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FRACTIONAL ORDER HARDY–TYPE INEQUALITY IN FRACTIONAL h–DISCRETE CALCULUS

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Abstract. We investigate the power weights fractional order Hardy-type inequality in the following form:

$$\left(\int_0^\infty\int_0^\infty \frac{|f(x)-f(y)|^p}{|x-y|^{1+p\alpha}}dxdy\right)^p \leqslant C\left(\int_0^\infty |f'(x)|^p x^{(1-\alpha)p}dx\right)^p$$

for $0<\alpha<1$ and $1<p<\infty$ in fractional h-discrete calculus, where $C=\frac{2^{\frac{1}{p}}\alpha^{-1}}{(p-p\alpha)^{\frac{1}{p}}}$. For h-fractional function we prove a discrete analogue of above inequality in fractional h-discrete calculus, is proved and discussed. Moreover, we prove that the same constant is sharp also in this case.

1. Introduction

Fractional h-discrete calculus has generated interest in recent years. It is a mathematical subject that has proved to be very useful in applied fields such as economics, engineering and physics (see, e.g. [3], [4], [25], [26], [34]). Concerning applications in various fields of mathematics we refer to [1], [2], [6], [7], [12], [14], [15], [19], [21], [24], [27], [28], [29], [33], [35], [36] and the references therein.

It is well known that integral inequalities play important roles in the research of qualitative as well as quantitative properties of solutions of differential equations, difference equations and dynamic equations. One of the examples is fractional Hardy-type inequalities. In [13], [16], [17], [18] and [20] a series of fractional order Hardy- type inequalities have been presented. We pronounce especially that even Chapter 5 in the new book [22] by A. Kufner, L.-E. Persson and N. Samko is completely devoted to this subject. In particular, it is proved there (see Theorem 5.3) that

$$\left(\int_0^\infty\int_0^\infty \frac{|f(x)-f(y)|^p}{|x-y|^{1+p\alpha}}dxdy\right)^p \leqslant C\left(\int_0^\infty |f'(x)|^p x^{(1-\alpha)p}\right)^p, \tag{1}$$

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for $0 < \alpha < 1$, $1 < p < \infty$, where $C = \frac{2^{\frac{1}{p}}\alpha^{-1}}{(p-p\alpha)^{\frac{1}{p}}}$ is the sharp constant. Moreover, in [5], [9], [10], [23], [31] and [37] some discrete Hardy-type inequalities have been established, which can be used as a handy tool in the research of solutions of difference equations. Up to now the discrete analogues of the fractional Hardy-type inequalities are not studied. The main aim of this paper is to establish the h-analogue of the fractional Hardy-type inequality (1) in fractional h-discrete calculus with sharp constants which is a discrete analogue of the inequality (1).

The paper is organized as follows: In order not to disturb our discussions later on some preliminaries are presented in Section 2. The main result (see Theorem 3.1) with the detailed proof can be found in Section 3.

2. Preliminaries

First we state some preliminary results of the h-discrete fractional calculus, which will be used throughout this paper.

Let $h > 0$ and $\mathbb{T}_a = \{a, a+h, a+2h, \cdots\}$, $\forall a \in \mathbb{R}$.

DEFINITION 1. Let $f : \mathbb{T}_a \to \mathbb{R}$. Then the h-derivative of the function $f = f(x)$ is defined by

$$D_h f(t) := \frac{f(\delta_h(t)) - f(t)}{h}, \quad t \in \mathbb{T}_a, \tag{2}$$

where $\delta_h(t) = t + h$.

See e.g. [8]. The chain rule formula that we will use in this paper is

$$D_h\left[x^\gamma(t)\right] := \gamma \int_0^1 \left[zx(\delta_h(t)) + (1-z)x(t)\right]^{\gamma-1} dz D_h x(t), \ \ \gamma \in \mathbb{R}, \tag{3}$$

which is a simple consequence of Keller's chain rule ([11, Theorem 1.90]).

DEFINITION 2. Let $f : \mathbb{T}_a \to \mathbb{R}$. Then the h-integral (h-difference sum) is given by

$$\int_a^b f(x) d_h x := \sum_{k=a/h}^{b/h-1} f(kh)h = \sum_{k=0}^{\frac{b-a}{h}-1} f(a+kh)h, \tag{4}$$

for $a, b \in \mathbb{T}_a, b > a$.

DEFINITION 3. We say that a function $g : \mathbb{T}_a \longrightarrow R$, is nonincreasing (respectively, nondecreasing) on $\mathbb{T}_a$ if and only if $D_h g(t) \leqslant 0$ (respectively, $D_h g(t) \geqslant 0$) whenever $t \in \mathbb{T}_a$.

Let $D_hF(x) = f(x)$. Then $F(x)$ is called a h-antiderivative of $f(x)$ and is denoted by $\int f(x)d_hx$. If $F(x)$ is a h-antiderivative of $f(x)$, for $a,b \in \mathbb{T}_a, b > a$, then we have that (see [21]):

$$\int_a^b f(x)d_hx = F(b) - F(a). \tag{5}$$

DEFINITION 4. Let $t,\alpha \in \mathbb{R}$. Then the h-fractional function $t_h^{(\alpha)}$ is defined by

$$t_h^{(\alpha)} := h^\alpha \frac{\Gamma(\frac{t}{h}+1)}{\Gamma(\frac{t}{h}+1-\alpha)},$$

where Γ is Euler gamma function, $\frac{t}{h} \notin \{-1,-2,-3,\cdots\}$ and we use the convention that division at a pole yields zero. Note that

$$\lim_{h\to 0} t_h^{(\alpha)} = t^\alpha.$$

Hence, by (2) we find that

$$t_h^{(\alpha-1)} = \frac{1}{\alpha} D_h \left[t_h^{(\alpha)}\right], \tag{6}$$

$$(a-t-h)_h^{(\alpha-1)} = -\frac{1}{\alpha} D_h \left[(a-t)_h^{(\alpha)}\right], \tag{7}$$

$$\frac{1}{(t+h)_h^{(\alpha+1)}} = -\frac{1}{\alpha} D_h \left[\frac{1}{t_h^{(\alpha)}}\right], \tag{8}$$

$$\frac{1}{(a-t)_h^{(\alpha+1)}} = \frac{1}{\alpha} D_h \left[\frac{1}{(a-t)_h^{(\alpha)}}\right]. \tag{9}$$

DEFINITION 5. The function $f:(0,\infty) \to \mathbb{R}$ is said to be log-convex if $f(ux+(1-u)y) \leqslant f^u(x)f^{1-u}(y)$, holds for all $x,y \in (0,\infty)$ and $0 < u < 1$.

Next, we will derive some properties of the h-fractional function, which we need for the proofs of the main results but which are also of independent interest.

PROPOSITION 1. *Let $t \in \mathbb{T}_0$. Then, for $\alpha,\beta \in \mathbb{R}$,*

$$t_h^{(\alpha+\beta)} = t_h^{(\alpha)}(t-\alpha h)_h^{(\beta)}, \tag{10}$$

$$t_h^{(p\alpha)} \leqslant \left[t_h^{(\alpha)}\right]^p \leqslant (t+\alpha(p-1)h)_h^{(p\alpha)}, \tag{11}$$

for $1 \leqslant p < \infty$.

Proof. By using Definition 4 we get that

$$t_h^{(\alpha+\beta)} = h^{\alpha+\beta}\frac{\Gamma(\frac{t}{h}+1)}{\Gamma(\frac{t}{h}+1-\alpha-\beta)}$$
$$= h^{\alpha}\frac{\Gamma(\frac{t}{h}+1)}{\Gamma(\frac{t}{h}+1-\alpha)}h^{\beta}\frac{\Gamma(\frac{t}{h}+1-\alpha)}{\Gamma(\frac{t}{h}+1-\alpha-\beta)} = t_h^{(\alpha)}(t-\alpha h)_h^{(\beta)}.$$

Therefore, (10) holds for $\alpha,\beta\in\mathbb{R}$.
It's well known that the gamma function is log-convex (see e.g [30, p. 21]). Hence,

$$\left[t_h^{(\alpha)}\right]^p = h^{p\alpha}\left[\frac{\Gamma(\frac{t}{h}+1)}{\Gamma(\frac{t}{h}+1-\alpha)}\right]^p$$
$$= h^{p\alpha}\left[\frac{\Gamma(\frac{1}{p}(\frac{t}{h}+1+\alpha(p-1))+(1-\frac{1}{p})(\frac{t}{h}+1-\alpha))}{\Gamma(\frac{t}{h}+1-\alpha)}\right]^p$$
$$\leqslant h^{p\alpha}\left[\frac{\Gamma^{\frac{1}{p}}(\frac{1}{h}+1+\alpha(p-1))\Gamma^{1-\frac{1}{p}}(\frac{t}{h}+1-\alpha)}{\Gamma(\frac{t}{h}+1-\alpha)}\right]^p$$
$$= h^{p\alpha}\frac{\Gamma(\frac{t}{h}+1+\alpha(p-1))}{\Gamma(\frac{t}{h}+1-\alpha)}$$
$$= (t+\alpha(p-1)h)_h^{(p\alpha)},$$

and

$$\left[t_h^{(\alpha)}\right]^p = h^{p\alpha}\left[\frac{\Gamma(\frac{t}{h}+1)}{\Gamma(\frac{t}{h}+1-\alpha)}\right]^p$$
$$= h^{p\alpha}\left[\frac{\Gamma(\frac{t}{h}+1)}{\Gamma((1-\frac{1}{p})(\frac{t}{h}+1)+\frac{1}{p}(\frac{t}{h}+1-p\alpha))}\right]^p$$
$$\geqslant h^{p\alpha}\left[\frac{\Gamma(\frac{t}{h}+1)}{\Gamma^{1-\frac{1}{p}}(\frac{t}{h}+1)\Gamma^{\frac{1}{p}}(\frac{t}{h}+1-p\alpha)}\right]^p$$
$$= h^{p\alpha}\frac{\Gamma(\frac{t}{h}+1)}{\Gamma(\frac{t}{h}+1-p\alpha)} = t_h^{(p\alpha)},$$

so we have proved that (11) holds whenever $1\leqslant p<\infty$. $\square$

Let $1\leqslant p\leqslant q<\infty$ and $\frac{1}{p}+\frac{1}{p'}=1$. Let $f=\{f_i\}_{i=0}^{\infty}$ be an arbitrary sequence of real numbers. Moreover, suppose that $\{u_i\}_{0=1}^{\infty}$, and $\{v_i\}_{i=0}^{\infty}$ are weight sequences, i.e., non-negative sequences. To prove our main result we use the following result for a standard weighted Hardy inequality, when $1\leqslant p\leqslant q<\infty$ (see [5, Theorem 4.1] and e.g. also [22]):

THEOREM B. *Let* $1\leqslant p\leqslant q<\infty$. *Then the inequality*

$$\left(\sum_{i=1}^{\infty}\left(\sum_{j=1}^{i}f_j\right)^q u_i^q\right)^{\frac{1}{q}} \leqslant C\left(\sum_{i=1}^{\infty}(f_iv_i)^p\right)^{\frac{1}{p}},$$

holds for all sequences $f=\{f_i\}_{i=0}^{\infty}$, $f_i\geqslant 0$, $i\geqslant 1$, *with the best constant* $C>0$ *if and only if* $B=\sup\limits_{k\geqslant 1}\left(\sum\limits_{i=k}^{\infty}u_i^q\right)^{\frac{1}{q}}\left(\sum\limits_{j=1}^{k}v_j^{-p'}\right)^{\frac{1}{p'}}<\infty$. *Moreover,* $B\leqslant C\leqslant p'q^{\frac{1}{q}}B$.

3. Main results

Our main result reads:

THEOREM 1. *Let* $1<p<\infty$, $0<\alpha<1$ *and* $f(x)=D_hF(x)$. *Then the following inequality*

$$\left[\int_0^{\infty}\int_0^{\infty}\frac{|F(x)-F(y)|^p d_hxd_hy}{\left[(|x-y|+3h)_h^{(\frac{1}{p}+\alpha)}\right]^p}\right]^{\frac{1}{p}}\leqslant C\left[\int_0^{\infty}\frac{|f(x)|^p\,d_hx}{\left[(x+h)_h^{(\alpha-1)}\right]^p}\right]^{\frac{1}{p}}, \tag{12}$$

holds with constant $C=\dfrac{2^{\frac{1}{p}}\alpha^{-1}}{(p-p\alpha)^{\frac{1}{p}}}$. *Moreover, this constant is sharp.*

The next lemma permits to shorten the proof of our main result:

LEMMA 1. *Let* $0<\alpha<1$, $1<p<\infty$ *and* $\frac{1}{p}+\frac{1}{p'}=1$. *Then*

$$B:=\sup_{z\in\mathbb{T}_0}\left(\int_z^{\infty}\frac{d_hx}{\left[(x+3h)_h^{(\frac{1}{p}+\alpha)}\right]^p}\right)^{\frac{1}{p}}\left(\int_0^{\delta(z)}\frac{t_h^{(\alpha-1)}d_ht}{\left[\delta(t)_h^{(\alpha)}\right]^{-\frac{p'}{p}}}\right)^{\frac{1}{p'}}<\frac{1}{\alpha}. \tag{13}$$

Proof. Let $0<\alpha<1$, $1\leqslant p<\infty$ and $\frac{1}{p}+\frac{1}{p'}=1$. Exploiting (2) we find that $D_h\left[x_h^{(\alpha)}\right]=\alpha x_h^{(\alpha-1)}\geqslant 0$ for $x\in\mathbb{T}_0$. Moreover, in view of Definition 3 we see that $x_h^{(\alpha)}\leqslant(x')_h^{(\alpha)}$ for $x,x'\in\mathbb{T}_0$ such that $x\leqslant x'$. Then, according to (2), (3), (8), (9) and (11), we obtain that

$$\begin{aligned}
D_h\left[\frac{1}{(x+2h)_h^{(\alpha)}}\right]^p&=p\int_0^1\left[\frac{z}{(x+3h)_h^{(\alpha)}}+\frac{1-z}{(x+2h)_h^{(\alpha)}}\right]^{p-1}dzD_h\left[\frac{1}{(x+2h)_h^{(\alpha)}}\right]\\
&=-p\alpha\frac{1}{(x+3h)_h^{(\alpha+1)}}\int_0^1\left[\frac{z}{(x+3h)_h^{(\alpha)}}+\frac{1-z}{(x+2h)_h^{(\alpha)}}\right]^{p-1}dz\\
&\leqslant-p\alpha\frac{1}{(x+3h)_h^{(\alpha+1)}}\left[\frac{1}{(x+3h)_h^{(\alpha)}}\right]^{p-1}
\end{aligned}$$

and

$$\left[(x+3h)_h^{(\frac{1}{p}+\alpha)}\right]^p = \left[(x+3h)_h^{(\alpha)}\right]^p \left[(x+3h-\alpha h)_h^{(\frac{1}{p})}\right]^p$$
$$\geqslant \left[(x+3h)_h^{(\alpha)}\right]^{p-1} (x+3h)_h^{(\alpha)} (x+3h-\alpha h)_h^{(1)}$$
$$= \left[(x+3h)_h^{(\alpha)}\right]^{p-1} (x+3h)_h^{(\alpha+1)}$$

i.e.

$$\left[\frac{1}{(x+3h)_h^{(\frac{1}{p}+\alpha)}}\right]^p \leqslant -\frac{1}{p\alpha} D_h \left[\frac{1}{(x+2h)_h^{(\alpha)}}\right]^p. \tag{14}$$

Next we note that, by Definition 3 and (6), $t_h^{(\alpha)} \leqslant (z+h)_h^{(\alpha)}$, for $t, z \in \mathbb{T}_0$ such that $t \leqslant z+h$ and then, by applying (5), (6) and (14), we get that

$$B^p \leqslant -\frac{1}{p\alpha} \sup_{z\in\mathbb{T}_0} (z+2h)_h^{(\alpha)} \int_z^\infty D_h \left[\frac{1}{(x+2h)_h^{(\alpha)}}\right]^p d_h x \left[\int_0^{\delta(z)} t_h^{(\alpha-1)} d_h t\right]^{\frac{p}{p'}}$$
$$\leqslant \frac{1}{p\alpha} \sup_{z\in\mathbb{T}_0} (z+2h)_h^{(\alpha)} \int_z^\infty D_h \left[\frac{1}{(x+2h)_h^{(\alpha)}}\right]^p d_h x \left[\frac{1}{\alpha}\int_0^{\delta(z)} D_h \left[t_h^{(\alpha)}\right] d_h t\right]^{\frac{p}{p'}}$$
$$\leqslant \frac{1}{\alpha^p} \sup_{z\in\mathbb{T}_0} (z+2h)_h^{(\alpha)} \frac{\left[(z+2h)_h^{(\alpha)}\right]^{\frac{p}{p'}}}{\left[(z+2h)_h^{(\alpha)}\right]^p} = \frac{1}{\alpha^p},$$

i.e. (13) holds so the proof is complete. □

Proof of Theorem 3.1. By using (4) we get that

$$L(F) := \int_0^\infty \int_0^\infty \frac{|F(x)-F(y)|^p d_h x d_h y}{\left[(|x-y|+3h)_h^{(\frac{1}{p}+\alpha)}\right]^p}$$
$$= \sum_{k=0}^\infty \sum_{i=0}^\infty h^2 \frac{|F(ih)-F(kh)|^p}{\left[(|ih-kh|+3h)_h^{(\frac{1}{p}+\alpha)}\right]^p}$$
$$\leqslant \sum_{k=0}^\infty \sum_{i=0}^k h^2 \frac{|F(ih)-F(kh)|^p}{\left[(|ih-kh|+3h)_h^{(\frac{1}{p}+\alpha)}\right]^p} + \sum_{k=0}^\infty \sum_{i=k}^\infty h^2 \frac{|F(ih)-F(kh)|^p}{\left[(|ih-kh|+3h)_h^{(\frac{1}{p}+\alpha)}\right]^p}$$
$$= 2\sum_{k=0}^\infty \sum_{i=k}^\infty h^2 \frac{|F(ih)-F(kh)|^p}{\left[(|ih-kh|+3h)_h^{(\frac{1}{p}+\alpha)}\right]^p}. \tag{15}$$

Let

$$\tilde{f}_m = h\,|f(mh)|\,, \quad \tilde{u}_i = \frac{h^{\frac{1}{p}}}{(|ih-kh|+3h)_h^{(\frac{1}{p}+\alpha)}},$$

$$\tilde{v}_m = \frac{h^{-\frac{1}{p'}}}{\left[\left[(mh-kh)_h^{(\alpha-1)}\right]^{p-1}(\delta(mh)-kh)_h^{(\alpha)}\right]^{\frac{1}{p}}}$$

and $f(x) = D_h F(x)$. Then, from (4) and (15) it follows that

$$L(F) \leqslant 2\sum_{k=0}^{\infty} h \sum_{i=k}^{\infty} \frac{h}{\left[(|ih-kh|+3h)_h^{(\frac{1}{p}+\alpha)}\right]^p}\left|\sum_{m=k}^{i-1} hf(mh)\right|^p$$

$$\leqslant 2\sum_{k=0}^{\infty} h\left[\sum_{i=k}^{\infty}\tilde{u}_i^p\left(\sum_{m=k}^{i}\tilde{f}_m\right)^p\right]. \tag{16}$$

Moreover, based on Theorem B we obtain that

$$\sum_{i=k}^{\infty}\tilde{u}_i^p\left(\sum_{m=k}^{i}\tilde{f}_m\right)^p \leqslant \tilde{B}_k^p\sum_{m=k}^{\infty}\tilde{f}_m^p\tilde{v}_m^{-p'}, \tag{17}$$

where

$$\tilde{B}_k^p := \sup_{n\geqslant k}\left(\sum_{i=n}^{\infty}\tilde{u}_i^q\right)^{\frac{1}{q}}\left(\sum_{j=k}^{n}\tilde{v}_j^{-p'}\right)^{\frac{1}{p'}}$$

$$= \sup_{n\geqslant k}\left[\int_{nh}^{\infty}\frac{d_hx}{\left[(x-kh+3h)_h^{(\frac{1}{p}+\alpha)}\right]^p}\right]^{\frac{1}{p}}\left[\int_{nh}^{\delta(z)}\frac{(t-kh)_h^{(\alpha-1)}\,d_ht}{\left[(\delta(t)-kh)_h^{(\alpha)}\right]^{-\frac{p'}{p}}}\right]^{\frac{1}{p'}} \leqslant B^p.$$

By combining (16) and (17) we have that

$$L(F) \leqslant 2\sum_{k=0}^{\infty} hB^p\sum_{m=k}^{\infty}\frac{h\,|f(mh)|^p}{\left[(mh-kh)_h^{(\alpha-1)}\right]^{p-1}(\delta(mh)-kh)_h^{(\alpha)}}. \tag{18}$$

Moreover, by using Definition 3 and (7) we obtain that

$$(\delta(mh)-t)_h^{(\alpha-1)} \leqslant (mh-t)_h^{(\alpha-1)},$$

for $t\in\mathbb{T}_0$.

Hence, in view of (2), (3) and (9) we get that

$$
\begin{aligned}
& D_{h,t}\left[\frac{1}{(\delta(mh)-t)_h^{(\alpha-1)}}\right]^p \\
& = p\int_0^1\left[\frac{z}{(mh-t)_h^{(\alpha-1)}}+\frac{(1-z)}{(\delta(mh)-t)_h^{(\alpha-1)}}\right]^{p-1} dz D_{h,t}\left[\frac{1}{(\delta(mh)-t)_h^{(\alpha-1)}}\right] \\
& = \frac{p(\alpha-1)}{(\delta(mh)-t)_h^{(\alpha)}}\int_0^1\left[\frac{z}{(mh-t)_h^{(\alpha-1)}}+\frac{(1-z)}{(\delta(mh)-t)_h^{(\alpha-1)}}\right]^{p-1} dz \\
& \leqslant \frac{p(\alpha-1)}{(\delta(mh)-t)_h^{(\alpha)}}\left[\frac{1}{(mh-t)_h^{(\alpha-1)}}\right]^{p-1}.
\end{aligned}
$$

Consequently,

$$
\frac{1}{(\delta(mh)-t)_h^{(\alpha)}}\left[\frac{1}{(mh-t)_h^{(\alpha-1)}}\right]^{p-1} \leqslant \frac{1}{p(\alpha-1)} D_{h,t}\left[\frac{1}{(\delta(mh)-t)_h^{(\alpha-1)}}\right]^p. \tag{19}
$$

Thus, by now using Lemma 3.1 and (18) and (19), we obtain that

$$
\begin{aligned}
L(F) & \leqslant 2B^p\sum_{k=0}^{\infty} h\sum_{m=k}^{\infty}\frac{h|f(mh)|^p}{\left[(mh-kh)_h^{(\alpha-1)}\right]^{p-1}(\delta(mh)-kh)_h^{(\alpha)}} \\
& \leqslant 2B^p\sum_{m=0}^{\infty} h|f(mh)|^p\sum_{k=0}^{m}\frac{h}{\left[(mh-kh)_h^{(\alpha-1)}\right]^{p-1}(\delta(mh)-kh)_h^{(\alpha)}} \\
& \leqslant \frac{2\alpha^{-p}}{p(\alpha-1)}\sum_{m=0}^{\infty} h|f(mh)|^p\int_0^{\delta(mh)} D_{h,t}\left[\frac{1}{(\delta(mh)-t)_h^{(\alpha-1)}}\right]^p d_h t \\
& \leqslant \frac{2\alpha^{-p}}{p(1-\alpha)}\sum_{m=0}^{\infty} h\frac{|f(mh)|^p}{\left[(mh+h)_h^{(\alpha-1)}\right]^p} \\
& \leqslant \frac{2\alpha^{-p}}{p(1-\alpha)}\int_0^{\infty}\frac{|f(x)|^p\, d_h x}{\left[(x+h)_h^{(\alpha-1)}\right]^p},
\end{aligned}
$$

which means that inequality (12) holds.

Finally, we will show that the constant $\frac{2^{\frac{1}{p}}\alpha^{-1}}{(p\alpha-p)^{\frac{1}{p}}}$ in (12) sharp. Let $x,y,a \in \mathbb{T}_0$ such that $y \leqslant a \leqslant x-4h$. By Definition 3 we obtain that

$$
\left(x-y+2h-\alpha h+\frac{1}{p'}h\right)_h^{(1)} \leqslant (x-y+3h-\alpha h)_h^{(1)},
$$

$$(x-y+2h-(\alpha-1)h)_h^{(1)} \leqslant (x+4h-\alpha h)_h^{(1)}.$$

Then, by using (3), (9), (10) and (11) we find that

$$\begin{aligned}
&\left[(|x-y|+3h)_h^{(\frac{1}{p}+\alpha)}\right]^p \\
&= \left[(x-y+3h)_h^{(\alpha-1)}\right]^p \left[(x-y+2h-\alpha h)_h^{(\frac{1}{p})}\right]^p \left[(x-y+3h-(\alpha-1)h)_h^{(1)}\right]^p \\
&\leqslant \left[(x-y+3h)_h^{(\alpha-1)}\right]^{p-1} (x-y+2h)_h^{(\alpha-1)} \left(x-y+2h-\alpha h+1/p'h\right)_h^{(1)} \\
&\quad \times \left[(x-y+3h-(\alpha-1)h)_h^{(1)}\right]^p \\
&\leqslant \left[(x-y+3h)_h^{(\alpha-1)}\right]^{p-1} (x-y+3h)_h^{(\alpha)} \left[(2x-a+4h-\alpha h)_h^{(1)}\right]^p
\end{aligned}$$

and

$$\begin{aligned}
&D_{h,y}\left[\frac{1}{(x-y+3h)_h^{(\alpha-1)}}\right]^p \\
&= p\int_0^1 \left[\frac{z}{(x-y+2h)_h^{(\alpha-1)}} + \frac{1-z}{(x-y+3h)_h^{(\alpha-1)}}\right]^{p-1} dz D_{h,y}\left[\frac{1}{(x-y+3h)_h^{(\alpha-1)}}\right] \\
&= \int_0^1 \left[\frac{z}{(x-y+2h)_h^{(\alpha-1)}} + \frac{1-z}{(x-y+3h)_h^{(\alpha-1)}}\right]^{p-1} dz \frac{p(\alpha-1)}{(x-y+3h)_h^{(\alpha)}} \\
&\geqslant -\left[\frac{1}{(x-y+3h)_h^{(\alpha-1)}}\right]^{p-1} \frac{p(1-\alpha)}{(x-y+3h)_h^{(\alpha)}} \frac{\left[(2x-a+4h-\alpha h)_h^{(1)}\right]^p}{\left[(x-a)_h^{(1)}\right]^p}.
\end{aligned}$$

Therefore,

$$\left[\frac{1}{(x-y+3h)_h^{(\frac{1}{p}+\alpha)}}\right]^p \geqslant -\frac{\left[(x-a)_h^{(1)}\right]^{-p}}{p(1-\alpha)} D_h\left[\frac{1}{(x-y+3h)_h^{(\alpha-1)}}\right]^p. \tag{20}$$

Assume now on the contrary that there exists a constant $C < \frac{2^{\frac{1}{p}}\alpha^{-1}}{(p\alpha-p)^{\frac{1}{p}}}$ such that (12) holds for all measurable functions where the right hand side is finite. We now consider the test function

$$f_0 := \chi_{[a,a']}(t)\,(t-a-h+\alpha h)_h^{(\alpha-1)},$$

for $a' \in \mathbb{T}_0$ such that $x \leqslant a'$. Then, by using (5), (6) and (10) we can deduce that

$$\begin{aligned}|F(x)-F(y)|^p &= \left|\int_a^x (t-a-h+\alpha h)_h^{(\alpha-1)}\, d_h t\right|^p \\ &= \frac{1}{\alpha^p}\left|\int_a^x D_h\left[(t-a-h+\alpha h)_h^{(\alpha)}\right] d_h t\right|^p \\ &= \frac{1}{\alpha^p}\left[(x-a-h+\alpha h)_h^{(\alpha)}\right]^p \\ &= \frac{1}{\alpha^p}\left[(x-a-h+\alpha h)_h^{(\alpha-1)}\right]^p\left[(t-a)_h^{(1)}\right]^p, \end{aligned} \tag{21}$$

where $(-h+\alpha h)_h^{(\alpha)} = h^{\alpha}\frac{\Gamma(\alpha)}{\Gamma(0)} = 0$ and

$$\int_0^\infty \frac{f_0^p(x)d_h x}{\left[(x+h)_h^{(\alpha-1)}\right]^p} \leqslant \int_a^{a'} \frac{\left[(x-a-h+\alpha h)_h^{(\alpha-1)}\right]^p d_h x}{\left[(x+h)_h^{(1-\alpha)}\right]^p} < \infty.$$

By combining (4) and (15) we obtain that

$$\begin{aligned} L(F) &:= \int_0^\infty\int_0^x \frac{|F(x)-F(y)|^p d_h x d_h y}{\left[(|x-y|+3h)_h^{(\frac{1}{p}+\alpha)}\right]^p} + \int_0^\infty\int_x^\infty \frac{|F(x)-F(y)|^p d_h x d_h y}{\left[(|x-y|+3h)_h^{(\frac{1}{p}+\alpha)}\right]^p} \\ &:= I_1 + I_2. \end{aligned} \tag{22}$$

From (20) and (21) it follows that

$$\begin{aligned} I_1 &\geqslant \int_a^{a'}\int_0^x \frac{|F(x)-F(y)|^p d_h x d_h y}{\left[(|x-y|+3h)_h^{(\frac{1}{p}+\alpha)}\right]^p} \\ &\geqslant -\frac{\alpha^{-p}}{p(1-\alpha)}\left[\int_a^{a'}\left[(x-a-h+\alpha h)_h^{(\alpha-1)}\right]^p \right. \\ &\quad \left. \times \int_0^{a+4h} D_{h,y}\left[\frac{1}{(x-y+3h)_h^{(\alpha-1)}}\right]^p d_h y\right] d_h x \\ &\geqslant \frac{\alpha^{-p}}{p(1-\alpha)}\int_0^\infty \frac{f_0^p(x)d_h x}{\left[(x+h)_h^{(1-\alpha)}\right]^p}, \end{aligned} \tag{23}$$

where $\frac{1}{(-h)_h^{(\alpha-1)}} = \frac{\Gamma(\alpha-1)}{\Gamma(0)} = 0$.

In the same way we can deduce that

$$I_2 \geqslant \frac{\alpha^{-p}}{p(1-\alpha)} \int_0^\infty \frac{f_0^p(x) d_h x}{\left[(x+h)_h^{(1-\alpha)}\right]^p}. \tag{24}$$

By now using (22), (23) and (24) we obtain that

$$C^{\frac{1}{p}} \geqslant \frac{L(F)}{\int_0^\infty \frac{f_0^p(x) d_h x}{\left[(x+h)_h^{(\alpha-1)}\right]^p}} = \frac{2\alpha^{-p}}{p(1-\alpha)},$$

which contradicts our assumption so we conclude that the constant $\frac{2^{\frac{1}{p}}\alpha^{-1}}{(p-p\alpha)^{\frac{1}{p}}}$ in (12) sharp. The proof is complete. □

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