positive constant C_∞ and $p_\infty \in \mathbb{R}$ such that, for all $x \in \mathbb{R}^n$,

$$|p(x) - p_\infty| \lesssim \frac{C_\infty}{\log(e + |x|)},$$

we call $p(\cdot)$ satisfies the globally log-Hölder continuous condition, denoted by $p(\cdot) \in C^{\log}(\mathbb{R}^n)$.

Let $\mathscr{P}^{\log}(\mathbb{R}^n)$ be the set of all measurable function $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ satisfying $\frac{1}{p(\cdot)} \in C^{\log}(\mathbb{R}^n)$.

Our main result is as follows.

THEOREM 1. *Let* $p(\cdot), p_1(\cdot), p_2(\cdot), q(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ *be such that* $\frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)} = \frac{1}{p(\cdot)}$. *Assume that* $s(\cdot) \in C^{\log}_{\mathrm{loc}} \cap L^\infty$, *and given an arbitrary* s_0, *such that* $0 < s_0 \leqslant s_-$. *Let* $N, R \in \mathbb{N}_0$ *be even numbers with* $N \geqslant R > \frac{q_+}{q_-}s_+$ *and* $N > \frac{2n}{\min(p_{1-}, p_{2-}, q_-, 1)} + \max(6C_{\log}(s), 6) + n$. *Let* $m(\xi, \eta)$ *be a* C^∞ *function on* $\mathbb{R}^n \times \mathbb{R}^n - \{(0,0)\}$ *such that* $|\partial_\xi^\alpha \partial_\eta^\beta m(\xi,\eta)| \leqslant C_{\alpha,\beta}(|\xi| + |\eta|)^{-|\alpha|-|\beta|}$ *for all multi-indices* α *and* β *with* $|\alpha| + |\beta| \leqslant N$. *Then*

$$\|T_m(f,g)\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \|f\|_{F^{s(\cdot)}_{p_1(\cdot),q(\cdot)}} \|g\|_{F^0_{p_2(\cdot),1}} + \|f\|_{F^0_{p_1(\cdot),1}} \|g\|_{F^{s(\cdot)}_{p_2(\cdot),q(\cdot)}}, \tag{1}$$

for every $f, g \in \mathscr{S}(\mathbb{R}^n)$. *In particular,*

$$\|T_m(f,g)\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \|f\|_{F^{s(\cdot)}_{p_1(\cdot),q(\cdot)}} \|g\|_{F^{s(\cdot)}_{p_2(\cdot),q(\cdot)}}, \tag{2}$$

for every $f, g \in \mathscr{S}(\mathbb{R}^n)$.

REMARK 1. Similar to Remark 1.3 of [32], using Theorem 2.11 of [22], it is enough to prove (1), since (2) can be obtained by (1).

2. Preliminaries

First, we introduce some definitions and notations, see [36], [41], [43], and the references therein.

Let $p(\cdot)$ and $q(\cdot)$ be variable exponents. Let $\{f_j\}_{j=0}^{\infty}$ be a sequence of measurable functions on $\mathbb{R}^n$. The quasi-norm $\|\cdot\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}$ is defined by

$$\|\{f_j\}_{j=0}^{\infty}\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} \equiv \left\| \left(\sum_{j=0}^{\infty} |f_j(\cdot)|^{q(\cdot)} \right)^{\frac{1}{q(\cdot)}} \right\|_{L^{p(\cdot)}}.$$

Now, we give a definition [36] which will be used in the sequel.

DEFINITION 1. [Littlewood-Paley Partitions of Unity] $\{\psi_k\}_{k\in\mathbb{N}_0} \subset \mathscr{S}(\mathbb{R}^n)$ is a Littlewood-Paley partition of unity in $\mathbb{R}^n$ if $\text{supp}(\psi_0) \subset \{\xi \in \mathbb{R}^n : |\xi| \leqslant 2\}$ and $\psi_0(\xi) = 1$ in $\{\xi \in \mathbb{R}^n : |\xi| \leqslant 1\}$ and for $k \in \mathbb{N}$,

$$\psi_k(\xi) = \psi(2^{-k}\xi), \quad \xi \in \mathbb{R}^n,$$

where $\psi(\xi) := \psi_0(\xi) - \psi_0(2\xi)$ for every $\xi \in \mathbb{R}^n$.

From the definition we know that

$$\text{supp}(\psi_k) \subset \left\{\xi \in \mathbb{R}^n : 2^{k-1} \leqslant |\xi| \leqslant 2^{k+1}\right\} \text{ for } k \in \mathbb{N} \text{ and } \sum_{k\in\mathbb{N}_0} \psi_k \equiv 1.$$

Set $\widetilde{\psi}_0 := \psi_0 + \psi_1$ and $\widetilde{\psi}_k := \psi_{k-1} + \psi_k + \psi_{k+1}$ for $k \in \mathbb{N}$, then we have that $\psi_k \widetilde{\psi}_k = \psi_k$ for $k \in \mathbb{N}_0$ and

$$\text{supp}(\widetilde{\psi}_k) \subset \left\{\xi \in \mathbb{R}^n : 2^{k-2} \leqslant |\xi| \leqslant 2^{k+2}\right\} \text{ for } k \geqslant 2,$$

$$\text{supp}(\widetilde{\psi}_k) \subset \left\{\xi \in \mathbb{R}^n : |\xi| \leqslant 2^{k+2}\right\} \text{ for } k = 0, 1.$$

Let us recall the definition of Triebel-Lizorkin spaces with variable exponents [41].

DEFINITION 2. Let $\{\psi_k\}_{k\in\mathbb{N}_0}$ be a Littlewood-Paley partition of unity, $p(\cdot), q(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$, and $s(\cdot) \in C_{\text{loc}}^{\log}(\mathbb{R}^n)$. The Triebel-Lizorkin space $F_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ with variable exponents is defined as

$$F_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n) := \{f \in \mathscr{S}'(\mathbb{R}^n) : \|f\|_{F_{p(\cdot),q(\cdot)}^{s(\cdot)}} < \infty\},$$

where

$$\|f\|_{F_{p(\cdot),q(\cdot)}^{s(\cdot)}} := \left\|\{2^{ks(\cdot)}\psi_k(D)f\}_{k\in\mathbb{N}_0}\right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} = \left\| \left(\sum_{k=0}^{\infty} |2^{ks(\cdot)}\psi_k(D)f|^{q(\cdot)} \right)^{\frac{1}{q(\cdot)}} \right\|_{L^{p(\cdot)}}.$$

The proof of the main result use the following contents of Triebel-Lizorkin spaces.

DEFINITION 3. Let $k \in \mathbb{N}_0$ and $a > 0$, the maximal function defined as follows:

$$f_k^{*a}(x) := \sup_{y \in \mathbb{R}^n} \frac{|(\psi_k(D)f)(x-y)|}{(1+2^k|y|)^a}, \quad x \in \mathbb{R}^n. \tag{3}$$

LEMMA 1. ([40, Theorem 4.21] and [41, Remark 2]) *Suppose that* $p(\cdot), q(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ *and* $s(\cdot) \in C_{\mathrm{loc}}^{\log}(\mathbb{R}^n)$. *If* $a > \frac{n+3C_{\log}(s)\min(p_-,q_-)}{\min(p_-,q_-)}$, *then for all* $f \in F_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$,

$$\left\|\{2^{ks(\cdot)} f_k^{*a}\}_{k\in\mathbb{N}_0}\right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} \lesssim \|f\|_{F_{p(\cdot),q(\cdot)}^{s(\cdot)}}.$$

LEMMA 2. ([41, Lemma 1]) *Let* f_k *and* h_k *be measurable functions on* $\mathbb{R}^n$. *If* $p(\cdot), q(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$, *then*

$$\|\{f_k+h_k\}_{k=0}^{\infty}\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}^{\min(p_-,q_-,1)} \leqslant \|\{f_k\}_{k=0}^{\infty}\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}^{\min(p_-,q_-,1)} + \|\{h_k\}_{k=0}^{\infty}\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}^{\min(p_-,q_-,1)}.$$

Define $\mathscr{U}_{p(\cdot)}(\mathbb{R}^n)$ as the collection of sequences of functions $\upsilon := \{\upsilon_k\}_{k\in\mathbb{N}_0}$ with $\upsilon_k \in L^{p(\cdot)}(\mathbb{R}^n) \bigcap \mathscr{S}'(\mathbb{R}^n)$ and $\mathrm{supp}(\widehat{\upsilon_k}) \subset \{\eta \in \mathrm{R}^n : |\eta| \leqslant 2^{k+1}\}$ for every $k \in \mathbb{N}_0$.

Using a similar argument with Theorem 8.1 of [5], we can have the following conclusion.

Let $p(\cdot), q(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ and $s(\cdot) \in C_{\mathrm{loc}}^{\log}(\mathbb{R}^n) \cap L^\infty$, with $s_- > 0$, then $f \in F_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ if and only if there exists $\upsilon = \{\upsilon_k\}_{k\in\mathbb{N}_0} \in \mathscr{U}_{p(\cdot)}(\mathbb{R}^n)$ such that

$$\lim_{k\to\infty} \upsilon_k = f \text{ in } \mathscr{S}'(\mathbb{R}^n),$$

and

$$\|f\|_{F_{p(\cdot),q(\cdot)}^{s(\cdot)}}^{\upsilon} := \|\upsilon_0\|_{L^{p(\cdot)}} + \left\|\{2^{ks(\cdot)}(f-\upsilon_k)\}_{k\in\mathbb{N}_0}\right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} < \infty.$$

Furthermore, we define that

$$\inf_{\upsilon} \|f\|_{F_{p(\cdot),q(\cdot)}^{s(\cdot)}}^{\upsilon} = \|f\|_{F_{p(\cdot),q(\cdot)}^{s(\cdot)}},$$

where the infimum is taken over all the sequences of functions υ as above.

In the end of this section, we give a notation. The symbol $A \lesssim B$ means that there exists a positive constant C such that $A \leqslant CB$.

3. Proof of Theorem 1

First, we introduce some notations.

Let $f,g \in \mathscr{S}(\mathbb{R}^n)$ and $\{\psi_k\}_{k\in\mathbb{N}_0}$ be a partition of unity in $\mathbb{R}^n$ as in Detinition 1. Then

$$\begin{aligned}
T_m(f,g)(x) &= \int_{\mathbb{R}^{2n}} m(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi ix\cdot(\xi+\eta)}d\xi d\eta \\
&= \sum_{j,k\in\mathbb{N}_0}\int_{\mathbb{R}^{2n}} m(\xi,\eta)\psi_k(\xi)\psi_j(\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi ix\cdot(\xi+\eta)}d\xi d\eta \\
&= \sum_{j,k\in\mathbb{N}_0}\int_{\mathbb{R}^{2n}}\left(\int_{\mathbb{R}^n}\widehat{m}^1(\zeta,\eta)e^{2\pi i\xi\cdot\zeta}d\zeta\right)\psi_k(\xi)\psi_j(\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi ix\cdot(\xi+\eta)}d\xi d\eta \\
&= T_1(x)+T_2(x),
\end{aligned}$$

where $\widehat{m}^1(\cdot,\cdot)$ denotes the Fourier transform of $m(\cdot,\cdot)$ with respect to the first variable and we denote

$$\begin{aligned}
T_1(x) := \sum_{\substack{j,k\in\mathbb{N}_0\\ j\leqslant k}}\int_{\mathbb{R}^{2n}}&\left(\int_{\mathbb{R}^n}\widehat{m}^1(\zeta,\eta)e^{2\pi i\xi\cdot\zeta}d\zeta\right) \\
&\times\psi_k(\xi)\psi_j(\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi ix\cdot(\xi+\eta)}d\xi d\eta,
\end{aligned}$$

and

$$\begin{aligned}
T_2(x) := \sum_{\substack{j,k\in\mathbb{N}_0\\ j> k}}\int_{\mathbb{R}^{2n}}&\left(\int_{\mathbb{R}^n}\widehat{m}^1(\zeta,\eta)e^{2\pi i\xi\cdot\zeta}d\zeta\right) \\
&\times\psi_k(\xi)\psi_j(\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi ix\cdot(\xi+\eta)}d\xi d\eta.
\end{aligned}$$

Because the estimate for T_2 is similiar with the one for T_1, so we only deal with T_1. By Definition 2.1, we have

$$T_1(x) = \sum_{\ell\in\mathbb{N}_0}\sum_{\substack{j,k\in\mathbb{N}_0\\ j\leqslant k}} T_{m_{j,k,\ell}}(f,g)(x)$$

with symbols denoted by

$$m_{j,k,0}(\xi,\eta) := \psi_k(\xi)\psi_j(\eta)\int_{\mathbb{R}^n}\left(\sum_{\nu=0}^{k}\psi_\nu(\zeta)\right)\widehat{m}^1(\zeta,\eta)e^{2\pi i\xi\cdot\zeta}d\zeta,$$

$$m_{j,k,\ell}(\xi,\eta) := \psi_k(\xi)\psi_j(\eta)\int_{\mathbb{R}^n}\psi_{k+\ell}(\zeta)\widehat{m}^1(\zeta,\eta)e^{2\pi i\xi\cdot\zeta}d\zeta, \quad \ell\geqslant 1,$$

where $j\leqslant k$.

From [32], we have the following two lemmas.

LEMMA 3. *Let $N,R \in \mathbb{N}_0$, with R even and $N \geqslant R$. Assume that $\alpha,\beta \in \mathbb{N}_0^n$ are multi-indices which satisfy $|\alpha|+|\beta| \leqslant N$. Let $m(\xi,\eta)$ be a C^∞ function on $\mathbb{R}^n \times \mathbb{R}^n - \{(0,0)\}$ such that $|\partial_\xi^\alpha \partial_\eta^\beta m(\xi,\eta)| \leqslant C_{\alpha,\beta}(|\xi|+|\eta|)^{-|\alpha|-|\beta|}$ for all multi-indices α and β with $|\alpha|+|\beta| \leqslant N$. Then*

$$|\partial_\xi^\alpha \partial_\eta^\beta m_{j,k,\ell}(\xi,\eta)| \lesssim 2^{-\ell R}$$

for all $\xi,\eta \in \mathbb{R}^n$, $j,k,\ell \in \mathbb{N}_0$, $j \leqslant k$, and in which the implicit constant depends only on N,R and n.

We set

$$G_{j,k,\ell}(y,z) := (\mathscr{F}_{2n} m_{j,k,\ell}(\cdot,\cdot))(y,z) \qquad y,z \in \mathbb{R}^n, \tag{4}$$

where $\mathscr{F}_{2n} m_{j,k,\ell}(\cdot,\cdot)$ represents the Fourier transform in $\mathbb{R}^{2n}$ of $m_{j,k,\ell}(\cdot,\cdot)$.

LEMMA 4. *Let $a>0$ and $N,R \in \mathbb{N}_0$ be even numbers with $N \geqslant R$ and $N > a+n$. Assume that the symbol m is the same as Lemma 3.1. Then*

$$\int_{\mathbb{R}^{2n}} |G_{j,k,\ell}(y,z)|(1+2^k|y|+2^j|z|)^a dydz \lesssim 2^{-\ell R} \tag{5}$$

for $j,k,\ell \in \mathbb{N}_0$ with $j \leqslant k$, and in which the implicit constant depends only on N,R,a and n.

Now, we prove the following two lemmas which will be used later.

LEMMA 5. *Let $H_k = \sum_{j=0}^k |T_{m_{j,k,\ell}}(f,g)|$. Given an arbitrary s_0, such that $0 < s_0 \leqslant s_-$, then*

$$\|\sum_{k\in\mathbb{N}_0} H_k\|_{L^{p(\cdot)}} \lesssim \left\| \left(\sum_{k\in\mathbb{N}_0} (2^{ks(\cdot)}H_k)^{q(\cdot)} \right)^{\frac{1}{q(\cdot)}} \right\|_{L^{p(\cdot)}}.$$

Proof. Since $q(\cdot) \leqslant q_+$, it is enough to prove

$$\|\sum_{k\in\mathbb{N}_0} H_k\|_{L^{p(\cdot)}} \lesssim \left\| \|2^{ks(\cdot)}H_k\|_{\ell^{q_+}} \right\|_{L^{p(\cdot)}}.$$

If $q_+ \leqslant 1$, then

$$\left(\sum_{k\in\mathbb{N}_0} H_k\right)^{q_+} \lesssim \left(\sum_{k\in\mathbb{N}_0} 2^{ks(\cdot)}H_k\right)^{q_+} \leqslant \sum_{k\in\mathbb{N}_0} 2^{ks(\cdot)q_+}H_k^{q_+}.$$

Thus,

$$\|\sum_{k\in\mathbb{N}_0} H_k\|_{L^{p(\cdot)}} \lesssim \left\| \|2^{ks(\cdot)}H_k\|_{\ell^{q_+}} \right\|_{L^{p(\cdot)}}.$$

If $q_+ > 1$, then

$$\begin{aligned}\Big\|\sum_{k\in\mathbb{N}_0} H_k\Big\|_{L^{p(\cdot)}} &= \Big\|\sum_{k\in\mathbb{N}_0} 2^{ks(\cdot)}H_k 2^{-ks(\cdot)}\Big\|_{L^{p(\cdot)}}\\ &\leqslant \Big\|\Big(\sum_{k\in\mathbb{N}_0}(2^{ks(\cdot)}H_k)^{q_+}\Big)^{1/q_+}\Big(\sum_{k\in\mathbb{N}_0}2^{-kq'_+s(\cdot)}\Big)^{1/q'_+}\Big\|_{L^{p(\cdot)}}\\ &\leqslant \Big(\frac{1}{1-2^{-s_0q'_+}}\Big)^{1/q'_+}\Big\|\Big(\sum_{k\in\mathbb{N}_0}(2^{ks(\cdot)}H_k)^{q_+}\Big)^{1/q_+}\Big\|_{L^{p(\cdot)}}\\ &\lesssim \Big\|\|2^{ks(\cdot)}H_k\|_{\ell^{q_+}}\Big\|_{L^{p(\cdot)}},\end{aligned}$$

where q'_+ is the conjugate exponent of q_+. This completes the proof. □

LEMMA 6. *Assume that* $s(\cdot) > 0, q_- > 0$, *then*

$$\Big(\sum_{k=0}^{\ell-1} 2^{ks_+q(\cdot)}\Big)^{\frac{1}{q(\cdot)}} \lesssim 2^{\frac{q_+}{q_-}s_+\ell}.$$

Proof. It is easy to see that

$$\Big(\sum_{k=0}^{\ell-1} 2^{ks_+q(x)}\Big)^{\frac{1}{q(x)}} = \Big(\frac{2^{\ell s_+q(x)}-1}{2^{s_+q(x)}-1}\Big)^{\frac{1}{q(x)}} \leqslant \Big(\frac{2^{\ell s_+q_+}}{2^{s_+q_-}-1}\Big)^{\frac{1}{q_-}} \lesssim 2^{\frac{q_+}{q_-}s_+\ell}. \quad \square$$

THEOREM 2. *Let* $p(\cdot), p_1(\cdot), p_2(\cdot), q(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ *be such that* $\frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)} = \frac{1}{p(\cdot)}$. *Assume that* $s(\cdot) \in C^{\log}_{\mathrm{loc}}$ *and the symbol* m *is the same as Lemma 3.1. Let* $N, R \in \mathbb{N}_0$ *be even numbers with* $N \geqslant R$. *If* $N > \frac{2n}{\min(p_{1-},p_{2-},q_-,1)} + \max(6C_{\log}(s), 6) + n$, *then*

$$\Big\|\{2^{ks(\cdot)}\sum_{j=0}^{k}|T_{m_{j,k,\ell}}(f,g)|\}_{k\in\mathbb{N}_0}\Big\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} \lesssim 2^{-\ell R}\|f\|_{F^{s(\cdot)}_{p_1(\cdot),q(\cdot)}}\|g\|_{F^{0}_{p_2(\cdot),1}},$$

for every $f, g \in \mathscr{S}(\mathbb{R}^n)$.

Proof. Let $\{\psi_k\}_{k\in\mathbb{N}_0}$ and $\{\tilde{\psi}_k\}_{k\in\mathbb{N}_0}$ be functions in $\mathbb{R}^n$ as defined in Definition 1. Set $f_k := \tilde{\psi}_k(D)f$ and $g_j := \tilde{\psi}_j(D)g$ for $j,k \in \mathbb{N}_0, j \leqslant k$. Then

$$\begin{aligned}m_{j,k,\ell}(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta) &= m_{j,k,\ell}(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)\tilde{\psi}_k(\xi)\tilde{\psi}_j(\eta)\\ &= m_{j,k,\ell}(\xi,\eta)\widehat{f_k}(\xi)\widehat{g_j}(\eta),\end{aligned}$$

and

$$T_{m_{j,k,\ell}}(f,g)(x) = \int_{\mathbb{R}^{2n}} m_{j,k,\ell}(\xi,\eta)\widehat{f_k}(\xi)\widehat{g_j}(\eta)e^{2\pi ix\cdot(\xi+\eta)}d\xi d\eta.$$

Using (4), (5) and the definition of the maximal functions in (3), for $a>0$ and $N>a+n$, we obtain that

$$\begin{aligned}
&|\int_{\mathbb{R}^{2n}} m_{j,k,\ell}(\xi,\eta)\widehat{f_k}(\xi)\widehat{g_j}(\eta)e^{2\pi ix\cdot(\xi+\eta)}d\xi d\eta| \\
&=|\int_{\mathbb{R}^{2n}} G_{j,k,\ell}(-y,-z)f_k(x-y)g_j(x-z)dydz| \\
&\leqslant \int_{\mathbb{R}^{2n}} |G_{j,k,\ell}(-y,-z)|(1+2^k|y|+2^j|z|)^a \frac{|f_k(x-y)||g_j(x-z)|}{(1+2^k|y|)^{a/2}(1+2^j|z|)^{a/2}}dydz \\
&\lesssim 2^{-\ell R} f_k^{*a/2}(x)g_j^{*a/2}(x).
\end{aligned}$$

Thus, for all $x\in\mathbb{R}^n$, we get that

$$|T_{m_{j,k,\ell}}(f,g)(x)| \lesssim 2^{-\ell R} f_k^{*a/2}(x)g_j^{*a/2}(x).$$

If $a>\frac{2n}{\min(p_{1-},p_{2-},q_-,1)}+\max(6C_{\log}(s),6)$, then

$$\begin{aligned}
&\Big\|\Big\{2^{ks(\cdot)}f_k^{*a/2}(x)\sum_{j=0}^{k} g_j^{*a/2}(x)\Big\}_{k\in\mathbb{N}_0}\Big\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} \\
&\leqslant \Big\|\Big\{2^{ks(\cdot)}f_k^{*a/2}(x)\sum_{j\in\mathbb{N}_0} g_j^{*a/2}(x)\Big\}_{k\in\mathbb{N}_0}\Big\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} \\
&=\Big\|\Big(\sum_{k\in\mathbb{N}_0}\big(2^{ks(\cdot)}f_k^{*a/2}(x)\sum_{j\in\mathbb{N}_0} g_j^{*a/2}(x)\big)^{q(\cdot)}\Big)^{\frac{1}{q(\cdot)}}\Big\|_{L^{p(\cdot)}} \\
&=\Big\|\Big(\sum_{k\in\mathbb{N}_0}\big(2^{ks(\cdot)}f_k^{*a/2}(x)\big)^{q(\cdot)}\Big)^{\frac{1}{q(\cdot)}}\sum_{j\in\mathbb{N}_0} g_j^{*a/2}(x)\Big\|_{L^{p(\cdot)}} \\
&\leqslant \Big\|\Big(\sum_{k\in\mathbb{N}_0}\big(2^{ks(\cdot)}f_k^{*a/2}(x)\big)^{q(\cdot)}\Big)^{\frac{1}{q(\cdot)}}\Big\|_{L^{p_1(\cdot)}}\Big\|\sum_{j\in\mathbb{N}_0} g_j^{*a/2}(x)\Big\|_{L^{p_2(\cdot)}} \\
&=\Big\|\Big\{2^{ks(\cdot)}f_k^{*a/2}(x)\Big\}_{k\in\mathbb{N}_0}\Big\|_{L^{p_1(\cdot)}(\ell^{q(\cdot)})}\Big\|\Big\{g_j^{*a/2}(x)\Big\}_{k\in\mathbb{N}_0}\Big\|_{L^{p_2(\cdot)}(\ell^{1})} \\
&\lesssim \|f\|_{F^{s(\cdot)}_{p_1(\cdot),q(\cdot)}}\|g\|_{F^{0}_{p_2(\cdot),1}}.
\end{aligned}$$

Therefore,

$$\|\{2^{ks(\cdot)}\sum_{j=0}^{k}|T_{m_{j,k,\ell}}(f,g)|\}_{k\in\mathbb{N}_0}\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} \lesssim 2^{-\ell R}\|f\|_{F^{s(\cdot)}_{p_1(\cdot),q(\cdot)}}\|g\|_{F^{0}_{p_2(\cdot),1}}. \quad \square$$

THEOREM 3. *Let $p(\cdot),p_1(\cdot),p_2(\cdot),q(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$ be such that $\frac{1}{p_1(\cdot)}+\frac{1}{p_2(\cdot)}=\frac{1}{p(\cdot)}$. Assume that $s(\cdot)\in C^{\log}_{\mathrm{loc}}\cap L^\infty$, and given an arbitrary s_0, such that $0<s_0\leqslant s_-$.*

Let $N,R \in \mathbb{N}_0$ be even numbers with $N \geqslant R > \frac{q_+}{q_-}s_+$. Assume that the symbol m is the same as Lemma 3.1. If $N > \frac{2n}{\min(p_{1-},p_{2-},q_-,1)} + \max(6C_{\log}(s),6)+n$, then

$$\Big\| \sum_{\substack{j,k\in\mathbb{N}_0\\ j\leqslant k}} T_{m_{j,k,\ell}}(f,g) \Big\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim 2^{(\frac{q_+}{q_-}s_+ - R)\ell} \|f\|_{F^{s(\cdot)}_{p_1(\cdot),q(\cdot)}} \|g\|_{F^0_{p_2(\cdot),1}}$$

for all $\ell \in \mathbb{N}_0$ and $f,g \in \mathscr{S}(\mathbb{R}^n)$.

Proof. For each fixed $\ell \in \mathbb{N}_0$, denote

$$h_\ell := \sum_{\substack{j,k\in\mathbb{N}_0\\ j\leqslant k}} T_{m_{j,k,\ell}}(f,g).$$

Now we will estimate $\|h_\ell\|^{\upsilon_\ell}_{F^s_{p,q}}$ for a proper sequence of functions υ_ℓ. Define the sequence $\upsilon_\ell := \{\upsilon_{k,\ell}\}_{k\in\mathbb{N}_0}$ as follows

$$\upsilon_{k,\ell} := \begin{cases} 0, & \text{if } k \leqslant \ell-1, \\ \sum_{\nu=0}^{k-\ell}\sum_{j=0}^{\nu} T_{m_{j,\nu,\ell}}(f,g), & \text{if } k \geqslant \ell. \end{cases}.$$

Using a similar argument to the proof of Lemma 3.3 of [32], we get that

$$\upsilon_{k,\ell} \in L^{p(\cdot)}(\mathbb{R}^n) \cap \mathscr{S}'(\mathbb{R}^n) \text{ and } \lim_{k\to\infty} \upsilon_{k,\ell} = h_\ell \text{ in } \mathscr{S}'(\mathbb{R}^n).$$

Using Lemma 5 and Theorem 2, we have

$$\begin{aligned} \|h_\ell\|_{L^{p(\cdot)}} &= \Big\| \sum_{\substack{j,k\in\mathbb{N}_0\\ j\leqslant k}} T_{m_{j,k,\ell}}(f,g) \Big\|_{L^{p(\cdot)}} \\ &\lesssim \Big\| \Big(\sum_{k\in\mathbb{N}_0} \big(2^{ks(\cdot)} \sum_{j=0}^{k} |T_{m_{j,k,\ell}}(f,g)|\big)^{q(\cdot)} \Big)^{\frac{1}{q(\cdot)}} \Big\|_{L^{p(\cdot)}} \\ &\lesssim 2^{-\ell R} \|f\|_{F^{s(\cdot)}_{p_1(\cdot),q(\cdot)}} \|g\|_{F^0_{p_2(\cdot),1}}. \end{aligned} \tag{6}$$

Observe that $\upsilon_{0,\ell} = 0$ if $\ell \in \mathbb{N}$, $\upsilon_{0,0} = T_{m_{0,0,0}}(f,g)$ and (6) implies that $\|T_{m_{0,0,0}}(f,g)\|_{L^{p(\cdot)}} \lesssim 2^{-\ell R}\|f\|_{F^{s(\cdot)}_{p_1(\cdot),q(\cdot)}} \|g\|_{F^0_{p_2(\cdot),1}}$. Thus,

$$\|\upsilon_{0,\ell}\|_{L^{p(\cdot)}} \lesssim 2^{-\ell R}\|f\|_{F^{s(\cdot)}_{p_1(\cdot),q(\cdot)}} \|g\|_{F^0_{p_2(\cdot),1}},$$

for all $\ell \in \mathbb{N}_0$.

Now we estimate $\big\|\{2^{ks(\cdot)}|h_\ell - \upsilon_{k,\ell}|\}_{k\in\mathbb{N}_0}\big\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}$ by breaking the sum in k into $k \leqslant \ell-1$ and $k \geqslant \ell$. Since $\upsilon_{k,\ell} = 0$ if $k \leqslant \ell-1$, about the first part, using Lemma 6, we obtain

$$\begin{aligned}\left\|\left(\sum_{k=0}^{\ell-1}(2^{ks(\cdot)}|h_\ell|)^{q(\cdot)}\right)^{1/q(\cdot)}\right\|_{L^{p(\cdot)}} &= \left(\sum_{k=0}^{\ell-1}2^{ks(\cdot)q(\cdot)}\right)^{1/q(\cdot)}\|h_\ell\|_{L^{p(\cdot)}}\\ &\leqslant \left(\sum_{k=0}^{\ell-1}2^{ks_+q(\cdot)}\right)^{\frac{1}{q(\cdot)}}\|h_\ell\|_{L^{p(\cdot)}}\\ &\lesssim 2^{(\frac{q_+}{q_-}s_+-R)\ell}\|f\|_{F^{s(\cdot)}_{p_1(\cdot),q(\cdot)}}\|g\|_{F^0_{p_2(\cdot),1}},\end{aligned}$$

in which the last inequality is due to (6).

For the second part (when $k\geqslant\ell$), we get

$$h_\ell-\upsilon_{k,\ell}=\sum_{\nu=k-\ell+1}^{\infty}\sum_{j=0}^{\nu}T_{m_{j,\nu,\ell}}(f,g)=\sum_{\nu=1}^{\infty}\sum_{j=0}^{k-\ell+\nu}T_{m_{j,k-\ell+\nu,\ell}}(f,g),$$

and then using Lemma 2 and Theorem 2, it follows that

$$\begin{aligned}&\left\|\{2^{ks(\cdot)}|h_\ell-\upsilon_{k,\ell}|\}_{k=\ell}^{\infty}\right\|^{\min(p_-,q_-,1)}_{L^{p(\cdot)}(\ell^{q(\cdot)})}\\ &\leqslant\left\|\left\{2^{ks(\cdot)}\sum_{\nu=1}^{\infty}\sum_{j=0}^{k-\ell+\nu}|T_{m_{j,k-\ell+\nu,\ell}}(f,g)|\right\}_{k=\ell}^{\infty}\right\|^{\min(p_-,q_-,1)}_{L^{p(\cdot)}(\ell^{q(\cdot)})}\\ &=\left\|\left\{\sum_{\nu=1}^{\infty}2^{(\ell-\nu)s(\cdot)}2^{(k-\ell+\nu)s(\cdot)}\sum_{j=0}^{k-\ell+\nu}|T_{m_{j,k-\ell+\nu,\ell}}(f,g)|\right\}_{k=\ell}^{\infty}\right\|^{\min(p_-,q_-,1)}_{L^{p(\cdot)}(\ell^{q(\cdot)})}\\ &\leqslant\sum_{\nu=1}^{\infty}2^{(\ell-\nu)s_+\min(p_-,q_-,1)}\left\|\left\{2^{(k-\ell+\nu)s(\cdot)}\sum_{j=0}^{k-\ell+\nu}|T_{m_{j,k-\ell+\nu,\ell}}(f,g)|\right\}_{k=\ell}^{\infty}\right\|^{\min(p_-,q_-,1)}_{L^{p(\cdot)}(\ell^{q(\cdot)})}\\ &\lesssim\sum_{\nu=1}^{\infty}\left(2^{(\ell-\nu)s_+}2^{-\ell R}\|f\|_{F^{s(\cdot)}_{p_1(\cdot),q(\cdot)}}\|g\|_{F^0_{p_2(\cdot),1}}\right)^{\min(p_-,q_-,1)}\\ &\lesssim\left(2^{(s_+-R)\ell}\|f\|_{F^{s(\cdot)}_{p_1(\cdot),q(\cdot)}}\|g\|_{F^0_{p_2(\cdot),1}}\right)^{\min(p_-,q_-,1)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|h_\ell\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}}&\lesssim\left\|\upsilon_{0,\ell}\right\|_{L^{p(\cdot)}}+\left\|\{2^{ks}|h_\ell-\upsilon_{k,\ell}|\}_{k\in\mathbb{N}_0}\right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}\\ &\lesssim 2^{(\frac{q_+}{q_-}s_+-R)\ell}\|f\|_{F^{s(\cdot)}_{p_1(\cdot),q(\cdot)}}\|g\|_{F^0_{p_2(\cdot),1}}.\end{aligned}$$

This ends our proof. □

Finally, we are ready to prove Theorem 1.

Proof of Theorem 1. From Remark 1, it suffices to prove (1). Because the estimate for T_2 is similiar with the one for T_1, so we only deal with T_1. Since

$$T_1(x) = \sum_{\ell\in\mathbb{N}_0} \sum_{\substack{j,k\in\mathbb{N}_0 \\ j\leqslant k}} T_{m_{j,k,\ell}}(f,g)(x),$$

if we choose $R > \frac{q_+}{q_-}s_+$, then Theorem 3 implies

$$\|T_1\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \|f\|_{F^{s(\cdot)}_{p_1(\cdot),q(\cdot)}} \|g\|_{F^0_{p_2(\cdot),1}}.$$

Interchanging the roles of j and k, f and g, ξ and η, we obtain that

$$\|T_2\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \|f\|_{F^{s(\cdot)}_{p_1(\cdot),q(\cdot)}} \|g\|_{F^0_{p_2(\cdot),1}},$$

and (1) is proved.

This completes the proof of Theorem 1. □

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