Mathematical
Inequalities
& Applications
Volume 22, Number 2 (2019), 703–718 doi:10.7153/mia-22-48

FRACTIONAL MAGNETIC SOBOLEV INEQUALITIES WITH TWO VARIABLES

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(*Communicated by I. Perić*)

Abstract. A fractional magnetic Sobolev inequality with two variables and critical exponents is considered in this paper, and the best constant in the inequality is determined. As an application of the inequality, we establish an existence result for the ground state solutions to a fractional magnetic critical system.

1. Introduction

Fractional magnetic problems are new. There are only a few results in the literature, such as [1, 2, 3, 4, 6, 9]. The so-called fractional magnetic Laplacian, which will be defined below and denoted $(-\Delta)^s_A$, can be considered as a fractional counterpart of the magnetic Laplacian $(\nabla - iA)^2$, with $A : \mathbb{R}^N \to \mathbb{R}^N$ being a vector potential. The motivations for this kind of problems rely essentially on the Lévy-Khintchine formula for the generator of a semigroup associated to a general Lévy process, which is more appropriate for some mathematical models in finance. For more details, we refer to d'Avenia and Squassina [4] and Ichinose [9].

Ambrosio and D'Avenia [2] studied a nonlinear fractional Schrödinger equation with magnetic field and a subcritical nonlinearity. Using variational methods and Ljusternick-Schnirelmann category, they got existence and multiplicity of solutions when the parameter is small. Binlin, Squassina and Xia [3] considered a singularly perturbed fractional Schrödinger equations involving critical frequency and critical growth in the presence of a magnetic field. Via variational methods, they obtained the existence of mountain pass solutions u_ε which tend to the trivial solution as $\varepsilon \to 0$. Fiscella, Pinamonti and Vecchi [6] investigated the existence of multiple solutions for a boundary value problem driven by the fractional magnetic Laplacian with a subcritical nonlinear term, under two different sets of conditions on the nonlinear term which are dual in a suitable sense. In a recent paper, d'Avenia and Squassina [4] proved the existence of solutions to $(-\Delta)^s_A u + u = |u|^{p-2}u$ in $\mathbb{R}^3$ for the subcritical and critical cases; therein $(-\Delta)^s_A$ is defined by the mid-point prescription (see below).

Mathematics subject classification (2010): 35R11, 35B25, 35B33, 35J61.
Keywords and phrases: Ground state, magnetic field, fractional Sobolev critical exponent.
Supported by the National Natural Science Foundation of China (Grant No. 11701248), the Doctoral Scientific Research Foundation of Liaoning Science & Technology Department (Grant No. 20180540028), and the Doctoral Scientific Research Foundation of Liaoning Normal University (Grant No. BS2018L004).

In the present paper, we study a fractional magnetic Sobolev inequality with two variables

$$\Lambda_{s,A}\left(\int_{\mathbb{R}^N}\left(\mu_1|u|^{2_s^*}+\mu_2|v|^{2_s^*}+\lambda|u|^{\alpha}|v|^{\beta}\right)\right)^{\frac{2}{2_s^*}} \leqslant \int_{\mathbb{R}^{2N}}\frac{\left|e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta}u(x)-u(y)\right|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^{2N}}\frac{\left|e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta}v(x)-v(y)\right|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y, \tag{1.1}$$

where $0<s<1, N>4s, 2_s^*:=\frac{2N}{N-2s}$ is fractional Sobolev critical exponent, μ_1,μ_2, $\alpha,\beta,\gamma>0$, $A:\mathbb{R}^N\to\mathbb{R}^N$ is a magnetic vector potential which is a continuous function with locally bounded gradient, $\Lambda_{s,A}$ is a constant and $u\in D_A^s(\mathbb{R}^N,\mathbb{C})$. Here $D_A^s(\mathbb{R}^N,\mathbb{C})$ is the completion of $C_c^\infty(\mathbb{R}^N,\mathbb{C})$ with respect to the so-called magnetic Gagliardo seminorm $[\cdot]_{D_A^s}$ given by

$$[u]_{D_A^s}^2:=\frac{c_{N,s}}{2}\int_{\mathbb{R}^{2N}}\frac{\left|e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta}u(x)-u(y)\right|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y$$

where

$$c_{N,s}:=\left(\int_{\mathbb{R}^N}\frac{1-\cos(\zeta_1)}{|\zeta|^{N+2s}}\mathrm{d}\zeta\right)^{-1}.$$

The scalar product in $D_A^s(\mathbb{R}^N,\mathbb{C})$ is

$$\langle u,v\rangle_{D_A^s}:=\frac{c_{N,s}}{2}\mathrm{Re}\int_{\mathbb{R}^{2N}}\frac{\left(e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta}u(x)-u(y)\right)}{|x-y|^{N+2s}}\cdot\overline{\left(e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta}v(x)-v(y)\right)}\mathrm{d}x\mathrm{d}y.$$

Although $[\cdot]_{D_A^s}$ is a seminorm, by fractional magnetic Sobolev embeddings (see Lemma 3.5 in [4]), we can view $[\cdot]_{D_A^s}$ as a norm $\|\cdot\|_{D_A^s}:=[\cdot]_{D_A^s}$ in space $D_A^s(\mathbb{R}^N,\mathbb{C})$. As Proposition 2.1 and 2.2 in [4], we can verify that $D_A^s(\mathbb{R}^N,\mathbb{C})$ is a Hilbert space.

The fractional magnetic inequality is related to fractional magnetic Laplacian defined by

$$(-\Delta)_A^s u(x)=c_{N,s}\lim_{\varepsilon\to 0^+}\int_{B_\varepsilon^c(x)}\frac{u(x)-e^{i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta}u(y)}{|x-y|^{N+2s}}\mathrm{d}y,\quad x\in\mathbb{R}^N.$$

For $N=3$ and with *mid-point presciption*, the fractional magnetic Laplacian was studied in [4]. More precisely, d'Avenia and Squassina [4] considered the operator

$$(-\Delta)_A^s u(x)=c_{N,s}\lim_{\varepsilon\to 0^+}\int_{B_\varepsilon^c(x)}\frac{u(x)-e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)}{|x-y|^{N+2s}}\mathrm{d}y,\quad x\in\mathbb{R}^3.$$

The operator $(-\Delta)_A^s$ treated in this paper can be regarded as a modification of the above-mentioned operator involving mid-point prescription.

Let $\mathscr{D}_A^s(\mathbb{R}^N,\mathbb{C}) := D_A^s(\mathbb{R}^N,\mathbb{C}) \times D_A^s(\mathbb{R}^N,\mathbb{C})$, endowed with norm $\|(u,v)\|^2_{\mathscr{D}_A^s} := \|u\|^2_{D_A^s} + \|v\|^2_{D_A^s}$. For similicity, we denote $|u|_{2_s^*} := \left(\int_{\mathbb{R}^N} |u|^{2_s^*}\right)^{1/2_s^*}$. Setting $S_A := c_{N,s}\Lambda_{s,A}/2$, then (1.1) is equivalent to the following minimization problem

$$S_A = \inf_{\substack{(u,v)\in\mathscr{D}_A^s(\mathbb{R}^N,\mathbb{C})\\ (u,v)\neq(0,0)}} \frac{\|(u,v)\|^2_{\mathscr{D}_A^s}}{\left(\int_{\mathbb{R}^N}\left(\mu_1|u|^{2_s^*}+\mu_2|v|^{2_s^*}+\lambda|u|^{\alpha}|v|^{\beta}\right)\right)^{\frac{2}{2_s^*}}}, \tag{1.2}$$

which can also be characterized as:

$$S_A = \inf_{u\in\mathscr{J}} \|(u,v)\|^2_{\mathscr{D}_A^s}, \tag{1.3}$$

where $A:\mathbb{R}^N\to\mathbb{R}^N$ is a continuous magnetic vector potential with locally bounded gradient, and

$$\mathscr{J} = \left\{(u,v)\in\mathscr{D}_A^s(\mathbb{R}^N,\mathbb{C}) : \int_{\mathbb{R}^N}\left(\mu_1|u|^{2_s^*}+\mu_2|v|^{2_s^*}+\lambda|u|^{\alpha}|v|^{\beta}\right)=1\right\}. \tag{1.4}$$

For the special case without magnetic fields, i.e., $A\equiv 0$, it was shown in [7] that, under the condition

$$(H) = \begin{cases} 1<\alpha,\beta<2, & \text{if } 4s<N<6s,\\ \alpha,\beta>1, & \text{if } N\geqslant 6s, \end{cases} \quad \text{with } \alpha+\beta=2_s^*,$$

S_0 is attained by (U,V), which is radially symmetric decreasing with the following decay condition

$$U(x),V(x)\leqslant C(1+|x|)^{2s-N}. \tag{1.5}$$

Our main result reads as follows:

THEOREM 1.1. *If Condition (H) holds and $A:\mathbb{R}^N\to\mathbb{R}^N$ is a continuous function with locally bounded gradient, then S_A is achieved by a nontrivial element $(U_A,V_A)\in\mathscr{D}_A^s(\mathbb{R}^N,\mathbb{C})$.*

As an application, we study the existence of ground state solutions to the following fractional magnetic critical system in a bounded set Ω of $\mathbb{R}^N$

$$\begin{cases} (-\Delta)_A^s u-\lambda_1 u=\mu_1|u|^{2_s^*-2}u+\frac{\alpha\gamma}{2_s^*}|u|^{\alpha-2}u|v|^{\beta},\\ (-\Delta)_A^s v-\lambda_2 v=\mu_2|v|^{2_s^*-2}v+\frac{\beta\gamma}{2_s^*}|u|^{\alpha}|v|^{\beta-2}v,\\ (u,v)\in\mathscr{D}_A^s(\Omega,\mathbb{C}):=D_A^s(\Omega,\mathbb{C})\times D_A^s(\Omega,\mathbb{C}), \end{cases} \tag{1.6}$$

where Ω is an open bounded Lipschitz domain in $\mathbb{R}^N$, $0<s<1, N>4s$, $\lambda_1,\lambda_2,\mu_1,\mu_2$, $\alpha,\beta,\gamma>0$, $A:\mathbb{R}^N\to\mathbb{R}^N$ is a magnetic vector potential,

$$D_A^s(\Omega,\mathbb{C}) := \left\{u\in D_A^s(\mathbb{R}^N,\mathbb{C}) : u=0 \text{ a.e. in } \Omega^c\right\},$$

equipped with the seminorm

$$\|u\|_{D^s_A(\Omega)} := \left(\frac{c_{N,s}}{2}\int_{\mathbb{R}^{2N}\setminus(\Omega^c\times\Omega^c)}\frac{\left|e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta}u(x)-u(y)\right|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}},$$

and Ω^c is the complement of Ω in $\mathbb{R}^N$. Since $u=0$ a.e. in Ω^c,

$$\begin{aligned}&\int_{\mathbb{R}^{2N}\setminus(\Omega^c\times\Omega^c)}\frac{\left|e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta}u(x)-u(y)\right|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\\ &\quad=\int_{\mathbb{R}^{2N}}\frac{\left|e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta}u(x)-u(y)\right|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y.\end{aligned}$$

We just denote $\|u\|_{D^s_A(\Omega)}$ by

$$\|u\|_{D^s_A} = \left(\frac{c_{N,s}}{2}\int_{\mathbb{R}^{2N}}\frac{\left|e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta}u(x)-u(y)\right|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}.$$

THEOREM 1.2. *If Condition (H) holds, $A:\mathbb{R}^N\to\mathbb{R}^N$ is a continuous function with locally bounded gradient and $\sigma((-\Delta)^s_A-\lambda_1),\sigma((-\Delta)^s_A-\lambda_2)\subset(0,+\infty)$, where $\sigma(\cdot)$ is the spectrum of $(\cdot)$ in $L^2(\mathbb{R}^N,\mathbb{C})$, then the system* (1.6) *possesses a nontrivial ground state solution.*

2. Preliminaries

We begin with the following diamagnetic inequality.

LEMMA 2.1. (Diamagnetic inequality) *For any $u\in H^s_A(\mathbb{R}^N,\mathbb{C})$, we have*

$$\big||u(x)|-|u(x)|\big| \leqslant \left|e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)d\theta}u(x)-u(y)\right|,\quad \text{for a.e. } x,y\in\mathbb{R}^N \tag{2.1}$$

and

$$\big\||u|\big\|_{D^s_0}\leqslant\|u\|_{D^s_A}, \tag{2.2}$$

which means $|u|\in D^s_0(\mathbb{R}^N,\mathbb{R})$.

Proof. For a.e. $x,y\in\mathbb{R}^N$, it holds

$$\mathrm{Re}\left(e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta}u(x)\overline{u(y)}\right)\leqslant|u(x)||u(y)|.$$

Then, we have

$$\begin{aligned}\big||u(x)|-|u(y)|\big|^2 &= |u(x)|^2+|u(y)|^2-2|u(x)||u(y)|\\ &\leqslant |u(x)|^2+|u(y)|^2-2\mathrm{Re}\left(e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta}u(x)\overline{u(y)}\right)\\ &= \left|e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta}u(x)-u(y)\right|^2,\end{aligned}$$

which implies (2.1) and (2.2). □

The following lemma follows from Theorem 6.5 and Corollary 7.2 in [5] and Lemma 2.1.

LEMMA 2.2. (Fractional magnetic Sobolev embeddings) *The embedding*

$$D_A^s(\mathbb{R}^N,\mathbb{C}) \hookrightarrow L^{2_s^*}(\mathbb{R}^N,\mathbb{C})$$

is continuous. For any bounded domain Ω in $\mathbb{R}^N$, the embedding

$$D_A^s(\Omega,\mathbb{C}) \hookrightarrow L^p(\Omega,\mathbb{C})$$

is compact for $1 \leqslant p < 2_s^*$.

The fractional magnetic Laplacian $(-\Delta)_A^s : D_A^s(\mathbb{R}^N,\mathbb{C}) \to D_A^{-s}(\mathbb{R}^N,\mathbb{C})$ is defined by duality as

$$\begin{aligned}\langle(-\Delta)_A^s u,v\rangle &:= \frac{c_{N,s}}{2}\mathrm{Re}\int_{\mathbb{R}^{2N}}\frac{\left(e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta}u(x)-u(y)\right)}{|x-y|^{N+2s}}\\ &\qquad\cdot\overline{\left(e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta}v(x)-v(y)\right)}\mathrm{d}x\mathrm{d}y\\ &= \frac{c_{N,s}}{2}\mathrm{Re}\int_{\mathbb{R}^{2N}}\frac{\left(u(x)-e^{i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta}u(y)\right)}{|x-y|^{N+2s}}\\ &\qquad\cdot\overline{\left(v(x)-e^{i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta}v(y)\right)}\mathrm{d}x\mathrm{d}y.\end{aligned}$$

3. Proof of Theorem 1.1

Analogously to Lemma 4.6 in [4], we have the following lemma.

LEMMA 3.1. *If Condition (H) holds, then $S_A = S_0$.*

Proof. By (1.3) and (1.4), for any $\varepsilon > 0$, there exists $u,v \in C_c^\infty(\mathbb{R}^N,\mathbb{R})$ such that

$$\|(u,v)\|_{\mathscr{D}_0^s}^2 \leqslant S_0+\varepsilon, \qquad \int_{\mathbb{R}^N}\left(\mu_1|u|^{2_s^*}+\mu_2|v|^{2_s^*}+\lambda|u|^\alpha|v|^\beta\right)=1. \tag{3.1}$$

For any $\varepsilon > 0$, consider the scaling

$$u_\varepsilon(x) := \varepsilon^{(2s-N)/2} u\left(\frac{x}{\varepsilon}\right), \quad v_\varepsilon(x) := \varepsilon^{(2s-N)/2} v\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}^N.$$

Then we have

$$\|u_\varepsilon\|^2_{D^s_A} = \frac{c_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{\left| e^{-i\varepsilon(x-y)\cdot \int_0^1 A[\varepsilon((1-\theta)x+\theta y)]\mathrm{d}\theta} u(x) - u(y)\right|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y$$

and the following invariance of scaling holds true:

$$\begin{aligned} \|(u_\varepsilon, v_\varepsilon)\|_{\mathscr{D}^s_0} &= \|(u,v)\|_{\mathscr{D}^s_0}, \\ |u_\varepsilon|_{2^*_s} = |u|_{2^*_s}, &\quad |v_\varepsilon|_{2^*_s} = |v|_{2^*_s}, \\ \int_{\mathbb{R}^N} |u_\varepsilon|^\alpha |v_\varepsilon|^\beta &= \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta. \end{aligned}$$

A direct computation yields that

$$\begin{aligned} &\|(u_\varepsilon, v_\varepsilon)\|^2_{\mathscr{D}^s_A} - \|(u,v)\|^2_{\mathscr{D}^s_0} \\ &= \frac{c_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{2\mathrm{Re}\left((1 - e^{-i\varepsilon(x-y)\cdot \int_0^1 A[\varepsilon((1-\theta)x+\theta y)]\mathrm{d}\theta}) u(x)u(y)\right)}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\ &\quad + \frac{c_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{2\mathrm{Re}\left((1 - e^{-i\varepsilon(x-y)\cdot \int_0^1 A[\varepsilon((1-\theta)x+\theta y)]\mathrm{d}\theta}) v(x)v(y)\right)}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\ &= c_{N,s} \int_{\mathbb{R}^{2N}} \frac{\left(1 - \cos\left(\varepsilon(x-y)\cdot \int_0^1 A[\varepsilon((1-\theta)x+\theta y)]\mathrm{d}\theta\right)\right)}{|x-y|^{N+2s}} \\ &\qquad \cdot \big(u(x)u(y) + v(x)v(y)\big)\mathrm{d}x\mathrm{d}y \\ &=: c_{N,s} \int_{\mathbb{R}^{2N}} \Upsilon_\varepsilon(x,y)\mathrm{d}x\mathrm{d}y \\ &= c_{N,s} \int_{K\times K} \Upsilon_\varepsilon(x,y)\mathrm{d}x\mathrm{d}y, \end{aligned}$$

where K is compact support of $|u|+|v|$. Obviously, $\Upsilon_\varepsilon(x,y) \to 0$ a.e. in $\mathbb{R}^{2N}$ as $\varepsilon \to 0$. Noticing that A is locally bounded, for $x,y \in K$, we get that

$$1 - \cos\left(\varepsilon(x-y)\cdot \int_0^1 A[\varepsilon((1-\theta)x+\theta y)]\mathrm{d}\theta\right) \leqslant C|x-y|^2.$$

By the boundedness of u and v, for $x,y \in K$, we have

$$|\Upsilon_\varepsilon(x,y)| \leqslant \begin{cases} \frac{C}{|x-y|^{N-2+2s}}, & \text{if } |x-y| < 1, \\ \frac{C}{|x-y|^{N+2s}}, & \text{if } |x-y| \geqslant 1. \end{cases}$$

Then, there exists a suitable constant $C > 0$ such that

$$|\Upsilon_\varepsilon(x,y)| \leqslant C \min\left\{\frac{1}{|x-y|^{N-2+2s}}, \frac{1}{|x-y|^{N+2s}}\right\} =: b(x,y), \quad x,y \in K.$$

The estimate

$$\begin{aligned}
&\int_{K\times K} b(x,y)\mathrm{d}x\mathrm{d}y \\
&= \int_{(K\times K)\cap\{|x-y|<1\}} b(x,y)\mathrm{d}x\mathrm{d}y + \int_{(K\times K)\cap\{|x-y|\geqslant 1\}} b(x,y)\mathrm{d}x\mathrm{d}y \\
&\leqslant C\int_{\{|z|<1\}} \frac{1}{|z|^{N-2+2s}}\mathrm{d}z + C\int_{\{|z|\geqslant 1\}} \frac{1}{|z|^{N+2s}}\mathrm{d}z \\
&< +\infty,
\end{aligned}$$

shows that $b \in L^1(K\times K)$. It follows from Lebesgue's Dominated Convergence Theorem that $\lim_{\varepsilon\to 0} \|(u_\varepsilon, v_\varepsilon)\|^2_{\mathscr{D}^s_A} = \|(u,v)\|^2_{\mathscr{D}^s_0}$. Thus, by (3.1), we derive that

$$S_A \leqslant \lim_{\varepsilon\to 0} \|(u_\varepsilon, v_\varepsilon)\|^2_{\mathscr{D}^s_A} = \|(u,v)\|^2_{\mathscr{D}^s_0} \leqslant S_0 + \varepsilon,$$

which implies that $S_A \leqslant S_0$. Lemma 2.1 guarantees the opposite inequality. □

Proof of Theorem 1.1. Since S_0 is achieved by nontrivial element $(U,V) \in \mathscr{D}^s_0(\mathbb{R}^N,\mathbb{R})$, the proof is completed by Lemma 3.1. □

4. Proof of Theorem 1.2

Define

$$S_A(\Omega) = \inf_{\substack{(u,v)\in\mathscr{D}^s_A(\Omega,\mathbb{C})\\(u,v)\neq(0,0)}} \frac{\|(u,v)\|^2_{\mathscr{D}^s_A} - |u|^2_{2,\Omega} - |v|^2_{2,\Omega}}{\left(\int_\Omega \left(\mu_1|u|^{2^*_s} + \mu_2|v|^{2^*_s} + \lambda|u|^\alpha|v|^\beta\right)\right)^{\frac{2}{2^*_s}}}.$$

Similar to Lemma 2.4 (iv) in [8], we have:

LEMMA 4.1. *$S_A(\Omega)$ is achieved by nontrivial element $(u,v) \in \mathscr{D}^s_A(\Omega,\mathbb{C})$ if and only if system* (1.6) *possesses a nontrivial ground state solution.*

Choose $\delta > 0$ satisfying $B_{4\delta} \subset \Omega$ and consider

$$u_\varepsilon(x) := \eta(x)U_\varepsilon(x), \qquad v_\varepsilon(x) := \eta(x)V_\varepsilon(x),$$

where η is a cut-off function such that $\eta|_{B_\delta} = 1$ and $\eta|_{B^c_{2\delta}} = 0$,

$$U_\varepsilon(x) := \varepsilon^{-\frac{N-2s}{2}} U\left(\frac{x}{\varepsilon}\right), \qquad V_\varepsilon(x) := \varepsilon^{-\frac{N-2s}{2}} V\left(\frac{x}{\varepsilon}\right).$$

Here, $(U,V) \in \mathscr{D}^s_0(\mathbb{R}^N,\mathbb{R})$ attains S_0 and satisfies (1.5). Then:

LEMMA 4.2.

$$\|u_\varepsilon\|^2_{D^s_A} \leqslant \|U_\varepsilon\|^2_{D^s_A} + O(\varepsilon^{N-2s}), \tag{4.1}$$

$$\|v_\varepsilon\|^2_{D^s_A} \leqslant \|V_\varepsilon\|^2_{D^s_A} + O(\varepsilon^{N-2s}), \tag{4.2}$$

$$|u_\varepsilon|^{2^*_s}_{2^*_s,\Omega} \geqslant |U_\varepsilon|^{2^*_s}_{2^*_s,\mathbb{R}^N} + O(\varepsilon^N) = |U|^{2^*_s}_{2^*_s,\mathbb{R}^N} + O(\varepsilon^N), \tag{4.3}$$

$$|v_\varepsilon|^{2^*_s}_{2^*_s,\Omega} \geqslant |V_\varepsilon|^{2^*_s}_{2^*_s,\mathbb{R}^N} + O(\varepsilon^N) = |V|^{2^*_s}_{2^*_s,\mathbb{R}^N} + O(\varepsilon^N), \tag{4.4}$$

$$\int_\Omega u_\varepsilon^\alpha v_\varepsilon^\beta \geqslant \int_{\mathbb{R}^N} U_\varepsilon^\alpha V_\varepsilon^\beta + O(\varepsilon^N) = \int_{\mathbb{R}^N} U^\alpha V^\beta + O(\varepsilon^N), \tag{4.5}$$

$$|u_\varepsilon|^2_{2,\Omega},\ |v_\varepsilon|^2_{2,\Omega} \geqslant \begin{cases} C\varepsilon^{2s} + O(\varepsilon^{N-2s}), & \text{if } N > 4s, \\ C\varepsilon^{2s}|\ln\varepsilon| + O(\varepsilon^{2s}), & \text{if } N = 4s, \\ C\varepsilon^{N-2s} + O(\varepsilon^{2s}), & \text{if } N < 4s, \end{cases} \tag{4.6}$$

where C is a positive constant relevant to s.

Proof. Noticing that $\|\cdot\|_{D^s_0}$ and $|\cdot|_{2^*_s,\mathbb{R}^N}$ are invariant under the scaling, the "$=$" signs in (4.3) – (4.5) follows. It is easy to see that the sign "$\geqslant$" in (4.3) – (4.6) hold. For the sign "$\leqslant$" in (4.1) and (4.2), we are inspired by Proposition 21 in [10].

Claim 1. We have that

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leqslant C\varepsilon^{\frac{N-2s}{2}}|x-y|,\ \forall x \in \mathbb{R}^N,\ y \in B^c_\delta \text{ with } |x-y| \leqslant \frac{\delta}{2}. \tag{4.7}$$

In fact, for $x \in \mathbb{R}^N$ and $y \in B^c_\delta$ with $|x-y| \leqslant \frac{\delta}{2}$, suppose that z is any point on the segment joining x and y, that is, $z = tx + (1-t)y$ for some $t \in [0,1]$. Then,

$$|z| = |y + t(x-y)| \geqslant |y| - t|x-y| \geqslant \delta - t\frac{\delta}{2} \geqslant \frac{\delta}{2}.$$

It follows from $U_\varepsilon(z) \leqslant C\varepsilon^{-\frac{N-2s}{2}}\left(1 + \frac{|z|}{\varepsilon}\right)^{2s-N}$ that

$$\begin{aligned}
|\nabla u_\varepsilon(z)| &= |U_\varepsilon(z)\nabla\eta(z) + \eta(z)\nabla U_\varepsilon(z)| \\
&\leqslant C\varepsilon^{-\frac{N-2s}{2}}\left(\left(1+\frac{|z|}{\varepsilon}\right)^{-(N-2s)} + \frac{1}{\varepsilon}\left(1+\frac{|z|}{\varepsilon}\right)^{-(N-2s)-1}\right) \\
&\leqslant C\varepsilon^{-\frac{N-2s}{2}}\left(\left(1+\frac{|z|}{\varepsilon}\right)^{-(N-2s)} + \frac{2}{\delta}\frac{|z|}{\varepsilon}\left(1+\frac{|z|}{\varepsilon}\right)^{-(N-2s)-1}\right) \\
&\leqslant C\varepsilon^{-\frac{N-2s}{2}}\left(1+\frac{2}{\delta}\right)\left(1+\frac{|z|}{\varepsilon}\right)^{-(N-2s)} \\
&\leqslant C\varepsilon^{\frac{N-2s}{2}},
\end{aligned}$$

which yields that

$$|u_\varepsilon(x) - u_\varepsilon(y)| = |\nabla u_\varepsilon(z)||x-y| \leqslant C\varepsilon^{\frac{N-2s}{2}}|x-y|.$$

Claim 2. The following inequality holds true:

$$|u_\varepsilon(x)-u_\varepsilon(y)| \leqslant C\varepsilon^{\frac{N-2s}{2}} \min\{1,|x-y|\},\ \forall x,y \in B_\delta^c. \tag{4.8}$$

Indeed, for $x,y \in B_\delta^c$ with $|x-y| \leqslant \frac{\delta}{2}$, (4.8) follows directly from (4.7). For $x,y \in B_\delta^c$ with $|x-y| > \frac{\delta}{2}$, since $U_\varepsilon(x) \leqslant C\varepsilon^{-\frac{N-2s}{2}}\left(1+\frac{|x|}{\varepsilon}\right)^{2s-N}$, we see that

$$u_\varepsilon(x) \leqslant U_\varepsilon(x) \leqslant C\varepsilon^{\frac{N-2s}{2}},\ \forall x \in B_\delta^c. \tag{4.9}$$

Then,

$$|u_\varepsilon(x)-u_\varepsilon(y)| \leqslant u_\varepsilon(x)+u_\varepsilon(y) \leqslant C\varepsilon^{\frac{N-2s}{2}},$$

which implies that (4.8).

Claim 3. For any $x,y \in B_\delta^c$, there exists $C>0$ such that

$$\left|e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta} u_\varepsilon(x)-u_\varepsilon(y)\right| \leqslant C\varepsilon^{\frac{N-2s}{2}} \min\{1,|x-y|\}. \tag{4.10}$$

Since A is locally bounded, there exists $C>0$ such that

$$\left|e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta} - 1\right| \leqslant C\min\{1,|x-y|\}. \tag{4.11}$$

Then, by (4.11), (4.9) and *Claim 2*, we derive that

$$\begin{aligned}
&\left|e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta} u_\varepsilon(x)-u_\varepsilon(y)\right| \\
\leqslant &\left|e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta} - 1\right| |u_\varepsilon(x)| + |u_\varepsilon(x)-u_\varepsilon(y)| \\
\leqslant &C\varepsilon^{\frac{N-2s}{2}} \min\{1,|x-y|\},
\end{aligned}$$

which proves (4.10).

It follows from *Claim 3* and $\eta|_{B_{2\delta}^c}=0$ that

$$\begin{aligned}
&\int_{B_\delta^c\times B_\delta^c} \frac{\left|e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta} u_\varepsilon(x)-u_\varepsilon(y)\right|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\
\leqslant &C\varepsilon^{N-2s}\int_{B_{2\delta}\times B_{2\delta}} \frac{\min\{1,|x-y|^2\}}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\
\leqslant &C\varepsilon^{N-2s}\left(\int_{\substack{|x|<2\delta\\|x-y|<1}} \frac{|x-y|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y + \int_{\substack{|x|<2\delta\\|x-y|>1}} \frac{1}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\right) \\
= &O\left(\varepsilon^{N-2s}\right).
\end{aligned} \tag{4.12}$$

Set

$$\mathbb{L} = \left\{(x,y)\in\mathbb{R}^{2N} : x\in B_\delta, y\in B_\delta^c \text{ and } |x-y| < \frac{\delta}{2}\right\},$$

$$\mathbb{G} = \left\{(x,y)\in\mathbb{R}^{2N} : x\in B_\delta, y\in B_\delta^c \text{ and } |x-y| > \frac{\delta}{2}\right\}.$$

For $(x,y) \in \mathbb{L}$, by (4.9), (4.11) and *Claim 1*, we have

$$\begin{aligned}
&\left|e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta}u_\varepsilon(x)-u_\varepsilon(y)\right| \\
\leqslant &\left|e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta}\right||u_\varepsilon(x)-u_\varepsilon(y)| \\
&+\left|e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta}-1\right||u_\varepsilon(y)| \\
\leqslant &|u_\varepsilon(x)-u_\varepsilon(y)|+C|x-y|\varepsilon^{\frac{N-2s}{2}} \\
\leqslant &C\varepsilon^{\frac{N-2s}{2}}|x-y|.
\end{aligned} \tag{4.13}$$

Then, by (4.13), we obtain that

$$\begin{aligned}
&\int_{\mathbb{L}}\frac{\left|e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta}u_\varepsilon(x)-u_\varepsilon(y)\right|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\
\leqslant &C\varepsilon^{N-2s}\int_{\substack{|x|<\delta\\|x-y|<\frac{\delta}{2}}}\frac{|x-y|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\
= &O\big(\varepsilon^{N-2s}\big).
\end{aligned} \tag{4.14}$$

By (4.9), we see that

$$U_\varepsilon(x)U_\varepsilon(y)\leqslant C\varepsilon^{-\frac{N-2s}{2}}\Big(1+\frac{|x|}{\varepsilon}\Big)^{2s-N}\varepsilon^{\frac{N-2s}{2}}=C\Big(1+\frac{|x|}{\varepsilon}\Big)^{2s-N} \tag{4.15}$$

for any $x\in\mathbb{R}^N$ and $y\in B_\delta^c$. For $(x,y)\in\mathbb{G}$, by (4.9) and (4.15), we deduce that

$$\begin{aligned}
&\int_{\mathbb{G}}\frac{\left|e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta}u_\varepsilon(x)-u_\varepsilon(y)\right|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\
\leqslant &\int_{\mathbb{G}}\frac{\left|e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta}U_\varepsilon(x)-U_\varepsilon(y)\right|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\
&+\int_{\mathbb{G}}\frac{|U_\varepsilon(y)-u_\varepsilon(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\
&+2\int_{\mathbb{G}}\frac{|e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta}U_\varepsilon(x)-U_\varepsilon(y)||U_\varepsilon(y)-u_\varepsilon(y)|}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\
\leqslant &\int_{\mathbb{G}}\frac{\left|e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta}U_\varepsilon(x)-U_\varepsilon(y)\right|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\
&+4\int_{\mathbb{G}}\frac{U_\varepsilon^2(y)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y+4\int_{\mathbb{G}}\frac{U_\varepsilon(x)U_\varepsilon(y)+U_\varepsilon^2(y)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \int_{\mathbb{G}} \frac{\left|e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta} U_\varepsilon(x) - U_\varepsilon(y)\right|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\
&\quad + C\varepsilon^{N-2s} \int_{\substack{|x|<\delta \\ |x-y|>\frac{\delta}{2}}} \frac{1}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\
&\quad + C \int_{\substack{|x|<\delta \\ |x-y|>\frac{\delta}{2}}} \left(1+\frac{|x|}{\varepsilon}\right)^{-N+2s} |x-y|^{-N-2s} \mathrm{d}x\mathrm{d}y \\
&= \int_{\mathbb{G}} \frac{\left|e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta} U_\varepsilon(x) - U_\varepsilon(y)\right|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\
&\quad + C\varepsilon^{N-2s} \int_{|\zeta|<\delta} \mathrm{d}\zeta \int_{|\xi|>\frac{\delta}{2}} |\xi|^{-N-2s} \mathrm{d}\xi \\
&\quad + C\varepsilon^{N} \int_{|\zeta|<\frac{\delta}{\varepsilon}} (1+|\zeta|)^{-N+2s} \mathrm{d}\zeta \int_{|\xi|>\frac{\delta}{2}} |\xi|^{-N-2s} \mathrm{d}\xi \\
&\leqslant \int_{\mathbb{G}} \frac{\left|e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta} U_\varepsilon(x) - U_\varepsilon(y)\right|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\
&\quad + C\varepsilon^{N-2s} + C\varepsilon^{N} \int_{|\zeta|<\frac{\delta}{\varepsilon}} |\zeta|^{-N+2s} \mathrm{d}\zeta \\
&= \int_{\mathbb{G}} \frac{\left|e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta} U_\varepsilon(x) - U_\varepsilon(y)\right|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y + O\big(\varepsilon^{N-2s}\big).
\end{aligned}
\tag{4.16}
$$

It follows from (4.12), (4.14) and (4.16) that

$$
\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{\left|e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta} u_\varepsilon(x) - u_\varepsilon(y)\right|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\
&\quad = \int_{B_\delta\times B_\delta} \frac{\left|e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta} U_\varepsilon(x) - U_\varepsilon(y)\right|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\
&\qquad + \int_{B_\delta^c\times B_\delta^c} \frac{\left|e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta} u_\varepsilon(x) - u_\varepsilon(y)\right|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\
&\qquad + 2\int_{\mathbb{L}} \frac{\left|e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta} u_\varepsilon(x) - u_\varepsilon(y)\right|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\
&\qquad + 2\int_{\mathbb{G}} \frac{\left|e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta} u_\varepsilon(x) - u_\varepsilon(y)\right|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y
\end{aligned}
$$

$$\leqslant \int_{B_\delta \times B_\delta} \frac{\left| e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta} U_\varepsilon(x) - U_\varepsilon(y)\right|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y$$

$$+2\int_{\mathbb{G}} \frac{\left| e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta} U_\varepsilon(x) - U_\varepsilon(y)\right|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y + O\big(\varepsilon^{N-2s}\big)$$

$$\leqslant \int_{\mathbb{R}^{2N}} \frac{\left| e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta} U_\varepsilon(x) - U_\varepsilon(y)\right|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y + O\big(\varepsilon^{N-2s}\big),$$

which proves (4.1). Similarly, (4.2) holds. □

REMARK 4.3. The result above has an analogue in Lemma 4.1 in [7].

Next we establish:

LEMMA 4.4.

$$\|U_\varepsilon\|^2_{D^s_A} \leqslant \|U\|^2_{D^s_0} + O(\varepsilon^2), \tag{4.17}$$

$$\|V_\varepsilon\|^2_{D^s_A} \leqslant \|V\|^2_{D^s_0} + O(\varepsilon^2). \tag{4.18}$$

Proof. We only prove (4.17). It is checked that

$$\begin{aligned}
\|U_\varepsilon\|^2_{D^s_A} &= \frac{\varepsilon^{-(N-2s)}C_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{\left| e^{-i(x-y)\cdot\int_0^1 A((1-\theta)x+\theta y)\mathrm{d}\theta} U\left(\frac{x}{\varepsilon}\right) - U\left(\frac{y}{\varepsilon}\right)\right|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\
&= \frac{C_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{\left| e^{-i\varepsilon(x-y)\cdot\int_0^1 A[\varepsilon((1-\theta)x+\theta y)]\mathrm{d}\theta} U(x) - U(y)\right|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\
&= \frac{C_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{U^2(x)+U^2(y)}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\
&\quad - \frac{C_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{2U(x)U(y)\cos\left(\varepsilon(x-y)\cdot\int_0^1 A[\varepsilon((1-\theta)x+\theta y)]\mathrm{d}\theta\right)}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Since $\|U\|^2_{D^s_0} = \frac{c_{N,s}}{2}\int_{\mathbb{R}^{2N}} \frac{U^2(x)+U^2(y)-2U(x)U(y)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y$, we have

$$\begin{aligned}
&\|U_\varepsilon\|^2_{D^s_A} - \|U\|^2_{D^s_0} \\
&= \frac{c_{N,s}}{2}\int_{\mathbb{R}^{2N}} \frac{2U(x)U(y)\left[1-\cos\left(\varepsilon(x-y)\cdot\int_0^1 A[\varepsilon((1-\theta)x+\theta y)]\mathrm{d}\theta\right)\right]}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\
&=: \int_{\mathbb{R}^{2N}} \Upsilon_\varepsilon(x,y)\mathrm{d}x\mathrm{d}y = \int_{K\times K} \Upsilon_\varepsilon(x,y)\mathrm{d}x\mathrm{d}y,
\end{aligned}$$

where K is the compact support of U. For ε small and $x,y \in K$, it follows from the local boundedness of A that

$$1-\cos\left(\varepsilon(x-y)\cdot\int_0^1 A[\varepsilon((1-\theta)x+\theta y)]\mathrm{d}\theta\right) \leqslant C\varepsilon^2|x-y|^2.$$

Moreover, noticing that $|x-y|$ is bounded for $x,y \in K$, we have

$$1-\cos\left(\varepsilon(x-y)\cdot\int_0^1 A[\varepsilon((1-\theta)x+\theta y)]\mathrm{d}\theta\right) \leqslant C\varepsilon^2.$$

Therefore, since U and V are bounded, there exists $C>0$ such that

$$|\Upsilon_\varepsilon(x,y)| \leqslant \begin{cases} \frac{C\varepsilon^2}{|x-y|^{N-2+2s}}, & \text{if } |x-y|<1, \\ \frac{C\varepsilon^2}{|x-y|^{N+2s}}, & \text{if } |x-y|\geqslant 1. \end{cases}$$

Then,

$$\begin{aligned} &\int_{K\times K}\Upsilon_\varepsilon(x,y)\mathrm{d}x\mathrm{d}y \\ =&\int_{(K\times K)\cap\{|x-y|<1\}}\Upsilon_\varepsilon(x,y)\mathrm{d}x\mathrm{d}y+\int_{(K\times K)\cap\{|x-y|\geqslant 1\}}\Upsilon_\varepsilon(x,y)\mathrm{d}x\mathrm{d}y \\ \leqslant& C\varepsilon^2\int_K \mathrm{d}\xi\int_{\{|\zeta|<1\}}\frac{1}{|\zeta|^{N-2+2s}}\mathrm{d}\zeta + C\varepsilon^2\int_K\mathrm{d}\xi\int_{\{|\zeta|\geqslant 1\}}\frac{1}{|\zeta|^{N+2s}}\mathrm{d}\zeta \\ =&O(\varepsilon^2). \quad \square \end{aligned}$$

Similar to Lemma 4.2 in [7], we obtain the following lemma.

LEMMA 4.5. *If* $\sigma((-\Delta)^s_A-\lambda_1), \sigma((-\Delta)^s_A-\lambda_2) \subset (0,+\infty)$, *then*

$$S_A(\Omega) < \min\left\{\mu_1^{-2/2^*_s}S_{A,\lambda_1}(\Omega),\ \mu_2^{-2/2^*_s}S_{A,\lambda_2}(\Omega)\right\},$$

where

$$S_{A,\lambda}(\Omega) := \inf_{u\in D^s_A(\Omega,\mathbb{C})\setminus\{0\}}\frac{\|u\|^2_{D^s_A}-\lambda|u|^2_{2,\Omega}}{|u|^2_{2^*_s,\Omega}}.$$

With these preparations we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. By Lemmas 4.2, 4.4 and 3.1, we have

$$\begin{aligned} S_A(\Omega) &\leqslant \frac{\|u_\varepsilon\|^2_{D^s_A}-\lambda_1|u_\varepsilon|^2_{2,\Omega}+\|v_\varepsilon\|^2_{D^s_A}-\lambda_2|v_\varepsilon|^2_{2,\Omega}}{\left(\mu_1|u_\varepsilon|^{2^*_s}_{2^*_s,\Omega}+\mu_2|v_\varepsilon|^{2^*_s}_{2^*_s,\Omega}+\gamma\int_\Omega u^\alpha_\varepsilon v^\beta_\varepsilon\right)^{\frac{2}{2^*_s}}} \\ &\leqslant \frac{\|U\|^2_{D^s_0}+\|V\|^2_{D^s_0}-C\varepsilon^{2s}+O(\varepsilon^2)+O(\varepsilon^{N-2s})}{\left(\mu_1|U|^{2^*_s}_{2^*_s,\mathbb{R}^N}+\mu_2|V|^{2^*_s}_{2^*_s,\mathbb{R}^N}+\gamma\int_{\mathbb{R}^N}U^\alpha V^\beta+O(\varepsilon^N)\right)^{\frac{2}{2^*_s}}} \\ &< S_0 = S_A. \end{aligned} \tag{4.19}$$

Choose a minimizing sequence $\{(u_n, v_n)\}$ for $S_A(\Omega)$ normalized by

$$\mu_1|u_n|_{2_s^*,\Omega}^{2_s^*} + \mu_2|v_n|_{2_s^*,\Omega}^{2_s^*} + \gamma\int_\Omega |u_n|^\alpha |v_n|^\beta = 1,$$

i.e.,

$$\|(u_n, v_n)\|_{\mathscr{D}_A^s}^2 - \lambda_1|u_n|_{2,\Omega}^2 - \lambda_2|v_n|_{2,\Omega}^2 = S_A(\Omega) + o(1). \tag{4.20}$$

Noticing that $\{u_n\}$ and $\{v_n\}$ are bounded in $D_A^s(\Omega,\mathbb{C})$, Lemma 2.2 ensures that there exist two subsequences-still denoted by $\{u_n\}$ and $\{v_n\}$-such that

$$\begin{aligned} u_n \rightharpoonup u,\ \ v_n \rightharpoonup v \quad &\text{weakly in } D_A^s(\Omega,\mathbb{C}) \\ u_n \to u,\ \ v_n \to v \quad &\text{strongly in } L^2(\Omega,\mathbb{C}), \\ u_n \to u,\ \ v_n \to v \quad &\text{a.e. on } \Omega, \end{aligned}$$

with

$$\mu_1|u|_{2_s^*,\Omega}^{2_s^*} + \mu_2|v|_{2_s^*,\Omega}^{2_s^*} + \gamma\int_\Omega |u|^\alpha |v|^\beta \leqslant 1.$$

Denoting $w_n := u_n - u$ and $z_n := v_n - v$, then $w_n \rightharpoonup 0$, $z_n \rightharpoonup 0$ weakly in $D_A^s(\Omega,\mathbb{C})$ and $w_n \to 0, z_n \to 0$ a.e. on Ω. By (4.20), we get that

$$S_A(\Omega) + \lambda_1|u_n|_{2,\Omega}^2 + \lambda_2|v_n|_{2,\Omega}^2 + o(1) \geqslant \|(u_n, v_n)\|_{\mathscr{D}_A^s}^2 \geqslant S_A.$$

It follows from (4.19) that $\lambda_1|u|_{2,\Omega}^2 + \lambda_2|v|_{2,\Omega}^2 \geqslant S_A - S_A(\Omega) > 0$, and then,

$$\begin{aligned} \|(u_n, v_n)\|_{\mathscr{D}_A^s}^2 =& S_A(\Omega) + \lambda_1|u_n|_{2,\Omega}^2 + \lambda_2|v_n|_{2,\Omega}^2 + o(1) \\ \geqslant& S_A(\Omega) + \lambda_1|u|_{2,\Omega}^2 + \lambda_2|v|_{2,\Omega}^2 > 0, \end{aligned}$$

which implies that $(u,v) \not\equiv (0,0)$. Since $w_n \rightharpoonup 0$, $z_n \rightharpoonup 0$ weakly in $D_A^s(\Omega,\mathbb{C})$, we obtain that

$$\begin{aligned} \|u_n\|_{D_A^s}^2 &= \|w_n\|_{D_A^s}^2 + \|u\|_{D_A^s}^2 + o(1), \\ \|v_n\|_{D_A^s}^2 &= \|z_n\|_{D_A^s}^2 + \|v\|_{D_A^s}^2 + o(1). \end{aligned}$$

Thus, by (4.20), we have

$$S_A(\Omega) = \|w_n\|_{D_A^s}^2 + \|u\|_{D_A^s}^2 - \lambda_1|u|_{2,\Omega}^2 + \|z_n\|_{D_A^s}^2 + \|v\|_{D_A^s}^2 - \lambda_2|v|_{2,\Omega}^2 + o(1). \tag{4.21}$$

The Brezis-Lieb Lemma yields

$$\begin{aligned} 1 =& \mu_1|u+w_n|_{2_s^*,\Omega}^{2_s^*} + \mu_2|v+z_n|_{2_s^*,\Omega}^{2_s^*} + \gamma\int_\Omega |u+w_n|^\alpha |v+z_n|^\beta \\ =& \mu_1|u|_{2_s^*,\Omega}^{2_s^*} + \mu_2|v|_{2_s^*,\Omega}^{2_s^*} + \gamma\int_\Omega |u|^\alpha |v|^\beta \\ &+ \mu_1|w_n|_{2_s^*,\Omega}^{2_s^*} + \mu_2|z_n|_{2_s^*,\Omega}^{2_s^*} + \gamma\int_\Omega |w_n|^\alpha |z_n|^\beta + o(1). \end{aligned}$$

Noticing that

$$\mu_1|u|_{2_s^*,\Omega}^{2_s^*} + \mu_2|v|_{2_s^*,\Omega}^{2_s^*} + \gamma\int_\Omega |u|^\alpha |v|^\beta \leqslant 1$$

and

$$\mu_1|w_n|_{2_s^*,\Omega}^{2_s^*}+\mu_2|z_n|_{2_s^*,\Omega}^{2_s^*}+\gamma\int_\Omega |w_n|^\alpha|z_n|^\beta \leqslant 1,$$

we derive that

$$\begin{aligned}1 &\leqslant \left(\mu_1|u|_{2_s^*,\Omega}^{2_s^*}+\mu_2|v|_{2_s^*,\Omega}^{2_s^*}+\gamma\int_\Omega |u|^\alpha|v|^\beta\right)^{\frac{2}{2_s^*}}\\ &\quad+\left(\mu_1|w_n|_{2_s^*,\Omega}^{2_s^*}+\mu_2|z_n|_{2_s^*,\Omega}^{2_s^*}+\gamma\int_\Omega |w_n|^\alpha|z_n|^\beta\right)^{\frac{2}{2_s^*}}+o(1)\\ &\leqslant \left(\mu_1|u|_{2_s^*,\Omega}^{2_s^*}+\mu_2|v|_{2_s^*,\Omega}^{2_s^*}+\gamma\int_\Omega |u|^\alpha|v|^\beta\right)^{\frac{2}{2_s^*}}\\ &\quad+\frac{1}{S_A}\|(w_n,z_n)\|_{\mathscr{D}_A^s}^2+o(1).\end{aligned} \tag{4.22}$$

Then, (4.21), (4.22) and (4.19) guarantee that

$$\begin{aligned}&\|u\|_{D_A^s}^2-\lambda_1|u|_{2,\Omega}^2+\|v\|_{D_A^s}^2-\lambda_2|v|_{2,\Omega}^2\\ &\leqslant S_A(\Omega)\left(\mu_1|u|_{2_s^*,\Omega}^{2_s^*}+\mu_2|v|_{2_s^*,\Omega}^{2_s^*}+\gamma\int_\Omega |u|^\alpha|v|^\beta\right)^{\frac{2}{2_s^*}}\\ &\quad+\left(\frac{S_A(\Omega)}{S_A}-1\right)\|(w_n,z_n)\|_{\mathscr{D}_A^s}^2+o(1)\\ &\leqslant S_A(\Omega)\left(\mu_1|u|_{2_s^*,\Omega}^{2_s^*}+\mu_2|v|_{2_s^*,\Omega}^{2_s^*}+\gamma\int_\Omega |u|^\alpha|v|^\beta\right)^{\frac{2}{2_s^*}}+o(1),\end{aligned}$$

which, together with $(u,v)\not\equiv(0,0)$, means that

$$\frac{\|u\|_{D_A^s}^2-\lambda_1|u|_{2,\Omega}^2+\|v\|_{D_A^s}^2-\lambda_2|v|_{2,\Omega}^2}{\left(\mu_1|u|_{2_s^*,\Omega}^{2_s^*}+\mu_2|v|_{2_s^*,\Omega}^{2_s^*}+\gamma\int_\Omega |u|^\alpha|v|^\beta\right)^{\frac{2}{2_s^*}}}\leqslant S_A(\Omega).$$

Hence, $S_A(\Omega)$ is achieved by (u,v). We next show that (u,v) can not be the type of $(u,0)$ or $(0,v)$. Suppose by contradiction that $S_A(\Omega)$ is achieved by $(u,0)$. Then,

$$S_A(\Omega)=\frac{\|u\|_{D_A^s}^2-\lambda_1|u|_{2,\Omega}^2}{\mu_1^{2/2_s^*}|u|_{2_s^*,\Omega}^2}\geqslant \mu_1^{-2/2_s^*}S_{A,\lambda_1}(\Omega),$$

which contradicts Lemma 4.5. Therefore, (u,v) cannot be of the type $(u,0)$. Similarly, it cannot be of the type $(0,v)$. By invoking Lemma 4.1, we prove that system (1.6) has a nontrivial ground state solution (u,v). □

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(Received June 15, 2018)

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