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ZERO–ORDER MEHLER–FOCK TRANSFORM AND SOBOLEV–TYPE SPACE

AKHILESH PRASAD, U. K. MANDAL AND S. K. VERMA

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Abstract. The present paper is devoted to the study of the Mehler-Fock transform with index as the Legendre function of first kind. Continuity property of the Mehler-fock transform on the test function spaces Λ_α and $\mathcal{G}_\alpha$ is given. Moreover pseudo-differential operator (p.d.o.) with symbol $\sigma(x,\tau) \in S^m$ in terms of Mehler-Fock transform is defined and also its continuity property from test function space $\mathcal{G}_\alpha$ into Λ_α is shown. The Mehler-Fock potential (MF-potential) $\mathscr{P}^s_\sigma$ is defined on $\mathcal{G}_\alpha(I)$ space and it is extended to the space of distribution. Also some properties of MF-potential are discussed. At the end Sobolev type space $V^{s,p}(I)$ is defined and it is shown that MF-potential is an isometry of $V^{s,p}(I)$.

1. Introduction

The Mehler-Fock transform was first introduced by F. G. Mehler [13] and then Mehler's investigation was substantially completed by V. A. Fock [2] by giving its inversion and some basic properties related to it. The generalization of the Mehler-Fock transform in terms of hypergeometric function was constructed by M. N. Olevskii [17] and N. Ya. Vilenkin [31]. Mehler-Fock transform belongs to a special class of integral transform, known as index transform. The kernel of the index transform depends on some of the parameter of special function involved in it. Mehler-Fock transform contains special function $P_{i\tau-\frac{1}{2}}(x)$ as kernel known as cone function or Mehler function or Legendre function of zero order. More details about the index transforms can be found in [33].

The Mehler-Fock transform has some important applications in mathematical physics and for solving some integral equations etc., [18, 11, 30, 25]. Apart from the applications area, investigation about this integral transform in the arena of pure mathematics was carried out by Lebedev [12, 11], Yakubovich and Luchko [34], Srivastava et al.[29] and many more can be found. Distinct forms of the Mehler-Fock transform were also

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introduced by various authors for instance see [6, 33, 4, 28]. In this paper, we consider the Mehler-Fock transform defined as [28, 23]:

$$(\mathfrak{M}\varphi)(\tau) = \int_1^\infty P_{i\tau-\frac{1}{2}}(x)\varphi(x)dx, \quad \tau > 0. \tag{1}$$

Its inversion is given by

$$\varphi(x) = \int_0^\infty \tau\tanh(\pi\tau)P_{i\tau-\frac{1}{2}}(x)(\mathfrak{M}\varphi)(\tau)d\tau, \quad x > 1, \tag{2}$$

where $P_{i\tau-\frac{1}{2}}(x)$ is cone function (Legendre function of first kind), represented in terms of Gaussian hypergeometric function ${}_2F_1$ as

$$P_{i\tau-\frac{1}{2}}(x) = P^0_{i\tau-\frac{1}{2}}(x) = {}_2F_1(1/2+i\tau, 1/2-i\tau; 1; (1-x)/2),$$

and it is an even function of the parameter τ, i.e.

$$P_{i\tau-\frac{1}{2}}(x) = P_{-i\tau-\frac{1}{2}}(x).$$

The asymptotic representation of Legendre function $P_{-\frac{1}{2}}(x)$ is given as [15, p. 171–173]

$$P_{-\frac{1}{2}}(x) \sim 1 \quad \text{as } x \to 1, \tag{3}$$

$$P_{-\frac{1}{2}}(x) \sim \frac{\sqrt{2}}{\pi}\frac{ln(x)}{\sqrt{x}} \quad \text{as} \quad x \to \infty. \tag{4}$$

The cone function $P_{i\tau-\frac{1}{2}}(x)$ is an eigen function for the self adjoint operator A_x as [4]

$$A_x = (x^2-1)D_x^2 + 2xD_x, \quad D_x = \frac{d}{dx} \tag{5}$$

and

$$A_x P_{i\tau-\frac{1}{2}}(x) = (-1)\left(\tau^2 + \frac{1}{4}\right)P_{i\tau-\frac{1}{2}}(x).$$

Moreover, for $k \in \mathbb{N}_0$, we have

$$A_x^k P_{i\tau-\frac{1}{2}}(x) = (-1)^k\left(\tau^2 + \frac{1}{4}\right)^k P_{i\tau-\frac{1}{2}}(x). \tag{6}$$

The series representation of A_x is given by

$$A_x^k\varphi(x) = \sum_{j=1}^{2k} p_j(x)D_x^j\varphi(x), \; k \in \mathbb{N}, \tag{7}$$

where $p_j(x)$ is the polynomial of j^{th} degree and $p_{2k}(x) = (x^2-1)^k$.

The product formula of the Legendre function [5, p. 112 (Lemma 1.9.10)] is given by

$$P_{i\tau-\frac{1}{2}}(x)P_{i\tau-\frac{1}{2}}(y) = \int_1^\infty K(x,y,z)P_{i\tau-\frac{1}{2}}(z)dz, \tag{8}$$

where

$$K(x,y,z) = \begin{cases} \frac{1}{\pi}(2xyz+1-x^2-y^2-z^2)^{-\frac{1}{2}}, & z \in I_{x,y}, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$I_{x,y} =: \left(xy - [(x^2-1)(y^2-1)]^{\frac{1}{2}},\ xy + [(x^2-1)(y^2-1)]^{\frac{1}{2}}\right).$$

Throughout the paper we will consider the Lebesgue space $L^p(I)$, $I = (1,\infty)$ as the class of measurable functions φ on I such that

$$\|\varphi\|_{L^p(I)} = \begin{cases} \left(\int_1^\infty |\varphi(x)|^p dx\right)^{\frac{1}{p}}, & \text{for} \quad 1 \leqslant p < \infty, \\ ess.\sup\limits_{x\in I} |\varphi(x)|, & \text{for} \quad p = \infty. \end{cases}$$

The present paper is classified into five sections, Section 1 is as introductory in which a brief introduction about the Mehler-Fock transform of zero order, Legendre function and its asymptotic behaviour etc., are given. In Section 2, translation and convolution operators in terms of Mehler-Fock transform is given and their some basic results are obtained. Section 3 is concerned with basic definitions of the test function spaces Λ_α and $\mathscr{G}_\alpha$. Moreover continuity of the differential operator A_x and Mehler-Fock transform on these function spaces have been discussed. In Section 4, symbol class S^m and pseudo-differential operator (p.d.o.) associated with the Mehler-Fock transform are defined and proved the continuity of p.d.o. from $\mathscr{G}_\alpha$ into Λ_α. Further in section 5, Mehler-Fock potential on $\mathscr{G}_\alpha(I)$ space is defined and extended on distribution space. Also Sobolev type space $V^{s,p}(I)$ is introduced and it is shown that MF-potential is an isometry of $V^{s,p}(I)$. Finally an $L^p(I)$ estimate of MF-potetinal is obtained.

2. The translation and convolution operator in classical framework of Mehler-Fock transform

Using the inversion formula (2) and product formula (8), the integral representation of $K(x,y,z)$, $x,\ y,\ z \in I$, can be written as:

$$K(x,y,z) = \int_0^\infty \tau\tanh(\pi\tau)\ P_{i\tau-\frac{1}{2}}(x)P_{i\tau-\frac{1}{2}}(y)P_{i\tau-\frac{1}{2}}(z)\ d\tau.$$

The product formula of the kernel (8) leads to define the translation operator for function φ on some suitable function space associated to Mehler-Fock transform as [4, 23]:

$$(\mathfrak{T}_x\varphi)(y) = \int_1^\infty K(x,y,z)\ \varphi(z)\ dz.$$

Consequently, the convolution operator is defined as

$$(\varphi * \psi)(x) = \int_1^\infty (\mathfrak{T}_x\varphi)(y)\psi(y)\,dy \\ = \int_1^\infty \int_1^\infty K(x,y,z)\ \varphi(z)\ \psi(y)\ dy\,dz. \tag{9}$$

From [23], we recall operational formula for the translation operator as:

$$(\mathfrak{M}(\mathfrak{T}_x\varphi))(\tau) = P_{i\tau-\frac{1}{2}}(x)(\mathfrak{M}\varphi)(\tau).$$

and for the convolution operator as:

$$(\mathfrak{M}(\varphi * \psi))(\tau) = (\mathfrak{M}\varphi)(\tau)(\mathfrak{M}\psi)(\tau). \tag{10}$$

LEMMA 1. *For $n \in \mathbb{N}_0$, we have the following inequality*

$$\left|\frac{d^n}{dx^n}P_{i\tau-\frac{1}{2}}(x)\right| \leqslant C(n,\tau)\ (x^2-1)^{-\frac{n}{2}}P_{-\frac{1}{2}}(x), \tag{11}$$

where

$$C(n,\tau) = \left|\frac{\Gamma(i\tau+n+1/2)}{\Gamma(i\tau+1/2)}\right|. \tag{12}$$

Also

$$C(n,\tau) \leqslant \begin{cases} \frac{\Gamma(n+1/2)}{\sqrt{\pi}}, & \textit{for } \tau \to 0, \\ \tau^n, & \textit{for } \tau \to \infty. \end{cases} \tag{13}$$

Proof. We recall from [16, 14.6.3], the following relation

$$\frac{d^n}{dx^n}P_{i\tau-\frac{1}{2}}(x) = (x^2-1)^{-\frac{n}{2}}P^n_{i\tau-\frac{1}{2}}(x), \tag{14}$$

here $P^n_{i\tau-\frac{1}{2}}(x)$ denotes associated Legendre function of order $n \in \mathbb{N}_0$. Now from [1, (14) p. 157], we have

$$P^n_{i\tau-\frac{1}{2}}(x) = \frac{\Gamma(i\tau+1/2+n)}{\pi\Gamma(i\tau+1/2)}\int_0^\pi [x+(x^2-1)^{\frac{1}{2}}\cos t]^{i\tau-\frac{1}{2}}\cos(nt)dt. \tag{15}$$

For $n = 0$, we see that

$$P^0_{i\tau-\frac{1}{2}}(x) = P_{i\tau-\frac{1}{2}}(x) = \frac{1}{\pi}\int_0^\pi [x+(x^2-1)^{\frac{1}{2}}\cos t]^{i\tau-\frac{1}{2}}dt,$$

which readily yield

$$|P_{i\tau-\frac{1}{2}}(x)| \leqslant P_{-\frac{1}{2}}(x). \tag{16}$$

Now from (15) and (16), we have

$$\begin{aligned} |P^n_{i\tau-\frac{1}{2}}(x)| &\leqslant C(n,\tau)\ \frac{1}{\pi}\int_0^{\pi}[x+(x^2-1)^{\frac{1}{2}}\cos t]^{-\frac{1}{2}}dt \\ &\leqslant C(n,\tau)P_{-\frac{1}{2}}(x), \end{aligned} \tag{17}$$

where $C(n,\tau)$ is defined as (12). Therefore from (14) and (17), we get the desired result (11). Further, for large value of τ, we have

$$\begin{aligned} C(n,\tau) &= \left|\frac{\Gamma(i\tau+n+1/2)}{\Gamma(i\tau+1/2)}\right| \\ &= \left|\big(i\tau+1/2+n-1\big)\big(i\tau+1/2+n-2\big)\cdots\big(i\tau+1/2\big)\right| \\ &= \tau^n\left[\left(1+\frac{[1/2+(n-1)]^2}{\tau^2}\right)\left(1+\frac{[1/2+(n-2)]^2}{\tau^2}\right)\cdots\left(1+\frac{1}{4\tau^2}\right)\right]^{\frac{1}{2}} \\ &\leqslant \tau^n \quad \text{as} \quad \tau\to\infty. \end{aligned}$$

Hence we get (13). □

An estimate for the derivative of the Legendre function $P_{i\tau-\frac{1}{2}}(x)$ with respect to τ can be viewed from [8] as:

$$|D^m_\tau P_{i\tau-\frac{1}{2}}(x)| \leqslant M\ [ln(x+(x^2-1)^{\frac{1}{2}}]^m\ P_{-\frac{1}{2}}(x), \tag{18}$$

where $M>0$ is a constant and $m\in\mathbb{N}_0$.

3. The test function spaces

We introduce the following function space Λ_α analogous to the function space defined in [7]:

DEFINITION 1. The function space Λ_α is the space of all infinitely differentiable complex valued function $\varphi(x)$, such that

$$\gamma_{\alpha,k}(\varphi) = \sup_{x\in I}|\lambda^-_\alpha(x)A^k_x\varphi(x)| < \infty, \tag{19}$$

where $\alpha>0$, $k\in\mathbb{N}_0$ and $\lambda^-_\alpha(x)$ denotes the continuous function on I, given by

$$\lambda^-_\alpha(x) = \begin{cases} e^{-\frac{\alpha}{x-1}}, & x\in(1,2], \\ e^{-\alpha(x-1)}, & x\in[2,\infty), \end{cases} \tag{20}$$

and A_x signifies the differential operator (5).

PROPOSITION 1. *The differential operator A_x is continuous linear mapping from Λ_α into itself.*

Proof. Proof is simple and thus avoided. □

REMARK 1. As the differential operator is self adjoint, thus we have

$$\langle A_x\varphi, \psi\rangle = \langle \varphi, A_x\psi\rangle,$$

where $\varphi \in \Lambda'_\alpha$ and $\psi \in \Lambda_\alpha$. Here Λ'_α denotes the dual space of Λ_α. Thus generalized operator A_x is continuous linear mapping from Λ'_α into itself.

Moreover from (6), (15) and the asymptotic expressions (3), (4), for some $\tau > 0$ we have

$$\gamma_{\alpha,k}(P_{i\tau-\frac{1}{2}}(x)) < \infty.$$

Thus the kernel $P_{i\tau-\frac{1}{2}}(x)$ of the Mehler-Fock transform belongs to the function space Λ_α.

Next we consider a new test function space defined as:

DEFINITION 2. The function space $\mathscr{G}_\alpha$ is the space of all infinitely differentiable complex valued function $\varphi(x)$, such that

$$\Gamma_{\alpha,k}(\varphi) = \sup_{x\in I} |\lambda_\alpha^+(x)A_x^k\varphi(x)| < \infty, \tag{21}$$

where $\alpha > 0$, $k \in \mathbb{N}_0$ and $\lambda_\alpha^+(x)$ denotes the continuous function on I, given by

$$\lambda_\alpha^+(x) = \begin{cases} e^{\frac{\alpha}{x-1}}, & x \in (1,2], \\ e^{\alpha(x-1)}, & x \in [2,\infty), \end{cases} \tag{22}$$

and the differential operator A_x is defined as (5).

Similar to Proposition 1, we remark that the differential operator A_x is also a continuous linear mapping from $\mathscr{G}_\alpha$ into itself.

Moreover for every $\varphi \in \mathscr{G}_\alpha$, we have

$$\begin{aligned} \gamma_{\alpha,k}(\varphi) &= \sup_{x\in I} |\lambda_\alpha^-(x)A_x^k\varphi(x)| \\ &= \sup_{x\in I} |(\lambda_\alpha^-(x))^2\lambda_\alpha^+(x)A_x^k\varphi(x)|, \end{aligned}$$

using (20) and Definition 2, we have

$$\gamma_{\alpha,k}(\varphi) \leqslant C\,\Gamma_{\alpha,k}(\varphi) < \infty.$$

Since $P_{i\tau-\frac{1}{2}}(x) \notin \mathscr{G}_\alpha$. Thus we can say that $\mathscr{G}_\alpha$ is proper subspace of Λ_α.

THEOREM 2. *For $\alpha > 0$, Mehler-Fock transform is continuous linear mapping from $\mathscr{G}_\alpha$ into Λ_α.*

Proof. The linearity of the transformation is obvious, thus we prove now its continuity. Consider $\varphi \in \mathscr{G}_\alpha$, then from (1) and the series representation of A_τ^k given as (7), we get

$$A_\tau^k(\mathfrak{M}\varphi)(\tau) = \sum_{j=1}^{2k} p_j(\tau) \int_1^\infty D_\tau^j P_{i\tau-\frac{1}{2}}(x)\varphi(x)dx.$$

Using the Definition 1, (18) and Definition 2, we have

$$\gamma_{\alpha,k}(\mathfrak{M}\varphi) \leqslant M \sup_{\tau\in I} \left|\lambda_\alpha^-(\tau)\sum_{j=1}^{2k} p_j(\tau)\right| \Gamma_{\alpha,0}(\varphi) \int_1^\infty [ln(x+(x^2-1)^{\frac{1}{2}})]^j \frac{P_{-\frac{1}{2}}(x)}{\lambda_\alpha^+}dx.$$

From (3) and (4) the asymptotic expressions of $P_{-\frac{1}{2}}(x)$ and (22), we have

$$\gamma_{\alpha,k}(\mathfrak{M}\varphi) \leqslant M \sup_{\tau\in I} \left|\lambda_\alpha^-(\tau)\sum_{j=1}^{2k} p_j(\tau)\right| \Gamma_{\alpha,0}(\varphi)\Big[\int_1^2 \frac{ln(x+(x^2-1)^{1/2})^j}{e^{\frac{\alpha}{x-1}}}dx$$
$$+\int_2^\infty \frac{\sqrt{2}}{\pi}\frac{[ln(x+(x^2-1)^{1/2})]^j ln(x)}{e^{\alpha(x-1)}\sqrt{x}}dx\Big],$$

thus for $\alpha > 0$ both integrals are convergent. Also using (20) for $\alpha > 0$ the supremum is bounded. Hence

$$\gamma_{\alpha,k}(\mathfrak{M}\varphi) \leqslant C\,\Gamma_{\alpha,0}(\varphi),$$

where $C > 0$ is a constant. Hence the Theorem is proved. $\square$

4. Pseudo-differential operators in terms of Mehler-Fock transform

The study of pseudo-differential operators (p.d.o.) began with the work of Kohn [10], Nirenberg [14], Hörmander [9] in terms of Fourier transform. These operators are extension of partial differential operators and now became a field of independent research. By using the theory of various integral transforms like Hankel transform, Fourier-Jacobi transform, Kontorovich-Lebedev transform, Fourier transform etc., pseudo-differential operators have been constructed and studied on several function and distribution spaces [26, 20, 19, 21, 22, 24, 32].

In this correspondence, the p.d.o. in terms of Mehler-Fock transform of zero order is defined as [23]:

DEFINITION 3. Let the complex valued function $\sigma(x,\tau) \in C^\infty(I\times\mathbb{R}_+)$, $\mathbb{R}_+ = (0,\infty)$. Then the pseudo-differential operator $\mathscr{P}_\sigma$ is defined as

$$(\mathscr{P}_\sigma\varphi)(x) = \int_0^\infty \tau\tanh(\pi\tau)P_{i\tau-\frac{1}{2}}(x)\sigma(x,\tau)(\mathfrak{M}\varphi)(\tau)d\tau. \tag{23}$$

We call complex valued function $\sigma(x,\tau)$ belongs to symbol S^m, $m\in\mathbb{R}$, as:

DEFINITION 4. The function $\sigma(x,\tau) : C^{\infty}(I\times\mathbb{R}_+) \to \mathbb{C}$ belongs to the symbol class S^m iff for $a,b,l \in \mathbb{N}_0$ and $m\in\mathbb{R}, \exists$ a constant $C = C_{m,a,b,l} > 0$, such that

$$(1+x)^l |D_x^a D_\tau^b \sigma(x,\tau)| \leqslant C\left(\frac{1}{4}+\tau^2\right)^{\frac{1}{2}(m-b)}. \tag{24}$$

THEOREM 3. *For $\alpha>0$, the pseudo-differential operator $\mathscr{P}_\sigma$ is a continuous linear mapping from $\mathscr{G}_\alpha$ into Λ_α.*

Proof. Applying A_x^k to (23) and using series representation (7), we get

$$\begin{aligned}
A_x^k(\mathscr{P}_\sigma\varphi)(x) &= \int_0^\infty \tau\tanh(\pi\tau)\sum_{j=1}^{2k} p_j(x) D_x^j[P_{i\tau-\frac{1}{2}}(x)\sigma(x,\tau)](\mathfrak{M}\varphi)(\tau)d\tau \\
&= \int_0^\infty \tau\tanh(\pi\tau)\sum_{j=1}^{2k} p_j(x)\sum_{r=0}^{j}\binom{j}{r} D_x^r P_{i\tau-\frac{1}{2}}(x) D_x^{j-r}\sigma(x,\tau) \\
&\quad \times(\mathfrak{M}\varphi)(\tau)d\tau. \\
&= \sum_{j=1}^{2k} p_j(x)\sum_{r=0}^{j}\binom{j}{r}\int_0^\infty \tau\tanh(\pi\tau) D_x^r P_{i\tau-\frac{1}{2}}(x) D_x^{j-r}\sigma(x,\tau) \\
&\quad \times\left(\frac{1}{4}+\tau^2\right)^{-n}(-1)^n\left[-\left(\frac{1}{4}+\tau^2\right)\right]^n(\mathfrak{M}\varphi)(\tau)d\tau.
\end{aligned} \tag{25}$$

Invoking (6) and Remark 1, we have

$$\begin{aligned}
\left[-\left(\frac{1}{4}+\tau^2\right)\right]^n(\mathfrak{M}\varphi)(\tau) &= \int_1^\infty \left[-\left(\frac{1}{4}+\tau^2\right)\right]^n P_{i\tau-\frac{1}{2}}(y)\varphi(y)dy \\
&= \int_1^\infty A_y^n P_{i\tau-\frac{1}{2}}(y)\varphi(y)dy \\
&= \int_1^\infty P_{i\tau-\frac{1}{2}}(y) A_y^n\varphi(y)dy.
\end{aligned}$$

Thus from (16) and Definition 2, we get

$$\left|\left[-\left(\frac{1}{4}+\tau^2\right)\right]^n(\mathfrak{M}\varphi)(\tau)\right| \leqslant \Gamma_{\alpha,n}(\varphi)\int_1^\infty \frac{1}{\lambda_\alpha^+(y)} P_{-\frac{1}{2}}(y)dy.$$

By using asymptotic expressions (3), (4) and (22) the above integral converges for $\alpha>0$. Thus

$$\left|\left[-\left(\frac{1}{4}+\tau^2\right)\right]^n(\mathfrak{M}\varphi)(\tau)\right| \leqslant C'\,\Gamma_{\alpha,n}(\varphi), \tag{26}$$

where $C'>0$ is a constant. Invoking Definition 1, (25) and (26), we have

$$\begin{aligned}
\gamma_{\alpha,k}(\mathscr{P}_\sigma\varphi) \leqslant C'\,\Gamma_{\alpha,n}(\varphi)\sup_{x\in I}\Bigg|\lambda_\alpha^-(x)\sum_{j=1}^{2k} p_j(x)\sum_{r=0}^{j}\binom{j}{r}\int_0^\infty \tau D_x^r P_{i\tau-\frac{1}{2}}(x) \\
\times D_x^{j-r}\sigma(x,\tau)\left(\frac{1}{4}+\tau^2\right)^{-n}d\tau.
\end{aligned} \tag{27}$$

Now using (11), (13) and (24), the integral (27) reduces to

$$\gamma_{\alpha,k}(\mathscr{P}_\sigma\varphi) \leqslant C\,C'\Gamma_{\alpha,n}(\varphi)\sup_{x\in I}\Bigg|\lambda_\alpha^-(x)P_{-\frac{1}{2}}(x)(1+x)^{-l}\sum_{j=1}^{2k}p_j(x)$$
$$\times\sum_{r=0}^{j}\binom{j}{r}(x^2-1)^{-\frac{r}{2}}\Bigg[\frac{\Gamma(r+1/2)}{\sqrt{\pi}}\int_0^1\tau\left(\frac{1}{4}+\tau^2\right)^{\frac{m}{2}}\left(\frac{1}{4}+\tau^2\right)^{-n}d\tau$$
$$+\int_1^\infty\tau^{r+1}\left(\frac{1}{4}+\tau^2\right)^{\frac{m}{2}}\left(\frac{1}{4}+\tau^2\right)^{-n}d\tau\Bigg|,$$

the integral converges for $r+m+2<0$. Again using (3), (4) and (20) the supremum is finite for $\alpha>0$. Thus

$$\gamma_{\alpha,k}(\mathscr{P}_\sigma\varphi) \leqslant C''\,\Gamma_{\alpha,n}(\varphi),$$

where $C''>0$ is a constant. This proves the Theorem. □

Special cases

Case (i): If we consider the symbol $\sigma(x,\tau)$, which can be explicitly represented as

$$\sigma(x,\tau)=w_1(x)w_2(\tau),$$

such that $w_1(x)\neq 0$ on I. Then from (23) and (2), we have

$$\begin{aligned}(\mathscr{P}_\sigma\phi)(x) &= \int_0^\infty \tau\tanh(\pi\tau)P_{i\tau-\frac{1}{2}}(x)w_1(x)w_2(\tau)(\mathfrak{M}\phi)(\tau)d\tau\\ \left(\frac{\mathscr{P}_\sigma\phi}{w_1}\right)(x) &= \mathfrak{M}^{-1}\left[w_2(\mathfrak{M}\phi)(\cdot)\right](x)\\ \left[\mathfrak{M}\left(\frac{\mathscr{P}_\sigma\phi}{w_1}\right)\right](\tau) &= w_2(\tau)(\mathfrak{M}\phi)(\tau).\end{aligned}$$

Further, if we consider $w_2(\tau)=C$ as a constant. Then

$$\left(\mathscr{P}_\sigma\phi\right)(x)=Cw_1(x)\phi(x).$$

Thus from here we can conclude that under certain circumstances pseudo-differential operator is just a product of two functions and independent of integral form.

Case (ii): If we consider the symbol $\sigma(x,\tau)$

$$\sigma(x,\tau)=\int_1^\infty P_{i\tau-\frac{1}{2}}(z)w(x,z)dz, \tag{28}$$

where $|w(x,z)| < K(x)$ and $K(x) \in L^1(I)$. Then from (23), (28), (8) and (9), we have

$$
\begin{aligned}
\big(\mathscr{P}_\sigma \phi\big)(x) &= \int_0^\infty \tau \tanh(\pi\tau) P_{i\tau-\frac{1}{2}}(x) \left(\int_1^\infty P_{i\tau-\frac{1}{2}}(z) w(x,z) dz \right) (\mathfrak{M}\phi)(\tau) d\tau \\
&= \int_1^\infty \int_0^\infty \tau \tanh(\pi\tau) \left(\int_1^\infty K(x,y,z) P_{i\tau-\frac{1}{2}}(y) dy \right) \\
&\quad \times w(x,z) \big(\mathfrak{M}\phi\big)(\tau) d\tau dz \\
&= \int_1^\infty \int_1^\infty \left(\int_0^\infty P_{i\tau-\frac{1}{2}}(y) \tau \tanh(\pi\tau) \big(\mathfrak{M}\phi\big)(\tau) d\tau \right) \\
&\quad \times K(x,y,z) w(x,z) dy dz \\
&= \int_1^\infty \int_1^\infty K(x,y,z) \phi(y) w(x,z) dy dz \\
&= [\phi * w(x,\cdot)](x).
\end{aligned}
$$

Thus we see that pseudo-differential operator $\mathscr{P}_\sigma$ can be represented in terms of convolution of the two functions.

5. Mehler-Fock potential and Sobolev type space

The potential operators have been discussed earlier associated with various integral transforms like Fourier transform, Jacobi transform, Hankel transform by the authors Wong [32], Salem et al. [27] and Pathak et al. [20] respectively. In the similar manner we defined potential operator associated with Mehler-Fock transform as:

For $s \in \mathbb{R}$ and $\varphi \in \mathscr{G}_\alpha(I)$, using (23) the pseudo-differential operator $\mathscr{P}_\sigma$ associated with the symbol $\sigma(\tau) = \left(\frac{1}{4} + \tau^2\right)^{-\frac{s}{2}} \in S^{-s}$ is

$$
\big(\mathscr{P}_\sigma^s \varphi\big)(x) = \int_0^\infty \tau \tanh(\pi\tau) P_{i\tau-\frac{1}{2}}(x) \left(\frac{1}{4} + \tau^2\right)^{-\frac{s}{2}} (\mathfrak{M}\varphi)(\tau) d\tau, \tag{29}
$$

which will be further known as Mehler-Fock potential (MF-potential) operator.

PROPOSITION 4. *For $\alpha > 0$, the MF-potential operator is a continuous linear mapping from $\mathscr{G}_\alpha$ into Λ_α.*

Proof. The proof can be carried out similar to Theorem 3. □

PROPOSITION 5. *Let G be a non-empty set of pseudo-differential operators defined as (29). Then (G, o) forms an abelian group, where "o" denotes composition of two operators.*

Proof. From (29), we see that for $t \in \mathbb{R}$

$$
(\mathscr{P}_\sigma^t \varphi)(x) = \mathfrak{M}^{-1}\left[\left(\frac{1}{4} + (\cdot)^2\right)^{-\frac{t}{2}} (\mathfrak{M}\varphi)(\cdot) \right](x) \in G
$$

such that

$$(\mathscr{P}_{\sigma}^{s} o \mathscr{P}_{\sigma}^{t} \varphi)(x) = (\mathscr{P}_{\sigma}^{s+t} \varphi)(x) = (\mathscr{P}_{\sigma}^{t} o \mathscr{P}_{\sigma}^{s} \varphi)(x).$$

Thus it satisfies the closure, associativity and commutativity properties.

Identity: From (29) and (2), it is clear that $(\mathscr{P}_{\sigma}^{0} \varphi)(x) = \varphi(x)$. Now, we have

$$(\mathscr{P}_{\sigma}^{s} o \mathscr{P}_{\sigma}^{0} \varphi)(x) = (\mathscr{P}_{\sigma}^{s} \varphi)(x) = (\mathscr{P}_{\sigma}^{0} o \mathscr{P}_{\sigma}^{s} \varphi)(x).$$

Hence $\mathscr{P}_{\sigma}^{0} \in G$ behaves as the identity.

Inverse: If the pseudo-differential operator

$$(\mathscr{Q}_{\sigma}^{s} \varphi)(x) = \mathfrak{M}^{-1}\left[\left(\frac{1}{4} + (\cdot)^2\right)^{\frac{s}{2}} (\mathfrak{M}\varphi)(\cdot)\right](x) \in G,$$

then

$$(\mathscr{P}_{\sigma}^{s} o \mathscr{Q}_{\sigma}^{s} \varphi)(x) = \varphi(x) = (\mathscr{Q}_{\sigma}^{s} o \mathscr{P}_{\sigma}^{s} \varphi)(x).$$

Thus $\mathscr{Q}_{\sigma}^{s}$ is the inverse element of $\mathscr{P}_{\sigma}^{s}$. Hence (G, o) is an abelian group. □

DEFINITION 5. The MF-potential $\mathscr{P}_{\sigma}^{s}$ is defined on Λ'_{α} as

$$\left\langle \mathscr{P}_{\sigma}^{s} \psi, \varphi \right\rangle = \left\langle \psi, \mathscr{P}_{\sigma}^{s} \varphi \right\rangle, \quad \psi \in \Lambda'_{\alpha},\ \varphi \in \mathscr{G}_{\alpha},$$

where $\left\langle \psi, \varphi \right\rangle = \int_1^{\infty} \psi(x)\varphi(x)dx$.

REMARK 2. In view of the Definition 5 and Proposition 4, it is clear that MF-potential $\mathscr{P}_{\sigma}^{s}$ maps Λ'_{α} into $\mathscr{G}'_{\alpha}$.

PROPOSITION 6. *For $\psi \in \Lambda'_{\alpha}$, we have*

$$(\mathscr{P}_{\sigma}^{s} \psi)(x) = \mathfrak{M}^{-1}\left[\left(\frac{1}{4} + (\cdot)^2\right)^{-\frac{s}{2}} (\mathfrak{M}\psi)(\cdot)\right](x). \tag{30}$$

Proof. Using Definition 5, proof can be obtained straightforward. □

THEOREM 7. *Let $\psi \in \Lambda'_{\alpha}$, then*

$$\begin{aligned} (i)\ & \mathscr{P}_{\sigma}^{s} o \mathscr{P}_{\sigma}^{t} \psi = \mathscr{P}_{\sigma}^{s+t} \psi, \quad s,t \in \mathbb{R} \\ (ii)\ & \quad \mathscr{P}_{\sigma}^{0} \psi = \psi. \end{aligned} \tag{31}$$

DEFINITION 6. (The Sobolev type space $V^{s,p}(I)$) For $s \in \mathbb{R}$ and $2 < p < \infty$, the space $V^{s,p}(I)$ is the collection of elements $\psi \in \Lambda'_{\alpha}$ such that $\mathscr{P}_{\sigma}^{-s} \psi$ is a function in $L^p(I)$. The norm on $V^{s,p}(I)$ is equipped with

$$\|\psi\|_{V^{s,p}} = \|\mathscr{P}_{\sigma}^{-s} \psi\|_{L^p(I)} = \left[\int_1^{\infty} |\mathscr{P}_{\sigma}^{-s} \psi|^p dx\right]^{\frac{1}{p}}. \tag{32}$$

In particular $V^{0,p}(I) = L^p(I)$.

THEOREM 8. *Let $s \in \mathbb{R}$ and $2 < p < \infty$. Then $V^{s,p}(I)$ is a Banach space with respect to the norm $\|\cdot\|_{V^{s,p}}$.*

Proof. It will be sufficient if we could prove that $V^{s,p}(I)$ is complete. Let ψ_k, $k \in \mathbb{N}$ is a Cauchy sequence in $V^{s,p}(I)$. Then by Definition 6, the sequence $\mathscr{P}_\sigma^{-s}\psi_k$ is a Cauchy sequence in $L^p(I)$. Since $L^p(I)$ is complete, it implies that there exists a function $\psi \in L^p(I)$ such that

$$\mathscr{P}_\sigma^{-s}\psi_k \to \psi, \quad \text{as} \quad k \to \infty.$$

Let $\varphi = \mathscr{P}_\sigma^{s}\psi$ then by (31), we have $\mathscr{P}_\sigma^{-s}\varphi = \psi$. Hence $\mathscr{P}_\sigma^{-s}\varphi \in L^p(I)$ which implies that $\varphi \in V^{s,p}(I)$. Then $\psi_k \to \varphi$ in $V^{s,p}(I)$ as $k \to \infty$. From above fact we can conclude that $V^{s,p}(I)$ is complete. Hence $V^{s,p}(I)$ is Banach space. $\square$

THEOREM 9. *The MF-potential $\mathscr{P}_\sigma^{t}$ is an isometry of $V^{s,p}(I)$ onto $V^{s+t,p}(I)$ and we have*

$$\|\mathscr{P}_\sigma^{t}\psi\|_{V^{s+t,p}} = \|\psi\|_{V^{s,p}},$$

where $s,t \in \mathbb{R}$, $2 < p < \infty$.

Proof. Let $\psi \in V^{s,p}(I)$. Then by using (32) and (31), we have

$$\begin{aligned}
\|\mathscr{P}_\sigma^{t}\psi\|_{V^{s+t,p}} &= \|\mathscr{P}_\sigma^{-s-t}(\mathscr{P}_\sigma^{t}\psi)\|_{L^p(I)} \\
&= \|\mathscr{P}_\sigma^{-s}\psi\|_{L^p(I)} \\
&= \|\psi\|_{V^{s,p}}.
\end{aligned}$$

Now, let $\varphi \in V^{s+t,p}(I)$. Then again using (32) and (31), we have

$$\begin{aligned}
\|\varphi\|_{V^{s+t,p}} &= \|\mathscr{P}_\sigma^{-s-t}\varphi\|_{L^p(I)} \\
&= \|\mathscr{P}_\sigma^{-t}\varphi\|_{V^{s,p}}.
\end{aligned}$$

Thus for each $\varphi \in V^{s+t,p}(I)$, $\exists$ $\mathscr{P}_\sigma^{-t}\varphi \in V^{s,p}(I)$ such that $\mathscr{P}_\sigma^{t}\mathscr{P}_\sigma^{-t}\varphi = \varphi$. Hence $\mathscr{P}_\sigma^{t}$ is onto. $\square$

REMARK 3. For $l,s \in \mathbb{R}$ and $2 < p < \infty$, following consequences can be drawn easily:

(i) $\mathscr{P}_\sigma^{l-s}$ is an isometry of $V^{s,p}$ onto $V^{l,p}$,
(ii) $\mathscr{P}_\sigma^{-l}$ is an isometry of $V^{l,p}$ onto $V^{0,p}$,
(iii) $\mathscr{P}_\sigma^{s-l}$ is an isometry of $V^{0,p}$ onto $V^{s-l,p}$.

Concluding from above Remarks, we have $\mathscr{P}_\sigma^{s-l}\mathscr{P}_\sigma^{-l}\mathscr{P}_\sigma^{l-s}$ is an isometry of $V^{s,p}$ onto $V^{s-l,p}$, that is $\mathscr{P}_\sigma^{-l} : V^{s,p} \to V^{s-l,p}$ is an isometry of $V^{s,p}$ onto $V^{s-l,p}$. The mapping can be represented pictorially as:

$$\begin{tikzcd}
V^{s,p} \arrow[rr, "\mathscr{P}_\sigma^{-l}"] \arrow[d, "\mathscr{P}_\sigma^{l-s}"'] & & V^{s-l,p} \\
V^{l,p} \arrow[rr, "\mathscr{P}_\sigma^{-l}"'] & & V^{0,p} \arrow[u, "\mathscr{P}_\sigma^{s-l}"']
\end{tikzcd}$$

LEMMA 2. *The function space* $\mathscr{G}_\alpha(I)$, $\alpha > 0$, *is contained in* $L^p(I)$, $1 \leqslant p < \infty$.

Proof. Let us consider $\varphi \in \mathscr{G}_\alpha(I)$. Then from (21) and (22), we have

$$\begin{aligned}
\|\varphi\|_{L^p(I)} &= \int_1^\infty |\varphi(x)|^p dx \\
&\leqslant \Gamma_{\alpha,0}(\varphi)\left(\int_1^\infty \frac{1}{|\lambda_\alpha^+(x)|^p} dx\right)^{\frac{1}{p}} \\
&= \Gamma_{\alpha,0}(\varphi)\left(\int_1^2 e^{\frac{-\alpha p}{x-1}} dx + \int_2^\infty e^{-\alpha p(x-1)}\right)^{\frac{1}{p}},
\end{aligned}$$

the integral converges. Therefore

$$\|\varphi\|_{L^p(I)} \leqslant C\,\Gamma_{\alpha,0}(\varphi) < \infty,$$

where $C > 0$ is a constant. $\square$

THEOREM 10. *Let* $2 < p < \infty$ *and* $s > 2$. *Then for* $\psi \in L^1(I)$, *we have*

$$\|\mathscr{P}_\sigma^s \psi\|_{L^p(I)} \leqslant C'' \|\psi\|_{L^1(I)},$$

where $C'' > 0$ *is a constant.*

Proof. Let us assume

$$\left(\frac{1}{4} + \tau^2\right)^{-\frac{s}{2}} = (\mathfrak{M}\varphi)(\tau). \tag{33}$$

Thus by an application of inversion formula (2) and (16), we get

$$|\varphi(x)| \leqslant M P_{-\frac{1}{2}}(x) \int_0^\infty \tau \left(\frac{1}{4} + \tau^2\right)^{-\frac{s}{2}} d\tau,$$

the integral converges for $s>2$. Thus

$$\|\varphi\|_{L^p(I)} \leqslant C\|P_{-\frac{1}{2}}(x)\|_{L^p(I)} \leqslant C', \tag{34}$$

where $C'>0$ is a constant.

From (2), (10), (33) and (29), we have

$$(\varphi*\psi)(x)=\Big[\mathfrak{M}^{-1}\Big(\Big(\frac{1}{4}+(\cdot)^2\Big)^{-\frac{s}{2}}(\mathfrak{M}\psi)(\cdot)\Big)\Big](x)=(\mathscr{P}^s_\sigma\psi)(x). \tag{35}$$

Now using [23, Theorem 2.3], (34) and (35), we have

$$\|\mathscr{P}^s_\sigma\psi\|_{L^p(I)} \leqslant C''\ \|\psi\|_{L^1(I)},$$

where $C''>0$ is a constant. Hence proved. □

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Akhilesh Prasad
Department of Applied Mathematics
Indian Institute of Technology (Indian School of Mines)
Dhanbad-826004, India
e-mail: apr_bhu@yahoo.com

U. K. Mandal
Department of Mathematics
Nalanda College Biharsharif, Patliputra University
Nalanda-803101, India
e-mail: upainmandal@gmail.com

S. K. Verma
Department of Applied Mathematics
Indian Institute of Technology (Indian School of Mines)
Dhanbad-826004, India
e-mail: sandeep16.iitism@gmail.com