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REMARKS ON TWO DETERMINANTAL INEQUALITIES

TIN-YAU TAM AND PINGPING ZHANG*

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Abstract. Denote by $\mathbb{P}_n$ the set of $n\times n$ positive definite matrices. Let $D = D_1 \oplus \dots \oplus D_k$, where $D_1 \in \mathbb{P}_{n_1}, \dots, D_k \in \mathbb{P}_{n_k}$ with $n_1 + \dots + n_k = n$. Partition $C \in \mathbb{P}_n$ according to $(n_1, \dots, n_k)$ so that $\operatorname{Diag} C = C_1 \oplus \dots \oplus C_k$. For any $p \geqslant 0$, we have

$$\det(I_{n_1} + (C_1^{-1}D_1)^p)\cdots\det(I_{n_k} + (C_k^{-1}D_k)^p) \leqslant \det(I_n + (C^{-1}D)^p).$$

This is a generalization of a determinantal inequality of Matic [6, Theorem 1.1]. In addition, we obtain a weak majorization result which is complementary to a determinantal inequality of Choi [2, Theorem 2] and ask a weak log majorization open question.

1. Introduction

Denote by $\mathbb{C}_{n\times n}$ the set of $n\times n$ complex matrices and $\mathbb{P}_n \subset \mathbb{C}_{n\times n}$ the set of $n\times n$ positive definite matrices. For $A \in \mathbb{C}_{n\times n}$, we denote by A^* and $|A| = (A^*A)^{\frac{1}{2}}$ the conjugate transpose and the positive semidefinite part of A, respectively. Given $n\times n$ Hermitian matrices A and B, $A \leqslant B$ means that $B - A$ is positive semidefinite.

For $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$, let $x^{\downarrow} = (x_{[1]}, x_{[2]}, \dots, x_{[n]})$ denote a rearrangement of the components of x such that $x_{[1]} \geqslant x_{[2]} \geqslant \dots \geqslant x_{[n]}$. The notation $x \leqslant y$ means that $x_{[i]} \leqslant y_{[i]}$, $i = 1, \dots, n$. We say that x is weakly majorized by y, denoted by $x \prec_w y$, if $\sum_{j=1}^k x_{[j]} \leqslant \sum_{j=1}^k y_{[j]}$ for all $1 \leqslant k \leqslant n$. We say that x is majorized by y, denoted by $x \prec y$, if $x \prec_w y$ and $\sum_{j=1}^n x_j = \sum_{j=1}^n y_j$.

Let $\mathbb{R}_+$ denote the set of all positive real numbers and $\mathbb{R}_+^n = (\mathbb{R}_+)^n$. Given $x, y \in (\mathbb{R}_+)^n$, we say that x is weakly log-majorized by y, written as $x \prec_{w\log} y$, if $\prod_{i=1}^k x_{[i]} \leqslant \prod_{i=1}^k y_{[i]}$, for $k = 1, \dots, n$; x is log-majorized by y, denoted by $x \prec_{\log} y$, if $x \prec_{w\log} y$ and $\prod_{j=1}^n x_j = \prod_{j=1}^n y_j$.

Let $A \in \mathbb{P}_n$. Denote by $\lambda(A) = (\lambda_1(A), \dots, \lambda_n(A)) \in \mathbb{R}_+^n$ the vector of eigenvalues of A and we may arrange the eigenvalues in non-increasing order $\lambda_1(A) \geqslant \dots \geqslant \lambda_n(A)$.

Matic [6, Theorem 1.1] proved the following determinantal inequality. Zhang [8] and Choi [2] gave two different proofs, respectively. We state the theorem using Choi's version.

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* Corresponding author.

THEOREM 1. *(Matic [6]) Let $C \in \mathbb{P}_n$ and $D = D_1 \oplus \ldots \oplus D_k$, where $D_1 \in \mathbb{P}_{n_1}, \ldots, D_k \in \mathbb{P}_{n_k}$ with $n_1 + \cdots + n_k = n$. Partition C according to $(n_1, \ldots, n_k)$ so that $\mathrm{Diag}C = C_1 \oplus \ldots \oplus C_k$ in which $\mathrm{Diag}C$ is the main block diagonal of C. Then*

$$\det(I_{n_1} + C_1^{-1}D_1) \cdots \det(I_{n_k} + C_k^{-1}D_k) \leqslant \det(I_n + C^{-1}D). \tag{1}$$

In this paper we generalize (1) to the power of p $(p \geqslant 0)$ as follows.

$$\det(I_{n_1} + (C_1^{-1}D_1)^p) \cdots \det(I_{n_k} + (C_k^{-1}D_k)^p) \leqslant \det(I_n + (C^{-1}D)^p), \quad p \geqslant 0, \tag{2}$$

where $(C_i^{-1}D_i)^p, 1 \leqslant i \leqslant k$ are well-defined because the eigenvalues of a product of positive matrices are positive real numbers. We will show by an example that (2) is not true when $p < 0$.

By looking at (2) as a generalization of (1), one might ask whether the following two possible generalizations of (1) are true or not:

$$\det(I_{n_1} + |C_1^{-1}D_1|^p) \cdots \det(I_{n_k} + |C_k^{-1}D_k|^p) \leqslant \det(I_n + |C^{-1}D|^p), \quad p \geqslant 0, \tag{3}$$

and

$$\det(I_{n_1} + C_1^{-p}D_1^p) \cdots \det(I_{n_k} + C_k^{-p}D_k^p) \leqslant \det(I_n + C^{-p}D^p), \quad p \geqslant 0. \tag{4}$$

We will show that both answers are negative.

Choi [2, Theorem 2] obtained the following determinantal inequality:

THEOREM 2. *(Choi [2]) Let $A_i \in \mathbb{P}_n$, $i = 1, \ldots, m$, and $\mathrm{Diag}A_i = A_i^{(1)} \oplus \cdots \oplus A_i^{(k)}$, where $A_i^{(j)} \in \mathbb{P}_{n_j}$ for $i = 1, \ldots, m$, $j = 1, \ldots, k$. Then*

$$\det\left(\sum_{i=1}^m (A_i^{(1)})^{-1}\right) \cdots \det\left(\sum_{i=1}^m (A_i^{(k)})^{-1}\right) \leqslant \det\left(\sum_{i=1}^m A_i^{-1}\right). \tag{5}$$

We will present a weak majorization inequality which is complementary to (5) and pose a weak log majorization open problem.

2. Some lemmas

In this section, we present some lemmas which are useful in the sequel.

LEMMA 1. *([4, p. 308]) If H and $\overline{H}$ are $n \times n$ Hermitian matrices of the form*

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, \qquad \overline{H} = \begin{bmatrix} H_{11} & 0 \\ 0 & H_{22} \end{bmatrix},$$

where H_{11} and H_{22} are square matrices, then

$$\lambda(\overline{H}) \prec \lambda(H).$$

LEMMA 2. *([4, p. 165]) For any convex function* $f:\mathbb{R}\to\mathbb{R}$, *if* $x \prec y$, *where* $x=(x_1,x_2,\dots,x_n), y=(y_1,y_2,\dots,y_n)\in\mathbb{R}^n$, *then*

$$(f(x_1),f(x_2),\dots,f(x_n)) \prec_w (f(y_1),f(y_2),\dots,f(y_n)).$$

LEMMA 3. *([4, p. 167]) For any increasing convex function* $f:\mathbb{R}\to\mathbb{R}$. *If* $x\prec_w y$, *where* $x=(x_1,x_2,\dots,x_n), y=(y_1,y_2,\dots,y_n)\in\mathbb{R}^n$, *then*

$$(f(x_1),f(x_2),\dots,f(x_n)) \prec_w (f(y_1),f(y_2),\dots,f(y_n)).$$

LEMMA 4. *([3, p. 441]) Let* $P=\begin{bmatrix} A & B\\ B^* & C\end{bmatrix}\in\mathbb{P}_n$, *where* A *and* C *are square matrices. Then* P *can be factorized as* $P=T^*T$ *with* $T=\begin{bmatrix} X & Y\\ 0 & Z\end{bmatrix}$ *being conformally partitioned as* P.

LEMMA 5. *([5, Theorem 3]) Let* $T=\begin{bmatrix} X & Y\\ 0 & Z\end{bmatrix}\in\mathbb{C}_{n\times n}$, *where* $X\in\mathbb{C}_{r\times r}$, $Z\in\mathbb{C}_{(n-r)\times(n-r)}$. *Then for any* $p>0$

$$\det(I_r+|X|^p)\cdot\det(I_{n-r}+|Z|^p)\leqslant\det(I_n+|T|^p). \tag{6}$$

Equality holds in (6) if and only if $Y=0$.

LEMMA 6. *([3, p. 18]) Let* $A\in\mathbb{C}_{n\times n}$, $D\in\mathbb{C}_{m\times m}$, $B\in\mathbb{C}_{n\times m}$, *and* $C\in\mathbb{C}_{m\times n}$. *Assume that* A *and* $S_A=D-CA^{-1}B$ *are invertible. Then*

$$\begin{bmatrix} A & B\\ C & D\end{bmatrix}^{-1}=\begin{bmatrix} A^{-1}+A^{-1}BS_A^{-1}CA^{-1} & -A^{-1}BS_A^{-1}\\ -S_A^{-1}CA^{-1} & S_A^{-1}\end{bmatrix}.$$

Similarly, if D *and* $S_D=A-BD^{-1}C$ *are invertible, then*

$$\begin{bmatrix} A & B\\ C & D\end{bmatrix}^{-1}=\begin{bmatrix} S_D^{-1} & -S_D^{-1}BD^{-1}\\ -D^{-1}CS_D^{-1} & D^{-1}+D^{-1}CS_D^{-1}BD^{-1}\end{bmatrix}.$$

LEMMA 7. *([1, p. 63]) Let* A,B *be* $n\times n$ *Hermitian matrices and* $B\leqslant A$. *Then*

$$\lambda_j(B)\leqslant\lambda_j(A),\quad j=1,2,\dots,n.$$

LEMMA 8. *([7, p. 219]) Let* $A\in\mathbb{C}_{n\times n}$ *be positive semidefinite and let* $[A]$ *be the principal submatrix of* A *corresponding to some fixed rows and columns. Assuming that the inverses involved exist, we have*

$$[A]^{-1}\leqslant[A^{-1}].$$

3. Main results

We begin this section with the following proposition.

PROPOSITION 1. *Under the conditions given in Theorem 1, we have*

$$\lambda(C_1^{-1}D_1 \oplus \cdots \oplus C_k^{-1}D_k) \prec_{w\log} \lambda(C^{-1}D). \tag{7}$$

Proof. It suffices to prove the case when $D = I_n$:

$$\lambda(C_1^{-1} \oplus \cdots \oplus C_k^{-1}) \prec_{w\log} \lambda(C^{-1}). \tag{8}$$

The reason is that

$$\lambda(C^{-1}D) = \lambda((D^{-1/2}CD^{-1/2})^{-1})$$

and

$$\mathrm{Diag}(D^{-1/2}CD^{-1/2}) = (D_1^{-1/2}C_1D_1^{-1/2}) \oplus \cdots \oplus (D_k^{-1/2}C_kD_k^{-1/2}).$$

Thus (7) is equivalent to

$$\lambda((\mathrm{Diag}(D^{-1/2}CD^{-1/2}))^{-1}) \prec_{w\log} \lambda((D^{-1/2}CD^{-1/2})^{-1})$$

so it is sufficient to show (8). By Lemma 1, we have

$$\lambda(C_1 \oplus \cdots \oplus C_k) \prec \lambda(C),$$

which together with Lemma 2 and the fact that $f(x) = \log\frac{1}{x}$ is convex on $(0,+\infty)$ leads to (8). Thus, we complete the proof. $\square$

We now have the following generalization of Theorem 1.

THEOREM 3. *Under the conditions given in Theorem 1, we have*

$$\det(I_{n_1} + (C_1^{-1}D_1)^p)\cdots\det(I_{n_k} + (C_k^{-1}D_k)^p) \leqslant \det(I_n + (C^{-1}D)^p), \qquad p \geqslant 0. \tag{9}$$

Proof. When $p = 0$, it is trivial. Here we assume $p > 0$. It suffices to prove the inequality

$$\det(I_{n_1} + C_1^{-p})\cdots\det(I_{n_k} + C_k^{-p}) \leqslant \det(I_n + C^{-p}) \tag{10}$$

by the following argument. Note that $\lambda((C^{-1}D)^p) = \lambda((D^{-1/2}CD^{-1/2})^{-p})$. So, if (10) is true, then we have

$$\begin{aligned}
\det(I_n + (C^{-1}D)^p) &= \det(I_n + (D^{-1/2}CD^{-1/2})^{-p}) \\
&\geqslant \det(I_{n_1} + (D_1^{-1/2}C_1D_1^{-1/2})^{-p})\cdots\det(I_{n_k} + (D_k^{-1/2}C_kD_k^{-1/2})^{-p}) \\
&= \det(I_{n_1} + (C_1^{-1}D_1)^p)\cdots\det(I_{n_k} + (C_k^{-1}D_k)^p).
\end{aligned}$$

For (10), we provide four different proofs.

Proof 1: Induction allows us to prove (10) for $k=2$. By Lemma 4, there exists a matrix $T=\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$ being conformally partitioned as C^{-1} such that $C^{-1}=T^*T$. By Lemma 5, we have

$$\begin{aligned}\det(I_n+C^{-p}) &= \det(I_n+(T^*T)^p)=\det(I_n+|T|^{2p}) \\ &\geqslant \det(I_{n_1}+|X|^{2p})\cdot\det(I_{n-n_1}+|Z|^{2p}) \\ &= \det(I_{n_1}+(X^*X)^p)\cdot\det(I_{n-n_1}+(Z^*Z)^p). \end{aligned} \tag{11}$$

From Lemma 6 and

$$C=(T^*T)^{-1}=\begin{bmatrix} X^*X & X^*Y \\ Y^*X & Y^*Y+Z^*Z \end{bmatrix}^{-1},$$

we have

$$C_1^{-1}\leqslant X^*X.$$

By Lemma 7, we have

$$0<\lambda_j(C_1^{-1})\leqslant\lambda_j(X^*X),\quad j=1,2,\ldots,n_1.$$

Thus,

$$\lambda_j(C_1^{-p})=(\lambda_j(C_1^{-1}))^p\leqslant(\lambda_j(X^*X))^p=\lambda_j((X^*X)^p),\quad j=1,2,\ldots,n_1. \tag{12}$$

Similarly, we have

$$C_2=(Y^*Y+Z^*Z-Y^*X(X^*X)^{-1}X^*Y)^{-1}=(Z^*Z)^{-1},$$

which together with (11) and (12) leads to (10) when $k=2$.

Proof 2: Consider the symmetric function $\phi:\mathbb{R}_+^n\to\mathbb{R}_+$ defined by $\phi(x_1,\ldots,x_n)=\prod_{i=1}^n(1+\frac{1}{x_i^p})$, where $p>0$. Clearly it is continuously differentiable. Now

$$\begin{aligned} f(x) &:= (x_i-x_j)\left(\frac{\partial\phi}{\partial x_i}-\frac{\partial\phi}{\partial x_j}\right) \\ &= (x_i-x_j)\left(-\frac{p}{x_i^{p+1}}\prod_{\ell\neq i}(1+\frac{1}{x_\ell^p})+\frac{p}{x_j^{p+1}}\prod_{\ell\neq j}(1+\frac{1}{x_\ell^p})\right) \\ &= p(x_i-x_j)\left(-(1+\frac{1}{x_j^p})\frac{1}{x_i^{p+1}}+(1+\frac{1}{x_i^p})\frac{1}{x_j^{p+1}}\right)\prod_{\ell\neq i,j}(1+\frac{1}{x_\ell^p}) \\ &= \frac{p(x_i-x_j)}{x_i^{p+1}x_j^{p+1}}[(x_i-x_j)+(x_i^{p+1}-x_j^{p+1})]\prod_{\ell\neq i,j}(1+\frac{1}{x_\ell^p})\geqslant 0. \end{aligned}$$

So ϕ is Schur-convex by Schur-Ostrowski criterion. Note that

$$\det(I_n+C^{-p})=\prod_{i=1}^n\left(1+\lambda_i(C^{-p})\right)=\prod_{j=1}^n\left(1+\frac{1}{\lambda_j(C^p)}\right)=\prod_{j=1}^n\left(1+\frac{1}{\lambda_j^p(C)}\right)=\phi(\lambda(C)).$$

Similarly,

$$\det(I_{n_1}+C_1^{-p})\cdots\det(I_{n_k}+C_k^{-p}) = \prod_{i=1}^{n_1}\left(1+\frac{1}{\lambda_i^p(C_1)}\right)\cdots\prod_{i=1}^{n_k}\left(1+\frac{1}{\lambda_i^p(C_k)}\right)$$
$$= \phi(\lambda(C_1),\cdots,\lambda(C_k)).$$

These together with Lemma 1 and Schur-convexity of ϕ imply (10).

Proof 3: Since the function $f(x) := \log(1+\frac{1}{x^p})$ is convex on $(0,+\infty)$ for $p>0$, by Lemma 1 and Lemma 2, we have

$$f(\lambda(C_1\oplus\cdots\oplus C_k)) \prec_w f(\lambda(C)).$$

Therefore,

$$\sum_{i=1}^{n}\log(1+\lambda_i(C_1^{-p}\oplus\cdots\oplus C_k^{-p})) = \sum_{i=1}^{n}\log(1+\lambda_i(C_1\oplus\cdots\oplus C_k)^{-p})$$
$$\leqslant \sum_{i=1}^{n}\log(1+\lambda_i(C)^{-p}).$$

We have (10), as desired, by taking exponential of the inequality.

Proof 4: Since the function $f(x)=\log(1+e^{px})$, where $p>0$, is an increasing convex function, by (8) and Lemma 3, the desired result follows. $\square$

REMARK 1. When $p=1$, Theorem 3 reduces to Theorem 1.

In the next example we show that (9) is not true when $p<0$.

EXAMPLE 1. Let $n=2$, $n_1=n_2=1$, $D=I_2$,

$$C=\begin{pmatrix}3&2\\2&3\end{pmatrix}\in\mathbb{P}_2,\quad C_1=C_2=3.$$

Direct computation gives $\lambda(C)=\{5,1\}$. Let $p<0$ and set $q:=-p$ so $q>0$. Then

$$\lambda(C^q)=\{5^q,1\},\quad C_1^q=C_2^q=3^q.$$

Let

$$f(q):=\det(I_2+C^q)=(1+5^q)(1+1)=2+2\cdot 5^q$$

and

$$g(q):=\det(1+C_1^q)\det(1+C_2^q)=(1+3^q)^2=1+2\cdot 3^q+3^{2q}.$$

We are going to show that $g(q)>f(q)$ for all $q>0$. Let $f(x)=2\cdot 3^x+3^{2x}-2\cdot 5^x-1$. Since

$$f'(x)=(3^x+9^x)\ln 9-2\cdot 5^x\ln 5\geqslant 2\sqrt{3^x\cdot 9^x}\ln 9-2\cdot 5^x\ln 5>0,\quad \text{for } x>0,$$

we have $f(x)>f(0)=0$ when $x>0$. Thus (9) is not true when $p<0$.

REMARK 2. From the example above, one may ask whether the reverse of (3) is true for all $p<0$. There are many cases refuting the claim.

We would like to point out that (9) is no longer true if $D\in\mathbb{P}_n$ is not in diagonal block form in general.

Next we will show that inequalities (3) and (4) are not true and we first give a counterexample to the following inequality.

$$\det(D_1^{-2}+C_1^{-2})\cdots\det(D_k^{-2}+C_k^{-2})\leqslant\det(D^{-2}+C^{-2}), \tag{13}$$

where $C,D,C_i,D_i,i=1,\ldots,k$, are given as in Theorem 1.

EXAMPLE 2. Let

$$C=\begin{pmatrix}16.25 & 21 & 10 & 12.5\\ 21 & 39.75 & 20.75 & 28.5\\ 10 & 20.75 & 22.5 & 27.75\\ 12.5 & 28.5 & 27.75 & 39.25\end{pmatrix},\quad C_1=\begin{pmatrix}16.25 & 21\\ 21 & 39.75\end{pmatrix},$$

$$C_2=\begin{pmatrix}22.5 & 27.75\\ 27.75 & 39.25\end{pmatrix},\quad D=\begin{pmatrix}14.7 & 15 & 0 & 0\\ 15 & 15.8 & 0 & 0\\ 0 & 0 & 0.25 & 0.4\\ 0 & 0 & 0.4 & 0.8\end{pmatrix},$$

and

$$D_1=\begin{pmatrix}14.7 & 15\\ 15 & 15.8\end{pmatrix},\quad D_2=\begin{pmatrix}0.25 & 0.4\\ 0.4 & 0.8\end{pmatrix}.$$

By Matlab

$$\det(D^{-2}+C^{-2})=51.0669<54.6523=\det(D_1^{-2}+C_1^{-2})\det(D_2^{-2}+C_2^{-2}).$$

Therefore, (13) is false.

Note that

$$\begin{aligned}\det(I_n+|C^{-1}D|^2)&=\det(I_n+DC^{-2}D)=\det(D(D^{-2}+C^{-2})D)\\&=\det(D^{-2}+C^{-2})\cdot(\det D)^2,\end{aligned}$$

and

$$\begin{aligned}&\det(I_{n_1}+|C_1^{-1}D_1|^2)\cdots\det(I_{n_k}+|C_k^{-1}D_k|^2)\\&=\det(D_1(D_1^{-2}+C^{-2})D_1)\cdots\det(D_k(D_k^{-2}+C_k^{-2})D_k)\\&=\det(D_1^{-2}+C_1^{-2})\cdot\det(D_1)^2\cdots\det(D_k^{-2}+C_k^{-2})\cdot(\det D_k)^2\\&=\det(D_1^{-2}+C_1^{-2})\cdots\det(D_k^{-2}+C_k^{-2})\cdot(\det D)^2.\end{aligned}$$

Since (13) is false, (3) is invalid.

Note that

$$\det(I_n + C^{-2}D^2) = \det((D^{-2} + C^{-2})D^2) = \det(D^{-2} + C^{-2}) \cdot (\det D)^2$$

and

$$\begin{aligned} &\det(I_{n_1} + C_1^{-2}D_1^2) \cdots \det(I_{n_k} + C_k^{-2}D_k^2) \\ &= \det(D_1^{-2} + C_1^{-2}) \cdot \det(D_1)^2 \cdots \det(D_k^{-2} + C_k^{-2}) \cdot (\det D_k)^2 \\ &= \det(D_1^{-2} + C_1^{-2}) \cdots \det(D_k^{-2} + C_k^{-2}) \cdot (\det D)^2. \end{aligned}$$

Since (13) is false, (4) is also invalid.

REMARK 3. Since (3) is not true, the vectors of eigenvalues cannot be replaced by the vectors of singular values in (7) from Proof 4 in Theorem 3. In other words,

$$s(C_1^{-1}D_1 \oplus \cdots \oplus C_k^{-1}D_k) \not\prec_{w \log} s(C^{-1}D),$$

where $C, D, C_i, D_i, i = 1, \ldots, k$, are given as in Theorem 1.

Next we present a weak majorization complementary to Choi's determinantal inequality as follows.

THEOREM 4. *Under the conditions given in Theorem 2, we have*

$$\lambda\left(\sum_{i=1}^m (A_i^{(1)})^{-1} \oplus \cdots \oplus \sum_{i=1}^m (A_i^{(k)})^{-1}\right) \prec_w \lambda\left(\sum_{i=1}^m A_i^{-1}\right). \tag{14}$$

Proof. By Lemma 1, we have

$$\lambda\left(\sum_{i=1}^m (A_i^{-1})^{(1)} \oplus \cdots \oplus \sum_{i=1}^m (A_i^{-1})^{(k)}\right) \prec \lambda\left(\sum_{i=1}^m A_i^{-1}\right). \tag{15}$$

By Lemma 7 and Lemma 8, we have

$$\lambda\left(\sum_{i=1}^m (A_i^{(1)})^{-1} \oplus \cdots \oplus \sum_{i=1}^m (A_i^{(k)})^{-1}\right) \leqslant \lambda\left(\sum_{i=1}^m (A_i^{-1})^{(1)} \oplus \cdots \oplus \sum_{i=1}^m (A_i^{-1})^{(k)}\right). \tag{16}$$

Now (15) and (16) lead to

$$\lambda\left(\sum_{i=1}^m (A_i^{(1)})^{-1} \oplus \cdots \oplus \sum_{i=1}^m (A_i^{(k)})^{-1}\right) \prec_w \lambda\left(\sum_{i=1}^m A_i^{-1}\right).$$

Thus, we complete the proof. □

From Theorem 2 and Theorem 4, it is natural to ask whether the following weak log-majorization inequality holds. We leave it as an open problem.

QUESTION 1. Under the conditions given in Theorem 2,

$$\lambda\left(\sum_{i=1}^{m}(A_i^{(1)})^{-1}\oplus\cdots\oplus\sum_{i=1}^{m}(A_i^{(k)})^{-1}\right)\prec_{w\log}\lambda\left(\sum_{i=1}^{m}A_i^{-1}\right)? \tag{17}$$

We would like to point out that when $n=2, k=2, n_1=n_2=1$, (17) holds by (5) and (14). The other cases are open. We performed computer experiments and the outcomes are consistent with the weak log majorization given in (17).

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Tin-Yau Tam
Department of Mathematics and Statistics
University of Nevada
Reno, NV 89557, USA
e-mail: ttam@unr.edu

Pingping Zhang
School of Science
Chongqing University of Posts and Telecommunications
Chongqing, 400065
e-mail: zhangpp@cqupt.edu.cn, zhpp04010248@126.com