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INEQUALITIES FOR CERTAIN POWERS OF POSITIVE DEFINITE MATRICES

FADI ALRIMAWI, OMAR HIRZALLAH AND FUAD KITTANEH*

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Abstract. Let A,B, and X be $n\times n$ matrices such that A,B are positive definite and X is Hermitian. If a and b are real numbers such that $0<a\leqslant s_n(A)$ and $0<b\leqslant s_n(B)$, then it is shown, among other inequalities, that

$$\left|\left|\left|A^bX+XB^a\right|\right|\right|\geqslant(1+\min(a^2,b^2))\,|||X|||$$

for every unitarily unitarily invariant norm.

1. Introduction

Let $\mathbb{M}_n(\mathbb{C})$ be the algebra of all $n\times n$ complex matrices. For a matrix $A\in\mathbb{M}_n(\mathbb{C})$, let $\lambda_1(A),\ldots,\lambda_n(A)$ be the eigenvalues of A repeated according to multiplicity. The singular values of A, denoted by $s_1(A),\ldots,s_n(A)$, are the eigenvalues of the positive semidefinite matrix $|A|=(A^*A)^{1/2}$ arranged in decreasing order and repeated according to multiplicity. A Hermitian matrix $A\in\mathbb{M}_n(\mathbb{C})$ is said to be positive semidefinite if $x^*Ax\geqslant 0$ for all $x\in\mathbb{C}^n$ and it is called positive definite if $x^*Ax>0$ for all $x\in\mathbb{C}^n$ with $x\neq 0$.

The spectral norm $||\cdot||$, the Schatten p-norm ($p\geqslant 1$), and the Ky Fan k-norms $||\cdot||_{(k)}$ $(k=1,\ldots,n)$ are, respectively, the norms defined on $\mathbb{M}_n(\mathbb{C})$ by $||A||=\max\{||Ax||:x\in\mathbb{C}^n,||x||=1\}$, $||A||_p=\left(\sum_{j=1}^n(s_j(A))^p\right)^{1/p}$, and $||A||_{(k)}=\sum_{j=1}^k s_j(A)$, $k=1,\ldots,n$. It is known that (see, e.g., [2, p. 76]) for every $A\in\mathbb{M}_n(\mathbb{C})$, we have

$$||A||=s_1(A) \tag{1.1}$$

and for each $k=1,\ldots,n$, we have

$$||A||_{(k)}=\max\left|\sum_{j=1}^k y_j^*Ax_j\right|, \tag{1.2}$$

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* Corresponding author.

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where the maximum is taken over all choices of orthonormal k-tuples $x_1,\ldots,x_k$ and $y_1,\ldots,y_k$. In fact, replacing each y_j by $z_j y_j$ for some suitable complex number z_j of modulus 1 for which $\bar{z}_j y_j^* A x_j = \left| y_j^* A x_j \right|$, implies that the k-tuple $z_1 y_1,\ldots,z_k y_k$ is still orthonormal, and so an identity equivalent the identity (1.2) can be seen as follows:

$$||A||_{(k)} = \max \sum_{j=1}^{k} \left| y_j^* A x_j \right|, \tag{1.3}$$

where the maximum is taken over all choices of orthonormal k-tuples $x_1,\ldots,x_k$ and $y_1,\ldots,y_k$.

A unitarily invariant norm $|||\cdot|||$ is a norm defined on $\mathbb{M}_n(\mathbb{C})$ that satisfies the invariance property $|||UAV||| = |||A|||$ for every $A \in \mathbb{M}_n(\mathbb{C})$ and every unitary matrices $U,V \in \mathbb{M}_n(\mathbb{C})$.

The direct sum of the matrices $A,B \in \mathbb{M}_n(\mathbb{C})$ is the matrix

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

It is known that

$$|||A \oplus A||| \geqslant |||B \oplus B||| \quad \text{for every unitarily invariant norm} \tag{1.4}$$

is equivalent to

$$|||A||| \geqslant |||B||| \quad \text{for every unitarily invariant norm.} \tag{1.5}$$

Also,

$$|||A \oplus B||| = |||A^* \oplus B||| = |||B \oplus A||| = \left|\left|\left| \begin{bmatrix} 0 & B \\ A^* & 0 \end{bmatrix} \right|\right|\right| \tag{1.6}$$

for every unitarily invariant norm. Typical examples of unitarily invariant norms are the spectral norm, the Schatten p-norms, and the Ky Fan k-norms. For further properties and examples of unitarily invariant norms, the reader is referred to [2], [4], or [5].

An elementary inequality (see, e.g., [3, p. 281]) for positive real numbers x,y, asserts that

$$x^y + y^x > 1. \tag{1.7}$$

It can be easily seen that the inequality (1.7) implies that if x and y are real numbers such that x is positive and y is nonnegative, then

$$x^y + y^x \geqslant 1 \tag{1.8}$$

with equality if and only if $y = 0$.

It has been shown in [1] that if x and y are two positive real numbers, then

$$x^x + y^y \geqslant 2e^{-e^{-1}}. \tag{1.9}$$

In this paper, we give further related inequalities for scalars and we extend some of them for matrices. In Section 2, we give a refinement of the inequality (1.7) and we introduce an inequality related to the inequality (1.9). In Section 3, we give matrix versions for our scalar inequalities related to the inequality (1.9).

2. Preliminary results

In this section, we give a refinement of the inequality (1.7) and we introduce an inequality related to the inequality (1.9). First, we start with the following lemma, which concerns Bernoulli's inequality (see, e.g., [3, p. 34]).

LEMMA 2.1. *Let x and y be real numbers, and assume that $x > -1$. Then the following statements hold:*

(a) If $y \geqslant 1$, then

$$(1+x)^y \geqslant 1+yx$$

with equality if and only if $x=0$ or $y=1$.

(b) If $0 \leqslant y \leqslant 1$, then

$$(1+x)^y \leqslant 1+yx$$

with equality if and only if $x=y=0$ or $y=1$.

The following theorem is our main result in this section.

THEOREM 2.2. *Let x and y be positive real numbers. Then*

$$x^y+y^x \geqslant 1+\min(x^2,y^2) \tag{2.1}$$

with equality if and only if $x=y=1$.

Proof. We prove the inequality (2.1) by dividing its proof into four cases:
Case 1: If $x,y \in [1,\infty)$, then $x=1+u$ and $y=1+v$ for some $u,v \in [0,\infty)$. Now

$$\begin{aligned}
x^y+y^x &= (1+u)^y+(1+v)^x \\
&\geqslant 1+yu+1+xv \qquad \text{(by Lemma 2.1(a))} \\
&= 2+u(1+v)+v(1+u) \\
&= 2+2uv+u+v \\
&\geqslant 1+(1+u)(1+v) \\
&= 1+xy \\
&\geqslant 1+\min(x^2,y^2).
\end{aligned}$$

Case 2: If $x,y \in (0,1]$, then $x = \frac{1}{1+u}$ and $y = \frac{1}{1+v}$ for some $u,v \in [0,\infty)$. Now

$$\begin{aligned}
x^y + y^x &= \frac{1}{(1+u)^y} + \frac{1}{(1+v)^x} \\
&\geqslant \frac{1}{1+yu} + \frac{1}{1+xv} \qquad \text{(by Lemma 2.1(b))} \\
&= \frac{1+v}{1+u+v} + \frac{1+u}{1+u+v} \\
&= 1 + \frac{1}{1+u+v} \\
&\geqslant 1 + \frac{1}{(1+u)(1+v)} \\
&= 1 + xy \\
&\geqslant 1 + \min(x^2, y^2).
\end{aligned}$$

Case 3: If $y \in (0,1)$ and $x \in [1,2]$, then it is clear that $x^y + y^x \geqslant 1 + y^2 = 1 + \min(x^2, y^2)$.

Case 4: If $y \in (0,1)$ and $x \in [2,\infty)$, then fix a number $y \in (0,1)$ and define a function f_y on the interval $[2,\infty)$ by $f_y(x) = x^y + y^x$. First, we show that this function is increasing on $[2,\infty)$, that is

$$f_y'(x) = yx^{y-1} + y^x \log y > 0 \text{ for } x > 2$$

which is equivalent to showing that

$$\frac{1}{-y\log y} e^{(y-1)\log x - (x-2)\log y} > 1 \text{ for } x > 2. \tag{2.2}$$

So, applying the logarithmic function to both sides of the inequality (2.2), our problem reduces to showing that

$$-\log(-y\log y) + (y-1)\log x - (x-2)\log y > 0 \text{ for } x > 2.$$

Define a function g_y on $[1,\infty)$ by

$$g_y(x) = (y-1)\log x - (x-2)\log y.$$

Then $g_y'(x) = \frac{y-1}{x} - \log y$ and $g_y''(x) = \frac{1-y}{x^2} > 0$, which implies that g_y is convex on $[1,\infty)$ and so its minimum value occurs at $x = \frac{y-1}{\log y}$. Hence, g_y is increasing on $\left[\frac{y-1}{\log y}, \infty\right)$. Since $\frac{y-1}{\log y} \in (0,1)$, it follows that g_y is increasing on $[2,\infty)$. Therefore, $g_y(x) \geqslant g_y(2) = (y-1)\log 2 > -\log 2 > -1$ for all $x \geqslant 2$, that is

$$(y-1)\log x - (x-2)\log y > -1 \text{ for } x \geqslant 2. \tag{2.3}$$

Adding $-\log(-y\log y)$ to both sides of the inequality (2.3), we have

$$\begin{aligned}
-\log(-y\log y) + (y-1)\log x - (x-2)\log y &> -\log(-y\log y) - 1 \\
&\geqslant 0.
\end{aligned}$$

This implies that $f_y'(x) > 0$ for $x > 2$, and so f_y is increasing on $[2,\infty)$. Consequently,

$$x^y + y^x = f_y(x) \geqslant f_y(2) = 2^y + y^2 > 1 + y^2 = 1 + \min(x^2, y^2),$$

which completes the proof of the inequality.

The equality conditions follow by direct computations and by the equality conditions of Lemma 2.1. $\square$

REMARK 2.3. It can be easily seen that the inequality (2.1) implies that if x and y are real numbers such that x is positive and y is nonnegative, then

$$x^y + y^x \geqslant 1 + \min(x^2, y^2) \tag{2.4}$$

with equality if and only if $x = y = 1$ or $y = 0$.

Based on Theorem 2.2, we have the following result.

COROLLARY 2.4. *Let x be positive a real number. Then $x^x \geqslant \frac{1+x^2}{2}$ with equality if and only if $x = 1$.*

An application of Corollary 2.4 can be seen in the following result related to Theorem 2.2.

COROLLARY 2.5. *Let x and y be positive real numbers. Then*

$$x^x + y^y \geqslant 1 + \frac{x^2 + y^2}{2} \tag{2.5}$$

with equality if and only if $x = y = 1$. Consequently,

$$x^x + y^y \geqslant 1 + xy \tag{2.6}$$

and

$$x^x + y^y \geqslant 1 + \min(x^2, y^2). \tag{2.7}$$

Proof. Corollary 2.4 implies that $x^x \geqslant \frac{1+x^2}{2}$ and $y^y \geqslant \frac{1+y^2}{2}$. The inequality (2.5) follows by adding these inequalities. The inequalities (2.6) and (2.7) follow from the inequality (2.5) and the fact that $\frac{x^2+y^2}{2} \geqslant xy \geqslant \min(x^2, y^2)$. $\square$

We conclude this section with the following remark.

REMARK 2.6. It should be mentioned here that if x and y are positive real numbers such that $\frac{x^2+y^2}{2} \geqslant 0.4$, then $1 + \frac{x^2+y^2}{2} > 2e^{-e^{-1}}$. So, in this case, the inequality (2.5) gives a better lower bound for $x^x + y^y$ than that given in the inequality (1.9).

3. Matrix versions of the inequalities (2.1) and (2.5)

In this section, we derive inequalities for matrices that present generalizations of the inequalities (2.1) and (2.5). Our results in this section can be considered as refinements of some results given in [1]. First, we need the following lemma (see, e.g., [2, p. 62]).

LEMMA 3.1. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ such that A and B are positive semidefinite. Then*

$$s_j(A+B) \geqslant s_k(A) + s_{j-k+n}(B)$$

for $j,k = 1,\ldots,n$ with $k \geqslant j$.

The following lemma is a direct consequence of the Weyl's Monotonicity Theorem (see, e.g., [2, p. 63]).

LEMMA 3.2. *Let $A, X \in \mathbb{M}_n(\mathbb{C})$ such that A is positive semidefinite. Then*

$$s_j(X^*AX) \geqslant s_j^2(X)\, s_n(A)$$

for $j = 1,\ldots,n$.

Based on Theorem 2.2 and Lemma 3.1, we have the following result. This result can be considered as a generalization of the inequality (2.1) in the setting of the singular values of matrices.

THEOREM 3.3. *Let $A,B,X,Y \in \mathbb{M}_n(\mathbb{C})$ such that A and B are positive definite. If a and b are real numbers such that $0 < a \leqslant s_n(A)$ and $0 < b \leqslant s_n(B)$, then*

$$s_j(X^*A^bX + Y^*B^aY) \geqslant \min(s_k^2(X), s_{j-k+n}^2(Y))(1 + \min(a^2, b^2)) \tag{3.1}$$

for $j,k = 1,\ldots,n$ with $k \geqslant j$.

Proof. Since A and B are positive definite, we have

$$\begin{aligned} s_j(X^*A^bX + Y^*B^aY) &\geqslant s_k(X^*A^bX) + s_{j-k+n}(Y^*B^aY) \text{ (by Lemma 3.1)} \\ &\geqslant s_k^2(X)s_n\left(A^b\right) + s_{j-k+n}^2(Y)s_n(B^a) \text{ (by Lemma 3.2)} \\ &\geqslant \min(s_k^2(X), s_{j-k+n}^2(Y))(s_n^b(A) + s_n^a(B)) \\ &\geqslant \min(s_k^2(X), s_{j-k+n}^2(Y))(a^b + b^a) \\ &\geqslant \min(s_k^2(X), s_{j-k+n}^2(Y))(1 + \min(a^2, b^2)) \\ &\qquad\text{(by Theorem 2.2)} \end{aligned}$$

for $j = 1,\ldots,n$ with $k \geqslant j$. $\square$

Applications of Theorem 3.3 can be seen in the following two results.

COROLLARY 3.4. *Let $A,B,X,Y \in \mathbb{M}_n(\mathbb{C})$ such that A and B are positive definite. If a and b are real numbers such that $0 < a \leqslant s_n(A)$ and $0 < b \leqslant s_n(B)$, then*

$$X^*A^bX + Y^*B^aY \geqslant \min(s_n^2(X), s_n^2(Y))(1+\min(a^2,b^2))I_n$$

and

$$A^b + B^a \geqslant (1+\min(a^2,b^2))I_n. \tag{3.2}$$

In particular,

$$A^{s_n(B)} + B^{s_n(A)} \geqslant (1+\min(s_n^2(A), s_n^2(B)))I_n$$

with equality if and only if $A = B = I_n$.

Proof. Since $X^*A^bX + Y^*B^aY$ is positive semidefinite, we have

$$\begin{aligned} X^*A^bX + Y^*B^aY &\geqslant s_n(X^*A^bX + Y^*B^aY)I_n \\ &\geqslant \min(s_n^2(X), s_n^2(Y))(1+\min(a^2,b^2)))I_n \\ &\qquad \text{(by the inequality (3.1)).} \quad \square \end{aligned}$$

COROLLARY 3.5. *Let $A,B,X,Y \in \mathbb{M}_n(\mathbb{C})$ such that A and B are positive definite. If a and b are real numbers such that $a \geqslant \|A\|$ and $b \geqslant \|B\|$, then*

$$s_j\left(X^*A^{-b^{-1}}X + Y^*B^{-a^{-1}}Y\right) \geqslant \min(s_k^2(X), s_{j-k+n}^2(Y))(1+\min(a^{-2},b^{-2}))$$

for $j,k = 1,\ldots,n$ with $k \geqslant j$,

$$X^*A^{-b^{-1}}X + Y^*B^{-a^{-1}}Y \geqslant \min(s_n^2(X), s_n^2(Y))(1+\min(a^{-2},b^{-2}))I_n,$$

and

$$A^{-b^{-1}} + B^{-a^{-1}} \geqslant (1+\min(a^{-2},b^{-2})I_n.$$

In particular,

$$A^{-\|B\|^{-1}} + B^{-\|A\|^{-1}} \geqslant (1+\min(\|A\|^{-2}, \|B\|^{-2})I_n$$

with equality if and only if $A = B = I_n$.

Proof. Since A and B are positive definite, the matrices A^{-1} and B^{-1} are positive definite. Also, the conditions $a \geqslant \|A\|$ and $b \geqslant \|B\|$ are equivalent to the conditions $0 < a^{-1} \leqslant s_n\left(A^{-1}\right)$ and $0 < b^{-1} \leqslant s_n\left(B^{-1}\right)$. So, the desired inequalities follow from Theorem 3.3 and Corollary 3.4 by replacing $A,B,a,$ and b by A^{-1}, B^{-1}, a^{-1}, and b^{-1}, respectively. $\square$

REMARK 3.6. In view of the proof of Theorem 3.3, a matrix version of the inequality (2.4) can be stated as follows: Let $A,B \in \mathbb{M}_n(\mathbb{C})$ such that A is positive definite and B is positive semidefinite. If $a,b \in \mathbb{R}$ with $0 < a \leqslant s_n(A)$ and $0 \leqslant b \leqslant s_n(B)$, then

$$A^b + B^a \geqslant (1+\min(a^2,b^2))I_n.$$

In particular,

$$A^{s_n(B)} + B^{s_n(A)} \geqslant (1+\min(s_n^2(A), s_n^2(B)))I_n$$

with equality if and only if $A = B = I_n$ or $B = 0$.

The following result presents a natural generalization of the inequality (2.1) in the setting of unitarily invariant norms.

THEOREM 3.7. *Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$ such that A, B are positive definite and X is Hermitian. If a and b are real numbers such that $0 < a \leqslant s_n(A)$ and $0 < b \leqslant s_n(B)$, then*

$$\left|\left|\left|A^b X + X B^a\right|\right|\right| \geqslant (1+\min(a^2, b^2))\, |||X||| \tag{3.3}$$

for every unitarily invariant norm.

Proof. Since X is Hermitian, it follows that there is an orthonormal basis $\{e_j\}$ of $\mathbb{C}^n$ consisting of eigenvectors corresponding to the eigenvalues $\{\lambda_j(X)\}$ arranged in such a way that $|\lambda_1(X)| \geqslant \cdots \geqslant |\lambda_n(X)|$. Since $s_j(X) = \left|\lambda_j(X)\right|$ for $j = 1, \ldots, n$, we have

$$\begin{aligned}
\left|\left|A^b X + X B^a\right|\right|_{(k)} &\geqslant \sum_{j=1}^{k} \left| e_j^* \left(A^b X + X B^a\right) e_j \right| \text{ (by the identity (1.3))} \\
&= \sum_{j=1}^{k} \left| e_j^* A^b X e_j + e_j^* X B^a e_j \right| \\
&= \sum_{j=1}^{k} \left| e_j^* A^b X e_j + (Xe_j)^* B^a e_j \right| \\
&= \sum_{j=1}^{k} \left| \lambda_j(X) e_j^* (A^b + B^a) e_j \right| \\
&= \sum_{j=1}^{k} |\lambda_j(X)| \ \left(e_j^* (A^b + B^a) e_j \right) \\
&= \sum_{j=1}^{k} s_j(X) \ \left(e_j^* (A^b + B^a) e_j \right) \\
&\geqslant (1+\min(a^2, b^2)) \sum_{j=1}^{k} s_j(X) \ \text{ (by the inequality (3.2))} \\
&= (1+\min(a^2, b^2))\, ||X||_{(k)}
\end{aligned}$$

for $k = 1, \ldots, n$. Now the inequality (3.3) follows by the Fan Dominance Theorem (see, e.g., [2, p. 93]). $\square$

An application of Theorem 3.7 can be seen in the following result in which the matrix X is not necessrarily Hermitian.

COROLLARY 3.8. *Let $A,B,X \in \mathbb{M}_n(\mathbb{C})$ such that A and B are positive definite. If a and b are real numbers such that $0 < a \leqslant s_n(A)$ and $0 < b \leqslant s_n(B)$, then*

$$\left|\left|\left|(A^bX + XB^a) \oplus (XA^b + B^aX)\right|\right|\right| \geqslant (1+\min(a^2,b^2))\,|||X \oplus X||| \tag{3.4}$$

for every unitarily invariant norm. In particular,

$$|||A^aX + XA^a||| \geqslant (1+a^2)\,|||X||| \tag{3.5}$$

for every unitarily invariant norm.

Proof. Let $\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \tilde{B} = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}$, and $\tilde{X} = \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix}$. Then $\tilde{A}$ and $\tilde{B}$ are positive definite and $\tilde{X}$ is Hermitian. Also, $0 < a \leqslant s_n\left(\tilde{A}\right)$ and $0 < b \leqslant s_n\left(\tilde{B}\right)$.

It follows, from Theorem 3.7, that

$$\begin{aligned}
&\left|\left|\left|(A^bX + XB^a) \oplus (XA^b + B^aX)\right|\right|\right| \\
&= \left|\left|\left|(XA^b + B^aX)^* \oplus (A^bX + XB^a)\right|\right|\right| \text{ (by the identities (1.6))} \\
&= \left|\left|\left|(A^bX^* + X^*B^a) \oplus (A^bX + XB^a)\right|\right|\right| \\
&= \left|\left|\left|\begin{bmatrix} A^bX^* + X^*B^a & 0 \\ 0 & A^bX + XB^a \end{bmatrix}\right|\right|\right| \\
&= \left|\left|\left|\begin{bmatrix} 0 & A^bX + XB^a \\ A^bX^* + X^*B^a & 0 \end{bmatrix}\right|\right|\right| \text{ (by the identities (1.6))} \\
&= \left|\left|\left|\tilde{A}^b\tilde{X} + \tilde{X}\tilde{B}^a\right|\right|\right| \\
&\geqslant (1+\min(a^2,b^2))\,|||\tilde{X}||| \text{ (by Theorem 3.7)} \\
&= (1+\min(a^2,b^2))\left|\left|\left|\begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix}\right|\right|\right| \\
&= (1+\min(a^2,b^2))\,|||X \oplus X|||,
\end{aligned}$$

which proves the inequality (3.4).

For the particular case, in the inequality (3.4), replacing B and b by A and a, respectively, we have

$$|||(A^aX + XA^a) \oplus (A^aX + XA^a)||| \geqslant (1+a^2)\,|||X \oplus X||| \tag{3.6}$$

for every unitarily invariant norm. So, the inequality (3.5) follows from the inequality (3.6) in view of the equivalence of the inequalities (1.4) and (1.5). $\square$

In the rest of this section, we give generalizations of the inequality (2.5). The proofs of our generalized results are similar to those given for the generalizations of the inequality (2.1). We start with the following generalization of the inequality (2.5) in the setting of the singular values of matrices.

THEOREM 3.9. *Let $A,B,X,Y \in \mathbb{M}_n(\mathbb{C})$ such that A and B are positive definite. If a and b are real numbers such that $0 < a \leqslant s_n(A)$ and $0 < b \leqslant s_n(B)$, then*

$$s_j(X^*A^aX + Y^*B^bY) \geqslant \min(s_k^2(X), s_{j-k+n}^2(Y))\left(1 + \frac{a^2+b^2}{2}\right)$$

for $j,k = 1,\dots,n$ with $k \geqslant j$ for $j = 1,\dots,n$.

Applications of Theorem 3.9 can be seen as follows.

COROLLARY 3.10. *Let $A,B,X,Y \in \mathbb{M}_n(\mathbb{C})$ such that A and B are positive definite. If a and b are real numbers such that $0 < a \leqslant s_n(A)$ and $0 < b \leqslant s_n(B)$, then*

$$X^*A^aX + Y^*B^bY \geqslant \min(s_n^2(X), s_n^2(Y))\left(1 + \frac{a^2+b^2}{2}\right) I_n$$

and

$$A^a + B^b \geqslant \left(1 + \frac{a^2+b^2}{2}\right) I_n.$$

In particular,

$$A^{s_n(A)} + B^{s_n(B)} \geqslant \left(1 + \frac{s_n^2(A)+s_n^2(B)}{2}\right) I_n$$

with equality if and only if $A = B = I_n$.

COROLLARY 3.11. *Let $A,B,X,Y \in \mathbb{M}_n(\mathbb{C})$ such that A and B are positive definite. If a and b are real numbers such that $a \geqslant \|A\|$ and $b \geqslant \|B\|$, then*

$$s_j\left(X^*A^{-a^{-1}}X + Y^*B^{-b^{-1}}Y\right) \geqslant \min(s_k^2(X), s_{j-k+n}^2(Y))\left(1 + \frac{a^{-2}+b^{-2}}{2}\right)$$

for $j,k = 1,\dots,n$ with $k \geqslant j$ and

$$X^*A^{-a^{-1}}X + Y^*B^{-b^{-1}}Y \geqslant \min(s_n^2(X), s_n^2(Y))\left(1 + \frac{a^{-2}+b^{-2}}{2}\right) I_n.$$

In particular,

$$A^{-\|A\|^{-1}} + B^{-\|B\|^{-1}} \geqslant \left(1 + \frac{\|A\|^{-2}+\|B\|^{-2}}{2}\right) I_n$$

with equality if and only if $A = B = I_n$.

Another generalization of the inequality (2.5) in the setting of unitarily invariant norms can be stated as follows.

THEOREM 3.12. *Let $A,B,X \in \mathbb{M}_n(\mathbb{C})$ such that A,B are positive definite and X is Hermitian. If a and b are real numbers such that $0 < a \leqslant s_n(A)$ and $0 < b \leqslant s_n(B)$, then*

$$\left|\left|\left|A^aX + XB^b\right|\right|\right| \geqslant \left(1 + \frac{a^2+b^2}{2}\right) |||X|||$$

for every unitarily invariant norm.

COROLLARY 3.13. *Let* $A,B,X \in \mathbb{M}_n(\mathbb{C})$ *such that* A *and* B *are positive definite. If* a *and* b *are real numbers such that* $0 < a \leqslant s_n(A)$ *and* $0 < b \leqslant s_n(B)$*, then*

$$\left|\left|\left|(A^aX+XB^b)\oplus(XA^a+B^bX)\right|\right|\right| \geqslant \left(1+\frac{a^2+b^2}{2}\right)|||X\oplus X|||$$

for every unitarily invariant norm.

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Fadi Alrimawi
Department of Basic Sciences
Al-Ahliyya Amman University
Amman, Jordan
e-mail: f.rimawi@ammanu.edu.jo

Omar Hirzallah
Department of Mathematics
The Hashemite University
Zarqa, Jordan
e-mail: o.hirzal@hu.edu.jo

Fuad Kittaneh
Department of Mathematics
The University of Jordan
Amman, Jordan
e-mail: fkitt@ju.edu.jo