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GENERALIZED WINTGEN INEQUALITY FOR LAGRANGIAN SUBMANIFOLDS IN QUATERNIONIC SPACE FORMS

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Abstract. In this paper, we obtain the generalized Wintgen inequality, also known as the DDVV inequality, in the case of a Lagrangian submanifold in quaternionic space forms. We also give a proof for the DDVV inequality in the case of slant submanifolds of a quaternionic space form.

1. Introduction

If $\tilde{M}$ is a $4m$-dimensional manifold with the Riemannian metric g, then $\tilde{M}$ is called *quaternion Kaehler manifold* if there exist a 3-dimensional vector bundle σ of type (1,1) with local basis of almost Hermitian structures J_1, J_2, J_3 such that

$$J_\alpha \circ J_{\alpha+1} = -J_{\alpha+1} \circ J_\alpha = J_{\alpha+2},\ J_\alpha^2 = -\mathrm{Id},$$

where $\alpha, \alpha+1, \alpha+2$ are taken modulo 3.

In this case, σ is called the *almost quaternionic structures on* $\tilde{M}$, $\{J_1, J_2, J_3\}$ is the *canonical local basis of* σ. So, $(\tilde{M}, \sigma)$ is called an *almost quaternionic manifold*, with $\dim \tilde{M} = 4m$, $m \geqslant 1$.

A Riemannian metric $\tilde{g}$ on $\tilde{M}$ is said to be *adapted to the almost quaternionic structure* σ if it satisfies

$$\tilde{g}(J_\alpha X, J_\alpha Y) = \tilde{g}(X, Y),\ \forall \alpha = \overline{1,3}.$$

$(\tilde{M}, \sigma, \tilde{g})$ is called *almost quaternionic Hermitian manifold*.

If σ is parallel with respect to $\tilde{\nabla}$ of $\tilde{g}$, then $(\tilde{M}, \sigma, \tilde{g})$ is called *quaternionic Kaehler manifold*. Equivalently, locally defined 1-forms $\omega_1, \omega_2, \omega_3$ exist such that $\forall \alpha = \overline{1,3}$, $\tilde{\nabla}_X J_\alpha = \omega_{\alpha+2}(X) J_{\alpha+1} - \omega_{\alpha+1} J_{\alpha+2}$, where $\alpha, \alpha+1, \alpha+2$ are taken modulo 3.

REMARK 1. Any quaternionic Kaehler manifold is an Einstein manifold ($\dim \tilde{M} \geqslant 4$).

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Let $(\tilde{M},\sigma,\tilde{g})$ be a quaternionic Kaehler manifold and X be a non-null vector on $\tilde{M}$. Then the 4-plane spanned by $\{X,J_1X,J_2X,J_3X\}$ denoted by $Q(X)$ is called a *quaternionic 4-plane*. Any 2-plane in $Q(X)$ is called a *quaternionic plane*. The sectional curvature of a quaternionic plane is called a *quaternionic sectional curvature*.

A quaternionic Kaehler manifold is called a *quaternionic space form* if its quaternionic sectional curvature is constant, say c. So, $(\tilde{M},\sigma,\tilde{g})$ is a quaternionic space form if and only if

$$\tilde{R}(X,Y)Z = \frac{c}{4}\Big\{\tilde{g}(Y,Z)X - \tilde{g}(X,Z)Y + \sum_{\alpha=1}^{3}[\tilde{g}(Z,J_\alpha Y)J_\alpha X - \tilde{g}(Z,J_\alpha X)J_\alpha Y + 2\tilde{g}(X,J_\alpha Y)J_\alpha Z]\Big\},$$

$\forall X,Y,Z \in \Gamma(T\tilde{M})$.

For a submanifold M of $\tilde{M}$, if $\{e_1,\ldots,e_n\}$ is an orthonormal basis of T_pM, $\{e_{n+1},\ldots,e_{4m}\}$ an orthonormal basis of $T_p^\perp M$, $p \in M$, the *mean curvature vector* is given by

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i,e_i).$$

We denote by

$$h_{ij}^r = g(h(e_i,e_j),e_r),\ i,j = \overline{1,n},\ r = \overline{n+1,4m}.$$

$$\|h\|^2(p) = \sum_{i,j=1}^{n} g(h(e_i,e_j),h(e_i,e_j)).$$

For a quaternionic Kaehler manifold, we have

$$\tilde{\nabla}_X J_\alpha = \sum_{\beta=1}^{3} Q_{\alpha\beta}(X)J_\beta,\ \alpha = \overline{1,3},\ \forall X \in \Gamma(T\tilde{M}),$$

where $Q_{\alpha\beta}$ are certain 1-forms locally defined on $\tilde{M}$ such that $Q_{\alpha\beta}+Q_{\beta\alpha}=0$.

A submanifold M of a quaternionic Kähler manifold $(\tilde{M},\sigma,\tilde{g})$ is said to be a *slant submanifold* if for each non-zero vector X tangent to M at p, the angle $\theta(X)$ between $J_\alpha(X)$ and T_pM, $\alpha \in \{1,2,3\}$ is constant, i.e. it does not depend on the choice of $p \in M$ and $X \in T_pM$. In this case, the slant submanifolds with $\theta = 0$ are called *quaternionic submanifolds* and those with $\theta = \frac{\pi}{2}$ are called *totally real submanifolds*. A slant submanifold of a quaternionic Kähler manifold is said to be *proper* (or θ*-slant proper*) if it is neither quaternionic nor totally real. An n-dimensional totally real submanifold of a quaternionic space form $\tilde{M}^{4m}(c)$ is said to be a *Lagrangian submanifold* if $n = m$.

2. Wintgen inequality

In 1979, P. Wintgen [9] proved that the Gauss curvature K, the squared mean curvature $\|H\|^2$ and the normal curvature $K^\perp$ of any surface M^2 in E^4 satisfy the inequality

$$K \leqslant \|H\|^2 - K^\perp.$$

The equality holds if and only if the ellipse of curvature of M^2 in E^4 is a circle.

An extension of the Wintgen inequality was given later by B. Rouxel [7] and by I. V. Guadalupe and L. Rodriguez [2] independently for surfaces M^2 of arbitrary codimension m in real space forms $\tilde{M}^{2+m}(c)$

$$K \leqslant \|H\|^2 - K^{\perp} + c.$$

In 2004, A. Mihai [5] found a corresponding inequality for totally real surfaces in n-dimensional complex space forms. Also, the equality case was studied and the author gived a non-trivial example of a totally real surface satisfying the equality case.

The conjecture of Wintgen inequality which is also known as the *DDVV conjecture* was formulated in 1999 by P. J. De Smet, F. Dillen, L. Verstraelen and L. Vrancken [8].

CONJECTURE. *Let $f : M^n \to \tilde{M}^{n+m}$ be an isometric immersion, where $\tilde{M}^{n+m}$ is a real space form of constant sectional curvature c. Then*

$$\rho \leqslant \|H\|^2 - \rho^{\perp} + c,$$

where ρ is the normalized scalar curvature and $\rho^{\perp}$ is the normalized normal scalar curvature.

Denoting by K and $R^{\perp}$ the sectional curvature function and the normal curvature tensor on M^n, respectively, the normalized scalar curvature and the normalized normal scalar curvature are given by

$$\rho = \frac{2\tau}{n(n-1)} = \frac{2}{n(n-1)} \sum_{1\leqslant i<j\leqslant n} K(e_i \wedge e_j),$$

$$\rho^{\perp} = \frac{2\tau^{\perp}}{n(n-1)} = \frac{2}{n(n-1)} \sqrt{\sum_{1\leqslant i<j\leqslant n}\sum_{1\leqslant \alpha<\beta\leqslant n} \left(R^{\perp}(e_i, e_j, \xi_\alpha, \xi_\beta)\right)^2},$$

where τ is the scalar curvature.

This conjecture was proven by the authors for submanifolds M^n of arbitrary dimension $n \geqslant 2$ and codimension 2 in real space forms $\tilde{M}^{n+2}(c)$ of constant sectional curvature c and a detailed characterization of the equality case in terms of the shape operators of M^n in $\tilde{M}^{n+2}(c)$ was given.

T. Choi and Z. Lu [1] proved that this conjecture is true for all 3-dimensional submanifolds M^3 of arbitrary codimension $m \geqslant 2$ in $\tilde{M}^{3+m}(c)$ and give also a characterization for the equality case.

Other extensions of the Wintgen inequality for invariant submanifolds in Kähler, nearly Kähler and Sasakian spaces have been studied by P. J. De Smet, F. Dillen, J. Fastenakels, A. Mihai, J. Van der Veken, L. Verstraelen and L. Vrancken.

Recently, Z. Lu and independently J. Ge and Z. Tang finally settled the general case of the DDVV-conjecture.

THEOREM 1. *The Wintgen inequality*

$$\rho \leqslant \|H\|^2 - \rho^{\perp} + c,$$

holds for every submanifold M^n *in any real space form* $\tilde{M}^{n+m}(c)$, $n \geqslant 2$, $m \geqslant 2$.

The equality case holds identically if and only if, with respect to suitable orthonormal frames $\{e_i\}$ *and* $\{\xi_\alpha\}$, *the shape operators of* M^n *in* $\tilde{M}^{n+m}(c)$ *take the forms*

$$A_{\xi_1} = \begin{pmatrix} \lambda_1 & \mu & 0 & \dots & 0 \\ \mu & \lambda_1 & 0 & \dots & 0 \\ 0 & 0 & \lambda_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_1 \end{pmatrix},$$

$$A_{\xi_2} = \begin{pmatrix} \lambda_2+\mu & 0 & 0 & \dots & 0 \\ 0 & \lambda_2-\mu & 0 & \dots & 0 \\ 0 & 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_2 \end{pmatrix},$$

$$A_{\xi_3} = \begin{pmatrix} \lambda_3 & 0 & 0 & \dots & 0 \\ 0 & \lambda_3 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_3 \end{pmatrix},$$

$A_{\xi_4} = \ldots = A_{\xi_m} = 0$, *where* $\lambda_1, \lambda_2, \lambda_3$ *and* μ *are real functions on* M^n.

Submanifolds satisfying the equality in the Wintgen inequality are called *Wintgen ideal submanifolds* [3].

3. Generalized Wintgen inequality for Lagrangian submanifolds in quaternionic space forms

I. Mihai [6] proved the following generalized Wintgen inequality for Lagrangian submanifolds in complex space forms.

THEOREM 2. *Let* M^n *a Lagrangian submanifold in a complex space form* $\tilde{M}^m(4c)$. *Then*

$$(\rho^\perp)^2 \leqslant (\|H\|^2 - \rho + c)^2 + \frac{4}{n(n-1)}(\rho - c)c + \frac{2c^2}{n(n-1)}.$$

In this paper we prove a similar inequality for Lagrangian submanifolds of quaternionic space forms.

Let M^n be an n-dimensional totally real submanifold of an $4m$-dimensional quaternionic space form $\tilde{M}^{4m}(4c)$ and $\{e_1, \ldots, e_n\}$ an orthonormal frame on M^n and $\{\xi_{n+1}, \ldots, \xi_{4m}\}$ an orthonormal frame in the normal bundle $T^\perp M^n$, respectively.

The *scalar normal curvature* of M^n is defined by

$$K_N = \frac{1}{4}\sum_{r,s=n+1}^{4m} \mathrm{Trace}[A_r, A_s]^2. \tag{1}$$

Then the normalized scalar normal curvature is given by $\rho_N = \frac{2\sqrt{K_N}}{n(n-1)}$.
From (1) we get

$$K_N = \frac{1}{2}\sum_{1\leqslant r<s\leqslant 4m-n} \mathrm{Trace}[A_r, A_s]^2 = \sum_{1\leqslant r<s\leqslant 4m-n}\sum_{1\leqslant i<j\leqslant n} (g([A_r, A_s]e_i, e_j))^2 \tag{2}$$

Denoting by $h^r_{ij} = g(h(e_i, e_j), \xi_r)$, $i, j = \overline{1,n}$, $r = \overline{1, 4m-n}$, we have

$$K_N = \sum_{1\leqslant r<s\leqslant 4m-n}\sum_{1\leqslant i<j\leqslant n}\left(\sum_{k=1}^{n}(h^r_{jk}h^s_{ik} - h^r_{ik}h^s_{jk})\right)^2. \tag{3}$$

The main result of this section is the following

THEOREM 3. (Main) *Let M^n be a Lagrangian submanifold of a quaternionic space form $\tilde{M}^{4n}(4c)$. Then*

$$\left(\rho^{\perp}\right)^2 \leqslant \left(\|H\|^2 - \rho + c\right)^2 + \frac{6}{n(n-1)}c^2 + \frac{4}{n(n-1)}c(\rho - c). \tag{4}$$

First we prove the following lemma

LEMMA 1. *Let M^n be a totally real submanifold of an $4m$-dimensional quaternionic space form $\tilde{M}^{4m}(4c)$. Then we have*

$$\|H\|^2 - \rho_N \geqslant \rho - c. \tag{5}$$

The equality case holds identically if and only if, with respect to suitable orthonormal frames $\{e_i\}$ and $\{\xi_\alpha\}$, the shape operators of M^n in $\tilde{M}^{4m}(4c)$ take the forms

$$A_{\xi_1} = \begin{pmatrix} \lambda_1 & \mu & 0 & \dots & 0 \\ \mu & \lambda_1 & 0 & \dots & 0 \\ 0 & 0 & \lambda_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_1 \end{pmatrix},$$

$$A_{\xi_2} = \begin{pmatrix} \lambda_2+\mu & 0 & 0 & \dots & 0 \\ 0 & \lambda_2-\mu & 0 & \dots & 0 \\ 0 & 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_2 \end{pmatrix},$$

$$A_{\xi_3} = \begin{pmatrix} \lambda_3 & 0 & 0 & \dots & 0 \\ 0 & \lambda_3 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_3 \end{pmatrix},$$

$A_{\xi_4} = \ldots = A_{\xi_{4m-n}} = 0$, *where* $\lambda_1, \lambda_2, \lambda_3$ *and* μ *are real functions on* M^n.

Proof of Lemma (1). First, we have

$$n^2\|H\|^2 = \sum_{r=1}^{4m-n}\left(\sum_{i=1}^{n} h_{ii}^r\right)^2 \tag{6}$$
$$= \frac{1}{n-1}\sum_{r=1}^{4m-n}\sum_{1\leqslant i<j\leqslant n}(h_{ii}^r - h_{jj}^r)^2 + \frac{2n}{n-1}\sum_{r=1}^{4m-n}\sum_{1\leqslant i<j\leqslant n} h_{ii}^r h_{jj}^r.$$

Using the following inequality [4]

$$\sum_{r=1}^{4m-n}\sum_{1\leqslant i<j\leqslant n}(h_{ii}^r - h_{jj}^r)^2 + 2n\sum_{r=1}^{4m-n}\sum_{1\leqslant i<j\leqslant n}(h_{ij}^r)^2 \tag{7}$$

$$\geqslant 2n\left[\sum_{1\leqslant r<s\leqslant 4m-n}\sum_{1\leqslant i<j\leqslant n}\left(\sum_{k=1}^{n}(h_{jk}^r h_{ik}^s - h_{ik}^r h_{jk}^s)\right)^2\right]^{\frac{1}{2}}.$$

Using the relations (6), (7) and (2), we obtain

$$\sum_{r=1}^{4m-n}\sum_{1\leqslant i<j\leqslant n}(h_{ii}^r - h_{jj}^r)^2 + 2n\sum_{r=1}^{4m-n}\sum_{1\leqslant i<j\leqslant n}(h_{ij}^r)^2 \geqslant 2n\sqrt{K_N} = n^2(n-1)\rho_N.$$

From this and using the relations (3) and (6) we have

$$\begin{aligned} n^2\|H\|^2 - n^2\rho_N &= \frac{1}{n-1}\sum_{r=1}^{4m-n}\sum_{1\leqslant i<j\leqslant n}(h_{ii}^r - h_{jj}^r)^2 + \frac{2n}{n-1}\sum_{r=1}^{4m-n}\sum_{1\leqslant i<j\leqslant n} h_{ii}^r h_{jj}^r \\ &\quad - \frac{2n}{n-1}\left[\sum_{1\leqslant r<s\leqslant 4m-n}\sum_{1\leqslant i<j\leqslant n}\left(\sum_{k=1}^{n}(h_{jk}^r h_{ik}^s h_{ik}^r h_{jk}^s)\right)^2\right]^{\frac{1}{2}} \\ &= \frac{1}{n-1}\sum_{r=1}^{4m-n}\sum_{1\leqslant i<j\leqslant n}(h_{ii}^r - h_{jj}^r)^2 + \frac{2n}{n-1}\sum_{r=1}^{4m-n}\sum_{1\leqslant i<j\leqslant n} h_{ii}^r h_{jj}^r - \frac{2n}{n-1}\sqrt{K_N} \\ &= \frac{1}{n-1}\left(\sum_{r=1}^{4m-n}\sum_{1\leqslant i<j\leqslant n}(h_{ii}^r - h_{jj}^r)^2 + 2n\sum_{r=1}^{4m-n}\sum_{1\leqslant i<j\leqslant n} h_{ii}^r h_{jj}^r - 2n\sqrt{K_N}\right) \\ &\geqslant \left[2n\sqrt{K_N} - 2n\sum_{r=1}^{4m-n}\left(h_{ij}^r\right)^2 + 2n\sum_{r=1}^{4m-n}\sum_{1\leqslant i<j\leqslant n} h_{ii}^r h_{jj}^r - 2n\sqrt{K_N}\right]. \end{aligned}$$

This gives us the relation

$$n^2\|H\|^2 - n^2\rho_N \geqslant \frac{2n}{n-1}\sum_{r=1}^{4m-n}\sum_{1\leqslant i<j\leqslant n}[h_{ii}^r h_{jj}^r - (h_{ij}^r)^2]. \tag{8}$$

From the Gauss equation we have

$$\tau = \frac{n(n-1)}{2}c + \sum_{r=1}^{4m-n}\sum_{1\leqslant i<j\leqslant n}[h_{ii}^r h_{jj}^r - (h_{ij}^r)^2]. \tag{9}$$

From the relations (8) and (9) we get

$$n^2\|H\|^2 - n^2\rho_N \geqslant \frac{2n}{n-1}\left[\tau - \frac{n(n-1)}{2}c\right].$$

This implies

$$n^2\|H\|^2 - n^2\rho_N \geqslant \frac{2n\tau}{n-1} - n^2 c,$$

so

$$\|H\|^2 - \rho_N \geqslant \frac{2\tau}{n(n-1)} - c,$$

which gives us

$$\|H\|^2 - \rho_N \geqslant \rho - c. \quad \square$$

Proof of Theorem (3). Choosing $\{e_1,\dots,e_n\}$ an orthonormal frame of the Lagrangian submanifold M^n, then $\{\xi_1 = J_1e_1,\dots,\xi_n = J_1e_n; \xi_{n+1} = J_2e_1,\dots,\xi_{2n} = J_2e_n; \xi_{2n+1} = J_3e_2,\dots,\xi_{3n} = J_3e_n; \xi_{3n+1},\dots,\xi_{4m}\}$ become an orthonormal frame of $T^\perp M^n$.

For $X = e_i, Y = e_j$ in T_pM and $i,j,k,l = \overline{1,n}$, $\beta,\gamma \in \{1,2,3\}$ we have

$$\tilde{R}(e_i,e_j,J_\beta e_k,J_\beta e_l) = c\sum_{\alpha=1}^{3}\left[g(J_\alpha e_i,J_\gamma e_l)g(J_\alpha e_j,J_\beta e_k) - g(J_\alpha e_i,J_\beta e_k)g(J_\alpha e_j,J_\gamma e_l)\right]. \tag{10}$$

This gives us

$$\tilde{R}(e_i,e_j,J_\beta e_k,J_\beta e_l) = c\sum_{\alpha=1}^{3}\left[\delta_{\alpha\gamma}\delta_{il}\delta_{\alpha\beta}\delta_{jk} - \delta_{\alpha\beta}\delta_{ik}\delta_{\alpha\gamma}\delta_{jl}\right], \tag{11}$$

from which we get

$$\tilde{R}(e_i,e_j,J_\beta e_k,J_\beta e_l) = c\sum_{\alpha=1}^{3}\delta_{\alpha\beta}\delta_{\alpha\gamma}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}). \tag{12}$$

From this relation and the Ricci equation, we have

$$g(R^\perp(e_i,e_j)J_\beta e_k,J_\gamma e_l) = c\sum_{\alpha=1}^{3}\delta_{\alpha\beta}\delta_{\alpha\gamma}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) + g([A_r,A_s]e_i,e_j), \tag{13}$$

where $A_r = A_{J_\beta e_k}$, $A_s = A_{J_\gamma e_l}$.

This relation gives us

$$g(R^\perp(e_i,e_j)J_\beta e_k, J_\gamma e_l) = c\delta_{\beta\gamma}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) + g([A_r,A_s]e_i,e_j). \tag{14}$$

From this, we get

$$\begin{aligned}\left(\tau^\perp\right)^2 &= \sum_{\beta,\gamma=1}^{3}\sum_{1\leqslant k<l\leqslant n}\sum_{1\leqslant i<j\leqslant n} g^2(R^\perp(e_i,e_j)J_\beta e_k, J_\gamma e_l) \\ &= \sum_{\beta,\gamma=1}^{3}\sum_{1\leqslant k<l\leqslant n}\sum_{1\leqslant i<j\leqslant n}\left[c\delta_{\beta\gamma}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) + g([A_{J_\beta e_k},A_{J_\gamma e_l}]e_i,e_j)\right]^2 \\ &= K_N + c^2\sum_{\beta,\gamma=1}^{3}\sum_{1\leqslant k<l\leqslant n}\sum_{1\leqslant i<j\leqslant n}\delta_{\beta\gamma}^2(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl})^2 \\ &\quad -2c\sum_{\beta,\gamma=1}^{3}\sum_{1\leqslant k<l\leqslant n}\sum_{1\leqslant i<j\leqslant n}\delta_{\beta\gamma}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl})g([A_{J_\beta e_k},A_{J_\gamma e_l}]e_i,e_j).\end{aligned} \tag{15}$$

From (15), we obtain

$$\left(\tau^\perp\right)^2 = \frac{n^2(n-1)^2}{4}\rho_N^2 + \frac{3n(n-1)}{2}c^2 - c\|h\|^2 + cn^2\|H\|^2. \tag{16}$$

From (9), we have

$$2\tau = n(n-1)c + n^2\|H\|^2 - \|h\|^2,$$

therefore

$$n^2\|H\|^2 - \|h\|^2 = 2\tau - n(n-1)c,$$

which implies

$$n^2\|H\|^2 - \|h\|^2 = n(n-1)\left(\rho - c\right). \tag{17}$$

Using the relations (16) and (17), we get

$$\left(\tau^\perp\right)^2 = \frac{n^2(n-1)^2}{4}\rho_N^2 + \frac{3n(n-1)}{2}c^2 + cn(n-1)(\rho - c). \tag{18}$$

The relation (18) implies that

$$\frac{n^2(n-1)^2}{4}\left(\rho^\perp\right)^2 = \frac{n^2(n-1)^2}{4}\rho_N^2 + \frac{3n(n-1)c^2}{2} + n(n-1)c\left(\rho - c\right).$$

This gives us

$$\left(\rho^\perp\right)^2 = \rho_N^2 + \frac{4}{n(n-1)}c\left(\rho - c\right) + \frac{6}{n(n-1)}c^2. \tag{19}$$

Using lemma (1), we get

$$\rho_N \leqslant \|H\|^2 - \rho + c. \tag{20}$$

Thus, from (19) and (20), we obtain

$$\left(\rho^{\perp}\right)^2 \leqslant \left(\|H\|^2 - \rho + c\right)^2 + \frac{4}{n(n-1)} c\,(\rho - c) + \frac{6}{n(n-1)} c^2, \tag{21}$$

which implies the relation (4). □

4. Slant submanifolds in quaternionic space forms

We recall the definition of a slant submanifold in a quaternionic space form.

DEFINITION 1. A submanifold M of a quaternionic Kähler manifold $(\tilde{M}, \sigma, \tilde{g})$ is said to be a *slant submanifold* if for each non-zero vector X tangent to M at p, the angle $\theta(X)$ between $J_\alpha(X)$ and T_pM, $\alpha \in \{1,2,3\}$ is constant, i.e. it does not depend on the choice of $p \in M$ and $X \in T_pM$.

THEOREM 4. *Let M^n be an n-dimensional θ-slant submanifold of an $4m$-dimensional quaternionic space form $\tilde{M}^{4m}(4c)$. Then, we have*

$$\|H\|^2 \geqslant \rho + \rho_N - c - \frac{9c}{n-1} \cos^2 \theta. \tag{22}$$

Proof. Let M^n be a θ-slant submanifold of a quaternionic space form $\tilde{M}^{4m}(4c)$, $\{e_1, \ldots, e_n\}$ an orthonormal basis for T_pM, $p \in M$ and $\{\xi_1, \ldots, \xi_{4m-n}\}$ an orthonormal basis for $T_p^{\perp}M^n$.

The Gauss equation is

$$\begin{aligned} &R(X,Y,Z,W) \\ &= c\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W) \\ &\quad + \sum_{\alpha=1}^{3} [g(P_\alpha Y,Z)g(P_\alpha X,W) - g(P_\alpha X,Z)g(P_\alpha Y,W) + 2g(X,P_\alpha Y)g(Z,P_\alpha W)] \\ &\quad + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)) \end{aligned} \tag{23}$$

$\forall X,Y,Z,W \in \Gamma(TM^n)$.

The Ricci equation is given by the relation

$$\begin{aligned} &R^{\perp}(X,Y,\xi,\eta) \\ &= c\left[\sum_{\alpha=1}^{3} g(J_\alpha X,\eta)g(J_\alpha Y,\xi) - g(J_\alpha X,\xi)g(J_\alpha Y,\eta) + 2g(J_\alpha X,Y)g(J_\alpha \xi,\eta)\right] \\ &\quad - g([A_\xi, A_\eta]X,Y),\ \forall X,Y \in \Gamma(TM^n),\ \xi,\eta \in \Gamma(T^{\perp}M^n). \end{aligned} \tag{24}$$

In the same way as in the proof of (1), we find

$$n^2\|H\|^2 - n^2\rho_N \geqslant \frac{2n}{n-1}\sum_{r=1}^{4m-n}\sum_{1\leqslant i<j\leqslant n}\left[h_{ii}^r h_{ij}^r - (h_{ij}^r)^2\right]. \tag{25}$$

From the relation (23), taking $X = Z = e_i$ and $Y = W = e_j$, we obtain

$$\sum_{1\leqslant i<j\leqslant n}\tilde{R}(e_i,e_j,e_i,e_j) = c\left[\cdot\frac{n(n-1)}{2} + \frac{3}{2}\sum_{\beta=1}^{3}\sum_{i,j=1}^{n} g^2(P_\beta e_i,e_j)\right], \tag{26}$$

where, for each $X \in T_pM$, $p \in M$, we have $J_\alpha X = P_\alpha X + F_\alpha X$, P_α and F_α being the tangential and the normal projections of J_α on T_pM, respectively on $T_p^\perp M$, $\alpha \in \{1,2,3\}$.

For $p \in M$, we choose an orthonormal basis $\{e_1,\dots,e_n\}$ of T_pM, such that $e_2 = \frac{1}{\cos\theta}P_1e_1$, $e_3 = \frac{1}{\cos\theta}P_2e_1$, $e_4 = \frac{1}{\cos\theta}P_3e_1,\dots,e_{4k-3}$, $e_{4k-2} = \frac{1}{\cos\theta}P_1e_{4k-3}$, $e_{4k-1} = \frac{1}{\cos\theta}P_2e_{4k-3}$, $e_{4k} = \frac{1}{\cos\theta}P_3e_{4k-3}$, with $4k = n$ and using the fact that

$$g^2(P_\beta e_i,e_{i+1}) = g^2(P_\beta e_{i+1},e_i) = \cos^2\theta, \tag{27}$$

for $i = 1,5,9,\dots,4k-3$ and

$$g(P_\beta e_i,e_j) = 0, \tag{28}$$

for (i,j) not in the cases mentioned above.

Thus, we get

$$\tau = c\left[\frac{n(n-1)}{2} + \frac{9n}{2}\cos^2\theta\right] + \sum_{r=1}^{4m-n}\sum_{1\leqslant i<j\leqslant n}\left[h_{ii}^r h_{ij}^r - (h_{ij}^r)^2\right]. \tag{29}$$

From (25) and (29) we get

$$n^2\|H\|^2 - n^2\rho_N \geqslant \frac{2n}{n-1}\left\{\tau - c\left[\frac{n(n-1)}{2} + \frac{9n}{2}\cos^2\theta\right]\right\}.$$

This gives us

$$\|H\|^2 - \rho_N \geqslant \frac{2}{n(n-1)}\left\{\tau - c\left[\frac{n(n-1)}{2} + \frac{9n}{2}\cos^2\theta\right]\right\},$$

thus

$$\|H\|^2 - \rho_N \geqslant \rho - c - \frac{9c}{n-1}\cos^2\theta.$$

This implies

$$\|H\|^2 \geqslant \rho_N + \rho - c - \frac{9c}{n-1}\cos^2\theta. \quad \square$$

REMARK 2. The result obtained in (4) taking M as a totally real submanifold is identic with (5) from lemma (1).

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