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ROUGH FRACTIONAL INTEGRAL OPERATORS AND BEYOND ADAMS INEQUALITIES

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Abstract. We consider the boundedness of fractional integral operators with rough kernel from Morrey spaces $L^{p,\lambda}$ to $L^{q,\mu}$. Our main concern is proving the boundedness property for $\mu < \lambda$ as an extension of Adams inequality on some special subsets of the operator's domain namely classes of A_p, simple function, and radial function respectively. For radial function, we prove the boundedness on local Morrey spaces. We also prove the boundedness property for $\mu \geqslant \lambda$ as well as the special case of $q \leqslant p$. It is interesting on its own term since the operator is not bounded from L^p to L^q if $q \leqslant p$. We also establish necessary conditions for boundedness. Our proposed condition for boundedness includes the sufficient conditions for both Adams inequality and Spanne inequality.

1. Introduction

Let Ω be a homogeneous function of degree zero on $\mathbb{R}^n$. For $0<\alpha<n$, *fractional integral operator with rough kernel* $T_{\Omega,\alpha}$ is defined as

$$T_{\Omega,\alpha}f(x)=\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}f(y)dy. \tag{1}$$

For $\Omega\equiv 1$, the operator $T_{1,\alpha}$ is the fractional integral operator I_α [2, 6].

Let $B(x,r)$ be an open ball on $\mathbb{R}^n$, centered at x, and with radius $r>0$. For $1\leqslant p<\infty$, $0\leqslant\lambda<n$, *Morrey spaces* $L^{p,\lambda}$ and *local Morrey spaces* $L^{p,\lambda}(0)$ are defined respectively as follows.

$$L^{p,\lambda}=\left\{f;\|f\|_{L^{p,\lambda}}=\sup_{x\in\mathbb{R}^n,r>0}\left(r^{-\lambda}\int_{B(x,r)}|f(y)|^p dy\right)^{\frac{1}{p}}<\infty\right\},$$

$$L^{p,\lambda}(0)=\left\{f;\|f\|_{L^{p,\lambda}(0)}=\sup_{r>0}\left(r^{-\lambda}\int_{B(0,r)}|f(y)|^p dy\right)^{\frac{1}{p}}<\infty\right\}.$$

One of important issue in the study of operators is their boundedness. Spanne and Adams proved the following boundedness properties.

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THEOREM A. [6, Theorem 5.4.] (Spanne inequality) *Suppose* $1 < p < \frac{n-\lambda}{\alpha}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *Then,*

$$\|I_\alpha f\|_{L^{q,\lambda q/p}} \lesssim \|f\|_{L^{p,\lambda}}{}^{1}.$$

THEOREM B. [2, Theorem 3.1.] (Adams inequality) *Suppose* $1 < p < \frac{n-\lambda}{\alpha}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$. *Then,*

$$\|I_\alpha f\|_{L^{q,\lambda}} \lesssim \|f\|_{L^{p,\lambda}}.$$

By Hölder inequality, one can observe that: if $\beta < \delta$, and $t = \frac{s(n-\beta)}{n-\delta} > s$, then $L^{t,\beta} \subset L^{s,\delta}$ (this inclusion property is proper, see [3]). Therefore, Adams inequality is stronger than Spanne inequality. Certainly, Adams inequality is the strongest boundedness property for I_α on Morrey spaces (see [7, Theorem 9.], [10, Proposition 4.2.]).

Our concern is proving the boundedness of $T_{\Omega,\alpha}$ from $L^{p,\lambda}$ to $L^{q,\mu}$. From different point of views of Adams and Spanne, we let the parameter μ to be arbitrary but controlled by the necessary condition for boundedness (see Theorem 2.1).

By classical method, we prove the boundedness of operator $T_{\Omega,\alpha}$ from $L^{p,\lambda}$ to $L^{q,\mu}$ where $\mu \geqslant \lambda$ (see Theorem 3.3). Theorem 3.3 is a stronger version of Proposition 1 in [8]. For $\mu \geqslant \lambda$, we have a special case of $q \leqslant p$ (see Corollary 3.4). It is interesting on its own term due to the operator $T_{\Omega,\alpha}$ can not be bounded from L^p to L^q for $q \leqslant p$.

Our main concern is investigating the behavior of $T_{\Omega,\alpha}$ for the case of $\mu < \lambda$, as an extension of Adams inequality. In the discussion, we restrict the domain of $T_{\Omega,\alpha}$ into subset of $L^{p,\lambda}$ such that A_p-condition holds (see Theorem 4.1), simple function (see Theorem 4.5), or radial function (see Theorem 4.7). For radial function, the boundedness property takes place from $L^{p,\lambda}(0)$ to $L^{q,\mu}(0)$. The reader can find Adams and Spanne type result for boundedness of I_α on local Morrey spaces in [9].

The discussion of this paper is delivered in 3 sections. We elaborate the necessary conditions for boundedness of $T_{\Omega,\alpha}$ in Section 2. We prove the boundedness of $T_{\Omega,\alpha}$ from $L^{p,\lambda}$ to $L^{q,\mu}$ for $\mu \geqslant \lambda$ in Section 3, and for $\mu < \lambda$ in Section 4.

2. Necessary conditions for boundedness

In order to have a better idea on the boundedness of $T_{\Omega,\alpha}$ from $L^{p,\lambda}$ to $L^{q,\mu}$, it is essential to know the necessary condition first.

THEOREM 2.1. *Let* $1 < p < \infty$, $1 \leqslant q < \infty$, *and* $0 \leqslant \lambda$, $\mu < n$. *If the operator* $T_{\Omega,\alpha}$ *is bounded from* $L^{p,\lambda}$ *to* $L^{q,\mu}$ *(or from* $L^{p,\lambda}(0)$ *to* $L^{q,\mu}(0)$*) then*

$$\frac{n-\mu}{q} = \frac{n-\lambda}{p} - \alpha \tag{2}$$

and

$$\max\left\{1, \frac{n-\lambda}{n-\mu+\alpha}\right\} < p < \frac{n-\lambda}{\alpha}. \tag{3}$$

[1] The symbol $a \lesssim b$ means that there is $c > 0$ essentially independent of a and b such that $a \leqslant cb$.

Proof. Let $t>0$ and $\delta_t f(x) = f(tx)$. Hence,

$$T_{\Omega,\alpha} f(x) = t^{\alpha} T_{\Omega,\alpha}(\delta_t f)(x/t), \text{ and } \|\delta_t f\|_{L^{p,\lambda}} = t^{-\frac{n-\lambda}{p}} \|f\|_{L^{p,\lambda}}.$$

Let $T_{\Omega,\alpha}$ is bounded from $L^{p,\lambda}$ to $L^{q,\mu}$, then

$$\begin{aligned} r^{-\frac{\mu}{q}} \|T_{\Omega,\alpha} f\|_{L^q(B(x,r))} &\leqslant t^{\alpha+\frac{n-\mu}{q}} \|T_{\Omega,\alpha}(\delta_t f)\|_{L^{q,\mu}} \\ &\lesssim t^{\alpha+\frac{n-\mu}{q}} \|\delta_t f\|_{L^{p,\lambda}} \lesssim t^{\alpha+\frac{n-\mu}{q}-\frac{n-\lambda}{p}} \|f\|_{L^{p,\lambda}}. \end{aligned} \tag{4}$$

Because $t>0$ is arbitrary, the exponent of t in inequality (4) should be zero. Hence, identity (2) holds and it follows that $p < \frac{n-\lambda}{\alpha}$. Since $q \geqslant 1$, by inequality (2)

$$0 \leqslant n-\mu-\frac{n-\mu}{q} = n-\mu+\alpha-\frac{n-\lambda}{p}. \tag{5}$$

Thus, inequality (3) holds.

The necessary condition for boundedness of $T_{\Omega,\alpha}$ from $L^{p,\lambda}(0)$ to $L^{q,\mu}(0)$ follows by the same argument since

$$\|\delta_t f\|_{L^{p,\lambda}(0)} = t^{-\frac{n-\lambda}{p}} \|f\|_{L^{p,\lambda}(0)}. \quad \square$$

Note that, identity (2) is the sufficient condition in Spanne inequality (Theorem A.) and Adams inequality (Theorem B.) if $\mu = \lambda q/p$ and $\mu = \lambda$ respectively.

3. Adams inequality and its weaker version

In this section, we prove the boundedness of $T_{\Omega,\alpha}$ from $L^{p,\lambda}$ to $L^{q,\mu}$ for $\mu \geqslant \lambda$. We use the classical method by Adams [2] that involve a maximal operator.

For $0<\alpha<n$, *maximal operator* $M_{\Omega,\alpha}$ is defined by

$$M_{\Omega,\alpha} f(x) = \sup_{r>0} r^{\alpha-n} \int_{B(x,r)} |\Omega(x-y)||f(y)|dy. \tag{6}$$

From the definition in (1) and (6), it is clear that $M_{\Omega,\alpha} f \leqslant T_{|\Omega|,\alpha}|f|$ where $\alpha \neq 0$. The following is obtained by boundedness properties of $T_{\Omega,\alpha}$ on Lebesgue spaces [1, Theorem 2] and the application of rotation method [4, Chapter 5, Section 3].

THOEREM C. *Let* $1<p<\infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, *and* $\Omega \in L^s(S^{n-1})$[2]. *If* $0 \leqslant \alpha < n$ *and* $s \geqslant \frac{n}{n-\alpha}$, *then*

$$\|M_{\Omega,\alpha} f\|_{L^q} \lesssim \|\Omega\|_{L^s(S^{n-1})} \|f\|_{L^p}.$$

We have the following estimation of $T_{\Omega,\alpha}$ in term of maximal operator.

[2]Set $S^{n-1} = \{x \in \mathbb{R}^n; |x| = 1\}$ is the unit sphere on $\mathbb{R}^n$

THEOREM 3.1. *Let* $0 \leqslant \lambda < n$, *and* $1 < p < \frac{n-\lambda}{\alpha}$. *Then for any* $x \in \mathbb{R}^n$,

$$|T_{\Omega,\alpha}f(x)| \lesssim (M_{\Omega,\frac{n-\lambda}{p}}f(x))^{\frac{\alpha p}{n-\lambda}}(M_{\Omega,0}f(x))^{1-\frac{\alpha p}{n-\lambda}}. \tag{7}$$

Proof. If f or Ω are identical to zero, then inequality (7) holds. Now assume f and Ω are not identical to zero. Fix $x \in \mathbb{R}^n$ and choose $R^{\frac{n-\lambda}{p}} = M_{\Omega,\frac{n-\lambda}{p}}f(x)/M_{\Omega,0}f(x)$. Then

$$\begin{aligned}|T_{\Omega,\alpha}f(x)| &\leqslant \sum_{j=-\infty}^{\infty}(2^{j-1}R)^{\alpha-n}\int_{B(x,2^jR)\setminus B(x,2^{j-1}R)}|\Omega(x-y)||f(y)|dy \\ &\lesssim R^{\alpha}M_{\Omega,0}f(x)\sum_{j=-\infty}^{0}2^{j\alpha} + R^{\alpha-\frac{n-\lambda}{p}}M_{\Omega,\frac{n-\lambda}{p}}f(x)\sum_{j=1}^{\infty}2^{j\left(\alpha-\frac{n-\lambda}{p}\right)} \\ &\lesssim (M_{\Omega,\frac{n-\lambda}{p}}f(x))^{\frac{\alpha p}{n-\lambda}}(M_{\Omega,0}f(x))^{1-\frac{\alpha p}{n-\lambda}}. \quad \square\end{aligned}$$

In preparation to prove the boundedness of $T_{\Omega,\alpha}$ from $L^{p,\lambda}$ to $L^{q,\mu}$ for the case of $\mu \geqslant \lambda$, let us prove the following lemma first.

LEMMA 3.2. *Let* $0 \leqslant \lambda, \mu < n$, *inequality* (3) *holds, and* $\Omega \in L^s(S^{n-1})$ *where* $s \geqslant p'$ [3]. *Let identity* (2) *holds. Then for any* $z \in \mathbb{R}^n$ *and* $r > 0$,

$$\|T_{\Omega,\alpha}(f\chi_{B^c(z,2r)})\|_{L^q(B(z,r))} \lesssim r^{\frac{\mu}{q}}\|\Omega\|_{L^s(S^{n-1})}\|f\|_{L^{p,\lambda}}. \tag{8}$$

Proof. If $x \in B(z,r)$, then $B^c(z,2r) \subset B^c(x,r)$ [4]. Hence, by Hölder inequality the following holds.

$$\begin{aligned}|T_{\Omega,\alpha}(f\chi_{B^c(z,2r)})(x)| &\leqslant \int_{B^c(x,r)}\frac{|\Omega(y-x)|}{|y-x|^{n-\alpha}}|f(y)|dy \\ &\leqslant \sum_{j=1}^{\infty}\left(2^{j-1}r\right)^{\alpha-n}\int_{B(x,2^jr)\setminus B(x,2^{j-1}r)}|\Omega(y-x)||f(y)|dy \\ &\lesssim r^{\alpha-\frac{n-\lambda}{p}}\|\Omega\|_{L^{p'}(S^{n-1})}\|f\|_{L^{p,\lambda}}\sum_{j=1}^{\infty}2^{j\left(\alpha-\frac{n-\lambda}{p}\right)}. \end{aligned}\tag{9}$$

The summation in inequality (9) converges. Since $s \geqslant p'$, we have the inequality $\|\Omega\|_{L^{p'}(S^{n-1})} \lesssim \|\Omega\|_{L^s(S^{n-1})}$. Thus,

$$\begin{aligned}\|T_{\Omega,\alpha}(f\chi_{B^c(z,2r)})\|_{L^q(B(z,r))} &\lesssim r^{\frac{n}{q}}r^{\alpha-\frac{n-\lambda}{p}}\|\Omega\|_{L^s(S^{n-1})}\|f\|_{L^{p,\lambda}} \\ &= r^{\frac{\mu}{q}}\|\Omega\|_{L^s(S^{n-1})}\|f\|_{L^{p,\lambda}}. \quad \square\end{aligned}$$

Now, we are ready to prove the boundedness of $T_{\Omega,\alpha}$ from $L^{p,\lambda}$ to $L^{q,\mu}$ for the case of $\mu \geqslant \lambda$.

[3] For $1 < p < \infty$, we have $p' = \frac{p}{p-1}$.
[4] The set $B^c(x,r)$ is $\mathbb{R}^n \setminus B(x,r)$.

THEOREM 3.3. *Let $0 \leqslant \lambda \leqslant \mu < n$, inequality (3) holds, and $\Omega \in L^s(S^{n-1})$ where $s \geqslant p'$. Identity (2) holds if and only if*

$$\|T_{\Omega,\alpha} f\|_{L^{q,\mu}} \lesssim \|\Omega\|_{L^s(S^{n-1})} \|f\|_{L^{p,\lambda}}.$$

Proof. ($\Leftarrow$) holds by Theorem 2.1.
($\Rightarrow$) For convenience, let $\|f\|_{L^{p,\lambda}} = 1$. By Hölder inequality, for any $x \in \mathbb{R}^n$

$$M_{\Omega, \frac{n-\lambda}{p}} f(x) \leqslant \|\Omega\|_{L^{p'}} \sup_{R>0} R^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x,R))} \leqslant \|\Omega\|_{L^s(S^{n-1})}. \tag{10}$$

Fix $B(z,r)$. Define $f_1 = f\chi_{B(z,2r)}$ and $f_2 = f - f_1$. Then,

$$\|T_{\Omega,\alpha} f\|_{L^q(B(z,r))} \leqslant \|T_{\Omega,\alpha} f_1\|_{L^q(B(z,r))} + \|T_{\Omega,\alpha} f_2\|_{L^q(B(z,r))}. \tag{11}$$

By Lemma 3.2, we can handle $\|T_{\Omega,\alpha} f_2\|_{L^q(B(z,r))}$. Now, let us handle $\|T_{\Omega,\alpha} f_1\|_{L^q(B(z,r))}$. By Theorem 3.1 and inequality (10),

$$\|T_{\Omega,\alpha} f_1\|_{L^q(B(z,r))} \lesssim \|\Omega\|_{L^s(S^{n-1})}^{1-u} \|(M_{\Omega,0} f_1)^u\|_{L^q(B(z,r))} \tag{12}$$

where $u = 1 - \frac{\alpha p}{n-\lambda}$. We note that $uq = p(n-\mu)/(n-\lambda) \leqslant p$. By Hölder inequality with order p/uq, and by Theorem C.

$$\begin{aligned}\|T_{\Omega,\alpha} f_1\|_{L^q(B(z,r))} &\lesssim r^{\frac{n}{q} - \frac{un}{p}} \|\Omega\|_{L^s(S^{n-1})}^{1-u} \|M_{\Omega,0} f_1\|_{L^p}^u \\ &\lesssim r^{\frac{n}{q} - \frac{un}{p}} \|\Omega\|_{L^s(S^{n-1})} \|f\|_{L^p(B(z,2r))}^u \lesssim r^{\frac{\mu}{q}} \|\Omega\|_{L^s(S^{n-1})}.\end{aligned} \tag{13}$$

From inequality (11), inequality (13), and Lemma 3.2, we conclude

$$r^{\frac{-\mu}{q}} \|T_{\Omega,\alpha} f\|_{L^q(B(z,r))} \lesssim \|\Omega\|_{L^s(S^{n-1})}. \tag{14}$$

The theorem is proved by taking supremum over $r > 0$ and $x \in \mathbb{R}^n$ on both side of inequality (14). □

REMARK 1. Proposition 1 in [8] is similar to Theorem 3.3. However, Proposition 1 in [8] holds for $\Omega \in L^s(S^{n-1})$ where $s > p'$. Meanwhile, Theorem 3.3 holds for $\Omega \in L^{p'}(S^{n-1})$. Since $L^s(S^{n-1}) \subset L^{p'}(S^{n-1})$ for $s > p'$, Theorem 3.3 is a stronger version of Proposition 1 in [8].

Let identity (2) and inequality (3) holds. Then, $p/q < (n-\lambda)/(n-\mu)$. As the consequence, if $\mu > \lambda$, we can consider the special case of $q \leqslant p$. In this special case, the following holds.

$$0 \leqslant \frac{n-\mu}{q} - \frac{n-\mu}{p} = \frac{\mu-\lambda}{p} - \alpha. \tag{15}$$

If $0 < \alpha < \mu - \lambda$, then by inequality (15) and inequality (3),

$$1 < \frac{n-\lambda}{n-\mu+\alpha} \leqslant p \leqslant \frac{\mu-\lambda}{\alpha} < \frac{n-\lambda}{\alpha}. \tag{16}$$

Therefore, Theorem 3.3 validates the following corollary.

COROLLARY 3.4. *Let $0<\lambda<\mu<n$, $1\leqslant q\leqslant p$, and $1<p$. Let $\Omega\in L^s(S^{n-1})$ where $s\geqslant p'$. If identity (2) holds, and $0<\alpha<\mu-\lambda$, then the operator $T_{\Omega,\alpha}$ is bounded from $L^{p,\lambda}$ to $L^{q,\mu}$.*

4. Beyond Adams inequality

In this section, we consider proving the boundedness of $T_{\Omega,\alpha}$ from $L^{p,\lambda}$ to $L^{q,\mu}$ where $\mu<\lambda$. The condition $\mu<\lambda$ implies that inequality (3) always has the following form.

$$1<p<\frac{n-\lambda}{\alpha}.$$

Let identity (2) holds and let us recall the inequality (7),

$$|T_{\Omega,\alpha}f(x)|\lesssim (M_{\Omega,\frac{n-\lambda}{p}}f(x))^{1-u}(M_{\Omega,0}f(x))^{u}$$

where $u=1-\frac{\alpha p}{n-\lambda}$. If $\mu<\lambda$, then $uq=p(n-\mu)/(n-\lambda)>p$. Hence, we can't use the method in the proof of Theorem 3.3.

The idea to handle this problem is by restricting the $T_{\Omega,\alpha}$ domain into class of functions such that reverse Hölder inequality holds (Subsection 4.1). Another idea is by reducing the value of u into v such that $vq<p$ (Subsection 4.2 and 4.3). In order to reduce the value of u, we involve the parameter $\gamma>\lambda$.

4.1. The A_p-condition

A nonnegative measurable function f is said to be in A_p if for any ball $B\subset\mathbb{R}^n$

$$\left(\frac{1}{|B|}\int_B f(x)dx\right)\left(\frac{1}{|B|}\int_B f(x)^{1-p'}dx\right)^{p-1}\lesssim 1,$$

where $|B|$ is the Lebesgue measure of B. In this case, we have the following reverse Hölder inequality.

THEOREM D. [4, Theorem 7.4.] *If $f\in A_p$, then there exist $\varepsilon^*>0$, such that for any small $0<\varepsilon\leqslant\varepsilon^*$, and any ball $B\subset\mathbb{R}^n$*

$$\left(\frac{1}{|B|}\int_B f(x)^{1+\varepsilon}dx\right)^{\frac{1}{1+\varepsilon}}\lesssim\left(\frac{1}{|B|}\int_B f(x)dx\right).$$

Let $g(x)=|x|^{\frac{\lambda-n}{p}}$, then $g\in L^{p,\lambda}$ and $|g|^p\in A_p$. From this fact, the following makes sense.

THEOREM 4.1. *Let $0<\lambda<n$, $1<p<\frac{n-\lambda}{\alpha}$, and $\Omega\in L^s(S^{n-1})$ where $s\geqslant p'$. If $|f|^p\in A_p$, then there exist $0<\mu<\lambda$ such that identity (2) holds and*

$$\|T_{\Omega,\alpha}f\|_{L^{q,\mu}}\lesssim\|\Omega\|_{L^s(S^{n-1})}\|f\|_{L^{p,\lambda}}.$$

Proof. Fix p, λ and α. Since $|f|^p \in A_p$, by Theorem D. there exist $0 < \mu < \lambda$ such that

$$\varepsilon = \frac{\lambda - \mu}{n - \lambda} < \varepsilon^*$$

and for any ball $B \subset \mathbb{R}^n$

$$\left(\frac{1}{|B|} \int_B |f(x)|^{p(1+\varepsilon)} dx \right)^{\frac{1}{1+\varepsilon}} \lesssim \left(\frac{1}{|B|} \int_B |f(x)|^p dx \right). \tag{17}$$

For any be choosen μ, we can always find q such that identity (2) holds due to

$$0 < \frac{n - \lambda}{p(n - \mu)} - \frac{\alpha}{n - \mu} < 1.$$

Let $\|f\|_{L^{p,\lambda}} = 1$ and fix $B(z,r)$. Define $f_1 = f\chi_{B(z,2r)}$ and $f_2 = f - f_1$. Since $T_{\Omega,\alpha}$ is a linear operator, then

$$\|T_{\Omega,\alpha} f\|_{L^q(B(z,r))} \leqslant \|T_{\Omega,\alpha} f_1\|_{L^q(B(z,r))} + \|T_{\Omega,\alpha} f_2\|_{L^q(B(z,r))}. \tag{18}$$

By Lemma 3.2, we can handle $\|T_{\Omega,\alpha} f_2\|_{L^q(B(z,r))}$.

Let us handle the first term of the right hand side of inequality (18). By the pointwise estimation from inequality (7) and inequality (10),

$$\|T_{\Omega,\alpha} f_1\|_{q(B(z,r))} \lesssim \|\Omega\|_{L^s(S^{n-1})}^{1-u} \|(M_{\Omega,0} f_1)^u\|_{L^q(B(z,r))},$$

where $u = 1 - \frac{\alpha p}{n-\lambda}$. We note that $qu = p(1+\varepsilon)$. By Theorem C. and inequality (17),

$$\begin{aligned} \|T_{\Omega,\alpha} f_1\|_{q(B(z,r))} &\lesssim \|\Omega\|_{L^s(S^{n-1})} \left(\int_{B(z,2r)} |f(y)|^{p(1+\varepsilon)} dy \right)^{\frac{1}{q}} \\ &\lesssim \|\Omega\|_{L^s(S^{n-1})} \left(r^{n\left(\frac{1}{1+\varepsilon} - 1\right)} \int_{B(z,2r)} |f(y)|^p dy \right)^{\frac{1+\varepsilon}{q}} \\ &\lesssim r^{\frac{n}{q} - \frac{un}{p} + \frac{\lambda u}{p}} \|\Omega\|_{L^s(S^{n-1})} = r^{\frac{\mu}{q}} \|\Omega\|_{L^s(S^{n-1})}. \end{aligned} \tag{19}$$

Finally, we conclude our proof by inequality (18), inequality (19), and Lemma 3.2. □

4.2. The simple function

We define a class of special simple functions as follows.

DEFINITION 1. Let $K \in \mathbb{N}$. The set $\mathscr{F}^K$ contains any functions f that can be written as

$$f = \sum_{j=1}^{K} c_j \chi_{B_j}$$

where c_j is a positive constant, and $B_j = B(x_j, r_j)$.

For any $f \in \mathscr{F}^K$, we can find $D \geqslant 1$ such that $\max_j\{r_j\} \leqslant D \min_j\{r_j\}$.

DEFINITION 2. Let $D \geqslant 1$. The set $\mathscr{F}^K_D$ contains any functions $f \in \mathscr{F}^K$ such that $\max_j\{r_j\} \leqslant D \min_j\{r_j\}$.

Let's start the discussion by investigating the case of $K = 1$.

LEMMA 4.2. *Let $B = B(x_b, r_b)$. Let $1 < p < \frac{n-\lambda}{\alpha}$ and $\Omega \in L^s(S^{n-1})$ where $s \geqslant p'$. If identity (2) holds, then*

$$\|T_{\Omega,\alpha}\chi_B\|_{L^{q,\mu}} \lesssim \|\Omega\|_{L^s(S^{n-1})}\|\chi_B\|_{L^{p,\lambda}}. \tag{20}$$

Proof. For $\mu \geqslant \lambda$, inequality (20) holds by Theorem 3.3.

For $\mu < \lambda$. Since identity (2) holds, we can choose $p^* = p(n-\mu)/(n-\lambda) > p$ such that

$$\frac{n-\mu}{q} = \frac{n-\mu}{p^*} - \alpha.$$

We also have $s \geqslant p' > (p^*)'$. Hence, by Theorem 3.3

$$\|T_{\Omega,\alpha}\chi_B\|_{L^{q,\mu}} \lesssim \|\Omega\|_{L^s(S^{n-1})}\|\chi_B\|_{L^{p^*,\mu}}. \tag{21}$$

We also note that

$$\|\chi_B\|_{L^{p^*,\mu}} \lesssim r_b^{\frac{n-\mu}{p^*}} = r_b^{\frac{n-\lambda}{p}} \lesssim \|\chi_B\|_{L^{p,\lambda}}. \tag{22}$$

As the consequence of inequality (21) and inequality (22), inequality (20) holds. □

Let us continue the discussion by investigating the case of $K > 1$. We recall the inequality (7) as

$$|T_{\Omega,\alpha}f(x)| \lesssim (M_{\Omega,\frac{n-\lambda}{p}}f(x))^{1-u}(M_{\Omega,0}f(x))^u$$

where $u = 1 - \frac{\alpha p}{n-\lambda}$. If identity (2) holds and $\mu < \lambda$, then $uq = p(n-\mu)/(n-\lambda) > p$. Hence, we can't use Hölder inequality as in the proof of Theorem 3.3. For that reason, we reduce the value of u into $v = 1 - \frac{\alpha p}{n-\gamma}$ where $\gamma > \lambda$ such that $vq < p$. Suppose $p < \frac{n-\gamma}{\alpha}$, by Theorem 3.1

$$|T_{\Omega,\alpha}f(x)| \lesssim (M_{\Omega,0}f(x))^v\left(M_{\Omega,\frac{n-\gamma}{p}}f(x)\right)^{1-v}. \tag{23}$$

At this moment, we need to estimate $M_{\Omega,\frac{n-\gamma}{p}}f$ as in the following lemma.

LEMMA 4.3. *Let $x \in B(z,r)$, $f \in \mathscr{F}^K$, $d = \min\{\min_j\{r_j\}, r\}$ and $\gamma > \lambda$. If $\Omega \in L^s(S^{n-1})$ where $s \geqslant p'$, then*

$$M_{\Omega,\frac{n-\gamma}{p}}\left(f\chi_{B(z,2r)}\right)(x) \lesssim K d^{\frac{\lambda-\gamma}{p}}\|\Omega\|_{L^s(S^{n-1})}\|f\|_{L^{p,\lambda}}. \tag{24}$$

Proof. By Hölder inequality,

$$M_{\Omega,\frac{n-\gamma}{p}}\left(f\chi_{B(z,2r)}\right)(x)\lesssim\|\Omega\|_{L^s(S^{n-1})}\sup_{R>0}(g_x(R))^{\frac{1}{p}} \tag{25}$$

where

$$g_x(R)=R^{-\gamma}\int_{B(x,R)}|f(y)\chi_{B(z,2r)}(y)|^p dy.$$

We use the following obvious observation. Let $J\in\mathbb{N}$, $a_i>0$ for any i, and $b>1$, then

$$\left(\frac{1}{J}\sum_{i=1}^{J}a_i\right)^b\leqslant\sum_{i=1}^{J}a_i^b<\left(\sum_{i=1}^{J}a_i\right)^b. \tag{26}$$

Since $f\in\mathscr{F}^K$, by inequality (26)

$$\begin{aligned} g_x(R)&=R^{-\gamma}\int_{B(x,R)}\left|\sum_{j=1}^{K}c_j\chi_{B_j\cap B(z,2r)}(y)\right|^p dy\\ &\leqslant K^p\sum_{j=1}^{K}R^{-\gamma}\int_{B(x_j,R)}\left|c_j\chi_{B_j\cap B(z,2r)}(y)\right|^p dy\\ &\leqslant K^p\sum_{j=1}^{K}\sup_{t>0}t^{-\gamma}\int_{B(x_j,t)}\left|c_j\chi_{B_j\cap B(z,2r)}(y)\right|^p dy. \end{aligned} \tag{27}$$

Since the value inside the supremum in inequality (27) is increasing for $t\in(0,\min\{r_j,r\})$ and decreasing for $t>2r$, then

$$\begin{aligned} g_x(R)&\leqslant K^p\sum_{j=1}^{K}\sup_{\min\{r_j,r\}<t<2r}t^{-\gamma}\int_{B(x_j,t)}\left|c_j\chi_{B_j\cap B(z,2r)}(y)\right|^p dy\\ &\leqslant K^p\sup_{d<t<2r,x\in\mathbb{R}^n}t^{-\gamma}\int_{B(x,t)}\left|\sum_{j=1}^{K}c_j\chi_{B_j}(y)\right|^p dy\\ &\leqslant K^p d^{\lambda-\gamma}\|f\|_{L^{p,\lambda}}^p. \end{aligned} \tag{28}$$

By inequality (28) and (25), we obtain inequality (24). □

We need the following lemma to deal with the term d in Lemma 4.3.

LEMMA 4.4. *Suppose $f\in\mathscr{F}_D^K$. Then*

$$\|f\|_{L^p(B(z,2r))}\leqslant KD^{\frac{\lambda}{p}}\min_j\{r_j\}^{\frac{\lambda}{p}}\|f\|_{L^{p,\lambda}}.$$

Proof. Since $f \in \mathscr{F}_D^K$, we have $\max_j\{r_j\} \leqslant D \min_j\{r_j\}$. By inequality (26)

$$\begin{aligned}
\|f\|^p_{L^p(B(z,2r))} &\leqslant K^p \sum_{j=1}^{K} \int_{B(z,2r)} |c_j \chi_{B_j}(y)|^p dy \\
&\leqslant \max_j \{r_j\}^{\lambda} K^p \sum_{j=1}^{K} r_j^{-\lambda} \int_{B_j} |c_j \chi_{B_j}(y)|^p dy \\
&\leqslant D^{\lambda} \min_j \{r_j\}^{\lambda} K^p \sum_{j=1}^{K} \sup_{t>0, x\in\mathbb{R}^n} t^{-\lambda} \int_{B(x,t)} |c_j \chi_{B_j}(y)|^p dy \\
&\leqslant D^{\lambda} \min_j \{r_j\}^{\lambda} K^p \sup_{t>0, x\in\mathbb{R}^n} t^{-\lambda} \int_{B(x,t)} \left| \sum_{j=1}^{K} c_j \chi_{B_j}(y) \right|^p dy \\
&\leqslant D^{\lambda} \min_j \{r_j\}^{\lambda} K^p \|f\|^p_{L^{p,\lambda}}. \qquad (29)
\end{aligned}$$

Raising by the power of $1/p$ for both side of inequality (29), we conclude the proof. □

THEOREM 4.5. *Let* $0 < \mu < \lambda < \gamma < n$,

$$\max\left\{1, \frac{(\lambda-\mu)(n-\gamma)}{(\gamma-\mu)(\alpha)}\right\} < p < \frac{\lambda(n-\gamma)}{\gamma\alpha}, \qquad (30)$$

$v = 1 - \frac{\alpha p}{n-\gamma}$, *and* $\Omega \in L^s(S^{n-1})$ *where* $s \geqslant p'$. *If identity* (2) *holds and* $f \in \mathscr{F}_D^K$, *then*

$$\|T_{\Omega,\alpha} f\|_{L^{q,\mu}} \lesssim K D^{\frac{v\lambda}{p}} \|\Omega\|_{L^s(S^{n-1})} \|f\|_{L^{p,\lambda}}.$$

Proof. Since $0 < \mu < \lambda < \gamma$, then $(\lambda-\mu)/(\gamma-\mu) < \lambda/\gamma$. Hence the existence of parameter p in inequality (30) is confirmed. Moreover, by the first inequality in (30) and identity (2), we note that

$$0 < \frac{\alpha(\gamma-\mu)}{n-\gamma} - \frac{\lambda-\mu}{p} = (n-\mu)\left(\frac{\alpha}{n-\gamma} - \frac{1}{p} + \frac{1}{q}\right) = \frac{(n-\mu)}{pq}(p - vq). \qquad (31)$$

Hence, $vq < p$.

Let $\|f\|_{L^{p,\lambda}} = 1$ and fix $B(z,r)$. Define $f_1 = f\chi_{B(z,2r)}$ and $f_2 = f - f_1$. Since $T_{\Omega,\alpha}$ is a linear operator, then

$$\|T_{\Omega,\alpha} f\|_{L^q(B(z,r))} \leqslant \|T_{\Omega,\alpha} f_1\|_{L^q(B(z,r))} + \|T_{\Omega,\alpha} f_2\|_{L^q(B(z,r))}. \qquad (32)$$

By Lemma 3.2, we can handle $\|T_{\Omega,\alpha} f_2\|_{L^q(B(z,r))}$.

Now, let us handle the $\|T_{\Omega,\alpha} f_1\|_{L^q(B(z,r))}$. By inequality (23), Lemma 4.3, the following is true.

$$\|T_{\Omega,\alpha} f_1\|_{L^q(B(z,r))} \lesssim K^{1-v} d^{\frac{(\lambda-\gamma)(1-v)}{p}} \|\Omega\|^{1-v}_{L^s(S^{n-1})} \|(M_{\Omega,0} f_1)^v\|_{L^q(B(z,r))}.$$

By Hölder inequality with order p/vq, and Theorem C,

$$\|T_{\Omega,\alpha}f_1\|_{L^q(B(z,r))} \leqslant K^{1-v} d^{\frac{(\lambda-\gamma)(1-v)}{p}} r^{\frac{n}{q}-\frac{nv}{p}} \|\Omega\|_{L^s(S^{n-1})} \|f\|^{v}_{L^p(B(z,2r))}. \tag{33}$$

Since $D > 1$, If $d = r$, inequality (33) become

$$\begin{aligned} \|T_{\Omega,\alpha}f_1\|_{L^q(B(z,r))} &\leqslant KD^{\frac{v\lambda}{p}} r^{\frac{(\lambda-\gamma)(1-v)}{p}} r^{\frac{n}{q}-\frac{nv}{p}} r^{\frac{\lambda u}{p}} \|\Omega\|_{L^s(S^{n-1})} \\ &\leqslant KD^{\frac{v\lambda}{p}} r^{\frac{\mu}{q}} \|\Omega\|_{L^s(S^{n-1})}. \end{aligned} \tag{34}$$

By the second inequality in (30) we note that

$$\frac{(\lambda-\gamma)\alpha}{n-\gamma} + \frac{\lambda v}{p} = \frac{\lambda}{p} - \frac{\gamma\alpha}{n-\gamma} > 0. \tag{35}$$

If $d \neq r$, by Lemma 4.4 and inequality (35), the following follows from inequality (33).

$$\begin{aligned} \|T_{\Omega,\alpha}f_1\|_{L^q(B(z,r))} &\leqslant KD^{\frac{v\lambda}{p}} \min_j\{r_j\}^{\frac{(\lambda-\gamma)\alpha}{n-\gamma}+\frac{\lambda v}{p}} r^{\frac{n}{q}-\frac{nv}{p}} \|\Omega\|_{L^s(S^{n-1})} \\ &\leqslant KD^{\frac{v\lambda}{p}} r^{\frac{\mu}{q}} \|\Omega\|_{L^s(S^{n-1})}. \end{aligned} \tag{36}$$

By inequality (32), inequality (34), inequality (36), and Lemma 3.2, the theorem is proved. □

Apart from being dependent on K and D, the boundedness properties in Theorem 4.5 is independent of the function value.

4.3. The radial function and local Morrey spaces

If f is a radial function on $\mathbb{R}^n$ (with $f(x) = f_0(|x|)$), then we have the following elementary observation for $r < |x|$ (see [5, Lemma 1.1.]).

$$\int_{B(x,r)} |f(y)|dy \lesssim r^{n-1} \int_{|x|-r}^{|x|+r} |f_0(t)|dt. \tag{37}$$

In this section, we prove the boundedness of $T_{\Omega,\alpha}$ on local Morrey spaces. Let us estimate operator $T_{\Omega,\alpha}$ as in inequality (23). Now, we need to estimate $M_{\Omega,\frac{n-\gamma}{p}}f$ for radial functions f as follows.

LEMMA 4.6. *Let $x \in B(0,r)\setminus\{0\}$. If $\Omega \in L^s(S^{n-1})$ where $s \geqslant p'$, $\lambda \leqslant \gamma < n-1$ and f be radial function, then*

$$M_{\Omega,\frac{n-\gamma}{p}}f(x) \lesssim \|\Omega\|_{L^s(S^{n-1})} |x|^{\frac{\lambda-\gamma}{p}} \|f\|_{L^{p,\lambda}(0)}.$$

Proof. By Hölder inequality,

$$M_{\Omega,\frac{n-\gamma}{p}} f(x) \lesssim \|\Omega\|_{L^s(S^{n-1})} \sup_{R>0} (h_x(R))^{\frac{1}{p}} \tag{38}$$

where

$$h_x(R) = R^{-\gamma} \int_{B(x,R)} |f(y)|^p dy.$$

For $R \geqslant \frac{|x|}{2}$, we have $|x| \leqslant |x| + R \lesssim R$ and

$$h_x(R) \lesssim (|x|+R)^{-\gamma} \int_{B(0,|x|+R)} |f(y)|^p dy \leqslant |x|^{\lambda-\gamma} \|f\|^p_{L^{p,\lambda}(0)}. \tag{39}$$

For $R < \frac{|x|}{2}$, by inequality (37), the value of $h_x(R)$ is bounded by

$$\begin{aligned} R^{n-\gamma-1} \int_{\frac{|x|}{2}}^{\frac{3|x|}{2}} |f_0(t)|^p dt \leqslant |x|^{n-\gamma-1} \int_{\frac{|x|}{2}}^{\frac{3|x|}{2}} |f_0(t)|^p dt \lesssim |x|^{-\gamma} \int_{\frac{|x|}{2}}^{\frac{3|x|}{2}} |f_0(t)|^p t^{n-1} dt \\ \lesssim |x|^{-\gamma} \int_{B(0,2|x|)} |f(y)|^p dy \lesssim |x|^{\lambda-\gamma} \|f\|^p_{L^{p,\lambda}(0)}. \end{aligned} \tag{40}$$

By inequality (39), inequality (40), and inequality (38), the lemma is valid. □

Acquiring the estimation of $M_{\Omega,\frac{n-\lambda}{p}} f$, let us prove the following theorem.

THEOREM 4.7. *Let* $0 < \mu < \lambda < \gamma < n-1$,

$$\max\left\{1, \frac{n(\lambda-\mu)(n-\gamma)}{(n\lambda-\mu\lambda+\mu\gamma-\mu n)(\alpha)}\right\} < p < \frac{n-\gamma}{\alpha}, \tag{41}$$

and $\Omega \in L^s(S^{n-1})$ *where* $s \geqslant p'$. *If identity* (2) *holds and* f *is a radial function, then*

$$\|T_{\Omega,\alpha} f\|_{L^{q,\mu}(0)} \lesssim \|\Omega\|_{L^s(S^{n-1})} \|f\|_{L^{p,\lambda}(0)}.$$

Proof. As the consequence of $\lambda < \gamma$, we obtain

$$n(\lambda-\mu) < n\lambda - \mu\lambda + \mu\gamma - n\mu < n(\gamma-\mu). \tag{42}$$

The first inequality in (42) confirms the existence of parameter p in inequality (41) and the second inequality in (42) gives us

$$p > \frac{(\lambda-\mu)(n-\gamma)}{(\gamma-\mu)(\alpha)}.$$

Let $v = 1 - \frac{\alpha p}{n-\gamma}$. By inequality (31), it is confirmed that $vq < p$.

Let $\|f\|_{L^{p,\lambda}(0)} = 1$, and fix $B(0,r)$. We define the function $f_1 = f\chi_{B(0,2r)}$ and $f_2 = f - f_1$. Then

$$\|T_{\Omega,\alpha} f\|_{L^q(B(0,r))} \leqslant \|T_{\Omega,\alpha} f_1\|_{L^q(B(0,r))} + \|T_{\Omega,\alpha} f_2\|_{L^q(B(0,r))}. \tag{43}$$

Let us handle $\|T_{\Omega,\alpha}f_1\|_{L^q(B(0,r))}$ first. By inequality (23), Lemma 4.6,

$$\|T_{\Omega,\alpha}f_1\|_{L^q(B(0,r))} \leqslant \|\Omega\|_{L^s(S^{n-1})}^{1-\nu} \left\| (M_{\Omega,0}f_1(\cdot))^\nu |\cdot|^{\frac{(\lambda-\gamma)(1-\nu)}{p}} \right\|_{L^q(B(0,r))}.$$

Let $t = \frac{p}{p-\nu q}$. By Hölder inequality with order $p/\nu q = t'$, and Theorem C

$$\|T_{\Omega,\alpha}f_1\|_{L^q(B(0,r))} \leqslant \|\Omega\|_{L^s(S^{n-1})}^{1-\nu} \|f_1\|_{L^p}^{\nu} \left\| |\cdot|^{\frac{(\lambda-\gamma)(1-\nu)}{p}q} \right\|_{L^t(B(0,r))}^{\frac{1}{q}}. \tag{44}$$

By the first inequality in (41),

$$\frac{(\lambda-\gamma)(1-\nu)}{p}qt + n = \frac{npq}{(p-q\nu)(n-\mu)} \left(\frac{(n\lambda - \mu\lambda + \mu\gamma - n\mu)\alpha}{n(n-\gamma)} - \frac{\lambda-\mu}{p} \right) > 0.$$

Hence,

$$\left\| |\cdot|^{\frac{(\lambda-\gamma)(1-\nu)}{p}q} \right\|_{L^t(B(0,r))}^{\frac{1}{q}} \lesssim \left(\int_0^r R^{\frac{(\lambda-\gamma)(1-\nu)}{p}qt+n-1} dr \right)^{\frac{1}{q}} \lesssim r^{\frac{(\lambda-\gamma)\alpha}{n-\gamma}+\frac{n}{q}-\frac{n\nu}{p}}. \tag{45}$$

By inequality (45) and inequality (44),

$$\|T_{\Omega,\alpha}f_1\|_{L^q(B(0,r))} \lesssim r^{\frac{(\lambda-\gamma)\alpha}{n-\gamma}+\frac{n}{q}-\frac{n\nu}{p}} \|\Omega\|_{L^s(S^{n-1})} \|f_1\|_{L^p}^{\nu} \leqslant r^{\frac{\mu}{q}} \|\Omega\|_{L^s(S^{n-1})}. \tag{46}$$

Now, we treat $\|T_{\Omega,\alpha}f_2\|_{L^q(B(0,r))}$. If $x \in B(0,r)$ then $B^c(0,2r) \subset B^c(x,r)$ and $B(x,2^j r) \subset B(0,2^{j+1}r)$. By Hölder inequality,

$$\begin{aligned} |T_{\Omega,\alpha}f_2(x)| &\leqslant \int_{B^c(x,r)} \frac{|\Omega(y-x)|}{|y-x|^{n-\alpha}} |f(y)| dy \\ &\leqslant \sum_{j=1}^{\infty} \left(2^{j-1}r\right)^{\alpha-n} \int_{B(x,2^j r)\setminus B(x,2^{j-1}r)} |\Omega(y-x)||f(y)| dy \\ &\leqslant \sum_{j=1}^{\infty} \left(2^{j-1}r\right)^{\alpha-n} \int_{B(0,2^{j+1}r)} |\Omega(y-x)||f(y)| dy \\ &\lesssim r^{\alpha-\frac{n-\lambda}{p}} \|\Omega\|_{L^s(S^{n-1})} \sum_{j=1}^{\infty} 2^{j(\alpha-\frac{n-\lambda}{p})}. \end{aligned} \tag{47}$$

Since the summation in inequality (47) converges,

$$\|T_{\Omega,\alpha}f_2\|_{L^q(B(0,r))} \lesssim r^{\frac{\mu}{q}} \|\Omega\|_{L^s(S^{n-1})}. \tag{48}$$

By inequality (46) and inequality (48), Theorem 4.7 is verified. □

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