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UNITARY CONGRUENCES AND POSITIVE BLOCK–MATRICES

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Abstract. In this note we give some two by two block matrices $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ where M and $M' = \begin{pmatrix} A & X^* \\ X & B \end{pmatrix}$ are unitarily congruent. We also generalize a class of positive semi-definite block-matrices satisfying the inequality $\|M\| \leqslant \|A+B\|$ for all symmetric norms.

1. Introduction and preliminaries

Let $\mathbb{M}_n^+$ denote the positive semi-definite part of the space of $n \times n$ complex matrices. For 2×2 positive semi-definite block-matrix M, we say that M is P.S.D. or $M \geqslant 0$ and we write $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \in \mathbb{M}_{n+m}^+$, with $A \in \mathbb{M}_n^+$, $B \in \mathbb{M}_m^+$.

A positive partial transpose matrix denoted by P.P.T. is a P.S.D. block matrix $M \in \mathbb{M}_{2n}^+$ such that both $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ and $M' = \begin{pmatrix} A & X^* \\ X & B \end{pmatrix}$ (its partial transpose) are positive semi-definite. Let $Im(X) := \dfrac{X - X^*}{2i}$ respectively $Re(X) := \dfrac{X + X^*}{2}$ be the imaginary part respectively the real part of a matrix X. If $W(X)$ denotes the numerical range of X then $W(Re(X)) = \Re(W(X))$ and $W(Im(X)) = \Im(W(X))$ see [1].

It is well known that if $M \in \mathbb{M}_{n+m}^+$ with $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ then

$$\|M\| \leqslant \|A\| + \|B\| \tag{1.1}$$

for all symmetric norms (see [2]). In the sequel any block-matrix have blocks in $\mathbb{M}_n$ of equal sizes. The identity matrix of any order is denoted by I.

Noting that $VM'V^* = \begin{pmatrix} B & -X \\ -X^* & A \end{pmatrix}$ with $V = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ we have $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \geqslant 0$ is P.P.T. if and only if $M \leqslant \begin{pmatrix} A+B & 0 \\ 0 & A+B \end{pmatrix}$. As a direct consequence if $M' \geqslant 0$, $\|M\|_s \leqslant \|A+B\|_s$ for the spectral norm and if $A + B = kI$, $k > 0$ (see [5]) $M' \geqslant 0$ if and only if $\|M\|_s \leqslant k$.

For any matrix X, the width of $W(X)$ is the one of the smallest strip in the plan containing it, in [3] the following was proved

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THEOREM 1.1. [3] *Let* $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \geqslant 0$, *if* $\omega(X)$ *is the width of the numerical range of* X *then* $\|M\| \leqslant \|A+B+\omega(X)I\|$ *for all symmetric norms.*

Lemma 1.2 is noted from [6] (see also Theorems 3.6 and 3.8 -[4]):

LEMMA 1.2. *Let* $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \geqslant 0$, $X = UH$, U *is unitary and* H *hermitian if:*

1. U *commutes with* A *and* U *commutes with* H *or*
2. U *commutes with* B *and* U *commutes with* H *or*
3. U *commutes with* A *and* U *commutes with* B

then M *and* M' *are unitarily congruent and* $\|M\| \leqslant \|A+B\|$ *for all symmetric norms.*

Proof. For 1. take $Q = \begin{pmatrix} (U^*)^2 & 0 \\ 0 & I \end{pmatrix}$ so $M' = QMQ^*$. For 2. take $Q = \begin{pmatrix} I & 0 \\ 0 & U^2 \end{pmatrix}$ so $M' = QMQ^*$ and for 3. take $Q = \begin{pmatrix} U^* & 0 \\ 0 & U \end{pmatrix}$ so $M' = QMQ^*$; $Q = \begin{pmatrix} U^* & 0 \\ 0 & I \end{pmatrix}$ or $Q = \begin{pmatrix} I & 0 \\ 0 & U \end{pmatrix}$ gives $\|M\| = \|QMQ^*\| = \left\| \begin{pmatrix} A & H \\ H & B \end{pmatrix} \right\| \leqslant \|A+B\|$ for all symmetric norms from Theorem 1.1. $\square$

PROPOSITION 1.3. *Suppose* $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \in \mathbb{M}_4^+$, $e^{i\zeta}X = Re(e^{i\zeta}X) + iH$ *for some real* ζ *with* $Re(e^{i\zeta}X)$ *diagonal. If* A *and* B *are diagonal then* M *and* M' *are unitarily congruent.*

Proof. Calculating the characteristic polynomials of M and M' proves that they are equal. $\square$

This property seems not to hold for $\mathbb{M}_{2n}^+$ when $n > 2$ see for example [5].

2. Main results

The next lemma is Hiroshima's majorization see [7] and the references therein:

LEMMA 2.1. [7] *Let* $M = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \in \mathbb{M}_{2n}^+$ *be a positive partial transpose matrix then*

$$\|M\| \leqslant \|A+B\| \tag{2.1}$$

for all symmetric norms.

Before stating the main Theorem we need the following lemmas:

LEMMA 2.2. [2] *For every matrix in $\mathbb{M}_{2n}^{+}$ written in blocks of the same size, we have the decomposition:*

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} = U \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} V^*$$

for some unitaries $U,V \in \mathbb{M}_{2n}$.

LEMMA 2.3. *The following system admits solutions over real numbers for any $(\alpha, v) \in \mathbb{R}^2$ fixed:* $\begin{cases} \cos(\theta)^2 - v\cos(\alpha)(\sin(\theta)\cos(\theta)) = \frac{1}{2} \\ \sin(\theta)^2 + v\cos(\alpha)(\sin(\theta)\cos(\theta)) = \frac{1}{2} \end{cases}$

Proof. If $v\cos(\alpha) = 0$ then we can take $\theta = \frac{\pi}{4}$. Otherwise since $\cos(\theta)^2 = \dfrac{1+\cos(2\theta)}{2}$ and $\sin(\theta)^2 = \dfrac{1-\cos(2\theta)}{2}$, θ satisfies $\tan(2\theta) = \dfrac{1}{v\cos(\alpha)}$. □

THEOREM 2.4. *Let $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \geqslant 0$, if for some modulus 1 complex number z the numerical range of $2zX - (B-A)$ is contained in a line segment then $\|M\| \leqslant \|A+B\|$ for all symmetric norms. Furthermore if this line is on the imaginary axis, M and M' are unitarily congruent and M is P.P.T.*

Proof. Set $e^{i(\zeta-\alpha)}X = \dfrac{B-A}{2} + H$ for some ζ and α in this form and for $Q = \begin{pmatrix} e^{i\zeta}\cos(\theta)I & \sin(\theta)I \\ -e^{i\zeta}\sin(\theta)I & \cos(\theta)I \end{pmatrix}$ we get the following matrix

$$QMQ^* := \begin{pmatrix} A\cos(\theta)^2 + B\sin(\theta)^2 + \sin(2\theta)Re(e^{i\zeta}X) & (B-A)\frac{\sin(2\theta)}{2} + e^{i\zeta}\cos(\theta)^2X - e^{-i\zeta}\sin(\theta)^2X^* \\ (B-A)\frac{\sin(2\theta)}{2} + e^{-i\zeta}\cos(\theta)^2X^* - e^{i\zeta}\sin(\theta)^2X & A\sin(\theta)^2 + B\cos(\theta)^2 - \sin(2\theta)Re(e^{i\zeta}X) \end{pmatrix}$$

By Lemma 2.3 and Lemma 2.2 putting $e^{i\zeta}X = e^{i\alpha}\left(\dfrac{B-A}{2}\right) + e^{i\alpha}H$ we can choose θ such that

$$QMQ^* = U \begin{pmatrix} \frac{A+B}{2} + \sin(2\theta)N_{\zeta,\alpha} & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & \frac{A+B}{2} - \sin(2\theta)N_{\zeta,\alpha} \end{pmatrix} V^* \tag{2.2}$$

where $N_{\zeta,\alpha} := Re(e^{i\zeta}X - e^{i\alpha}(\frac{B-A}{2}))$ for some reals ζ, α. $N_{\zeta,\alpha} = rI$ for some scalar r if and only if $W(2zX - (B-A))$ is on a line segment with $z = e^{i(\zeta-\alpha)}$ and since the blocks in the decomposition orbits are positive semi-definite the proof follows by applying Ky-Fan dominance theorem ([1], Sec 10.7). If $Re(W(2zX - (B-A))) = 0$ i.e. $Re(e^{i\rho}X) = \frac{B-A}{2}$ then $QMQ^* = M'$ with the matrix $Q = \dfrac{1}{\sqrt{2}}\begin{pmatrix} -e^{i2\rho}I & e^{i\rho}I \\ e^{i\rho}I & I \end{pmatrix}$. □

We can construct matrices in $\mathbb{M}_{2n}^{+}$ that follows Theorem 2.4 conditions exclusively. Take any complex z of modulus 1 ($z = e^{i\alpha}$) different from ± 1 and $\pm i$ for a certain A, B and a triangular matrix X: $M = \begin{pmatrix} A & X+rI \\ X^*+rI & A+aI+bJ \end{pmatrix} \geqslant 0$, J is the matrix whose entries are all one and X is a triangular matrix whose all non zero entries are equal to $-b\Re(z)$.

EXAMPLE 2.5. Let $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ with $A = \frac{1}{10}\begin{pmatrix} 21 & 22 \\ 22 & 41 \end{pmatrix}$, $B = \frac{1}{10}\begin{pmatrix} 41 & 42 \\ 42 & 61 \end{pmatrix}$ and $z = e^{i\frac{\pi}{4}}$; $Re(z(\frac{B-A}{2})) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. For $X = \begin{pmatrix} 0 & 0 \\ -\sqrt{2} & 0 \end{pmatrix}$ we see that $M \geqslant 0$ is not a P.P.T. matrix and X is not normal with $Re(2X + z(B-A)) = \sqrt{2}I$.

Theorem 2.4 can be generalized as Theorem 2.1 in [3]:

COROLLARY 2.6. *Let* $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \geqslant 0$, z *a complex number of modulus one and* $\omega_{A,B}(X)$ *the width of* $W(X + z\frac{B-A}{2})$ *then* $\|M\| \leqslant \|A + B + \omega_{A,B}(X)I\|$ *for all symmetric norms.*

Proof. The proof is the same as given in Theorem 2.1 in [3] we consider $\delta := \omega(\sin(2\theta)X + z\sin(2\theta)\frac{B-A}{2}) \leqslant \omega_{A,B}(X)$.

$$rI \leqslant Re(e^{i\kappa}(\sin(2\theta)X + z\sin(2\theta)\,\frac{B-A}{2})) \leqslant (r+\delta)I$$

for some reals r and κ; from (2.2) we get

$$\|M\| \leqslant \|\frac{A+B}{2} + (r+\delta)I\| + \|\frac{A+B}{2} - rI\| = \|A + B + \delta I\|$$

for all symmetric norms. $\square$

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