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ON THE CONCENTRATION OF A FUNCTION AND ITS LAGUERRE–BESSEL TRANSFORM

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Abstract. This paper deals with uncertainty principle related to Laguerre-Bessel transform invoking smallness of the support. In particular, we obtain a Benedichs-Amrein-Berthier type theorem related to Laguerre-Bessel transform. As a consequence, we get a global uncertainty inequality and a Heisenberg uncertainty inequality for Laguerre-Bessel transform. Furthermore, invoking essential support, we prove analogous of Donoho-Stark theorem in $L^1(\mathbb{K})$ and $L^2(\mathbb{K})$, where $\mathbb{K} = [0,+\infty) \times [0,+\infty)$.

1. Introduction

Several works have been interested in the uncertainty principle in different occurrences. The most known is due to Heisenberg [11] in 1927. His famous inequality, appearing in Weyl paper [16] who credits to Pauli, states as follows

$$\left(\int_{\mathbb{R}} x^2|f(x)|^2dx\right)\left(\int_{\mathbb{R}} \xi^2|\hat{f}(\xi)|^2d\xi\right) \geqslant \frac{1}{4}\left(\int_{\mathbb{R}} |f(x)|^2dx\right)^2.$$

In quantum mechanics, this means that we cannot simultaneously and precisely localize the values of the position and the momentum of a particle. Its equivalent in signal theory deals with time frequency localization. It tells us about the loss of the precision of frequency of a signal observed for a finite period of time.

Another approach of this physical idea is to consider the concentration measured with smallness of the support: Benedicks [2], Amrein and Berthier [1] proved the corresponding result for classical Fourier transform which states that for finite supports S and Σ, any function $f \in L^2(\mathbb{R})$ vanishes as soon as f is supported in S and $\hat{f}$ is supported in Σ.

Recently, Ghobber S. and Jaming Ph. have given a generalization of this result for Bessel transform in [9] and for an integral operator with a bounded kernel and defined on the Euclidean space $\mathbb{R}^d$ verifying Plancherel formula in [7].

A quantitative version of the uncertainty principle was given by Donoho and Stark [5] when they have replaced the exact support by the essential support. Note that a

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measurable subset S is considered as the essential support of f if f is ε-concentrated on S. i.e

$$\left(\int_{\mathbb{R}^d\setminus S}|f(x)|^2dx\right)^{\frac{1}{2}}=\|f-\chi_S f\|_2\leqslant\varepsilon\|f\|_2,$$

where χ_E is the characteristic function of the set E.
When $\varepsilon=0$, S is the exact support of f.

Donoho and Stark's theorem states that if f of unit L_2 norm is ε_T-concentrated on a measurable set T and its Fourier transform $\widehat{f}$ is ε_W-concentrated on a measurable set W, then

$$|W||T|\geqslant(1-\varepsilon_T-\varepsilon_W)^2.$$

Here $|T|$ is the Lebesgue measure of the set T. A generalization of this theorem has been proven in other settings, one can cite [3, 4, 14, 15].

In this paper, we extend a Benedicks-Amrein-Berthier theorem and a Donoho-Stark theorem for Laguerre-Bessel transform denoted $\mathscr{F}_{LB}$. It is already known that Fourier Bessel or Hankel transform is obtained by considering Fourier transform of radial functions on Euclidean group. Its analogue for Heisenberg group is the Laguerre transform $\mathscr{F}_L$. (One can see [6] for more details). $\mathscr{F}_{LB}$ is a compound of these two transforms (cf. [12]). It is related to the Laguerre-Bessel system of partial differential operators given by, for all $(x,t)\in\mathbb{K}=[0,+\infty)\times[0,+\infty)$ and $\alpha\geqslant 0$,

$$\begin{cases} D_1 = \dfrac{\partial^2}{\partial t^2}+\dfrac{2\alpha}{t}\dfrac{\partial}{\partial t}\\ D_2 = \dfrac{\partial^2}{\partial x^2}+\dfrac{2\alpha+1}{x}\dfrac{\partial}{\partial x}+x^2D_1\end{cases}$$

Equipped with the convolution $*_\alpha$, $\mathbb{K}$ has the structure of a hypergroup in the sense of Jewett [13] with the involution the identity and the Haar measure defined by

$$d\mu_\alpha(x,t)=\frac{x^{2\alpha+1}t^{2\alpha}}{\Gamma(\alpha+\frac{1}{2})\Gamma(\alpha+1)}dx\,dt. \tag{1}$$

The translation operator, given by

$$(\delta_{(x,t)}*_\alpha\delta_{(y,s)})(f)=T^\alpha_{x,t}f(y,s) \tag{2}$$

has the integral form : for $\alpha=0$,

$$T^\alpha_{x,t}f(y,s)=\frac{1}{4\pi}\sum_{i,j=0}^{1}\int_0^\pi f(\sqrt{x^2+y^2+2xy\cos\theta},(-1)^it+(-1)^js+xy\sin\theta)\,d\theta$$

and, for $\alpha>0$,

$$T^\alpha_{x,t}f(y,s)=\frac{\alpha\Gamma(\alpha+\frac{1}{2})}{\pi^{\frac{3}{4}}\Gamma(\alpha)}\int_{[0,\pi]^3}f(\Delta_\theta(x,y),\Delta_\xi(X,xy\sin\theta)d\nu_\alpha(\xi,\psi,\theta).$$

where

$$\Delta_\theta(x,y)=\sqrt{x^2+y^2+2xyr\cos\theta}\quad,\quad X=\Delta_\psi(t,s)$$

and

$$d\nu_\alpha(\xi,\psi,\theta) = (\sin\xi)^{2\alpha-1}(\sin\psi)^{2\alpha-1}(\sin\theta)^{2\alpha}d\xi\,d\psi\,d\theta. \tag{3}$$

The convolution product of two functions is given by

$$(f *_\alpha g)(x,t) = \int_{\mathbb{K}} T^\alpha_{x,t} f(y,s)g(y,s)\,d\mu_\alpha(y,s). \tag{4}$$

The outline of the content of the paper is given as follows :
Section 2 is devoted to give some useful relations about the Fourier Laguerre Bessel transform $\mathscr{F}_{LB}$. In section 3, we consider two orthogonal projections P_T and P_W; we prove analogous of Benedicks-Amrein-Berthier theorem for the Laguerre-Bessel transform. Consequently we show that: for $T \subset \mathbb{K}$, $W \subset \hat{\mathbb{K}}$ a pair of measurable subsets of finite measure, there exists a constant $C(T,W)$ such that for all $f \in L^2(\mathbb{K})$,

$$\|f\|^2_{2,\mu_\alpha} \leqslant C(T,W)\left(\|f\chi_{T^c}\|^2_{2,\mu_\alpha} + \|\mathscr{F}_{LB} f\chi_{W^c}\|^2_{2,\hat{\mu}_\alpha}\right).$$

We say that the pair (T,W) is strongly annihilating. The analog of Donoho-Stark's theorems in $L^2(\mathbb{K})$ and $L^1(\mathbb{K})$ are given in section 4.

2. Preliminaries

We denote $\hat{\mathbb{K}} = [0,+\infty)\times\mathbb{N}$. For $(\lambda,m) \in \hat{\mathbb{K}}$, the initial problem

$$\begin{cases} D_1 u = -\lambda^2 u, \\ D_2 u = -4|\lambda|(m + \dfrac{\alpha+1}{2})u \\ u(0,0) = 1,\ \dfrac{\partial u}{\partial x}(0,0) = \dfrac{\partial u}{\partial t}(0,0) = 0 \end{cases}$$

has a unique solution $\varphi_{\lambda,m}$ given by

$$\forall (x,t) \in \mathbb{K}, \qquad \varphi_{\lambda,m}(x,t) = j_{\alpha-\frac{1}{2}}(\lambda t)\mathscr{L}^{(\alpha)}_m(\lambda x^2), \tag{5}$$

where j_α is the spherical Bessel function given by

$$j_\alpha(x) = \Gamma(\alpha+1)\sum_{k=0}^{+\infty}\frac{(-1)^k(\frac{x}{2})^{2k}}{k!\Gamma(\alpha+k+1)} \tag{6}$$

and $\mathscr{L}^{(\alpha)}_m$ is the Laguerre function defined on $\mathbb{R}_+$ by

$$\mathscr{L}^{(\alpha)}_m(x) = e^{-\frac{x}{2}}\frac{L^\alpha_m(x)}{L^\alpha_m(0)}, \tag{7}$$

where L^α_m is the Laguerre polynomial of degree m and order α.

$\varphi_{\lambda,m}$ is called Laguerre-Bessel kernel and verfies the following property

$$\forall (\lambda,m) \in \mathbb{R}\times\mathbb{N}, \qquad \sup_{(x,t)\in\mathbb{K}} |\varphi_{\lambda,m}(x,t)| = 1. \tag{8}$$

The Fourier Laguerre Bessel transform of a suitable function is given by

$$\mathscr{F}_{LB}f(\lambda,m)=\int_{\mathbb{K}} f(x,t)\,\varphi_{\lambda,m}(x,t)d\mu_\alpha(x,t). \tag{9}$$

From [12], the Fourier Laguerre Bessel transform can be inverted to

$$\mathscr{F}_{LB}^{-1}f(x,t)=\int_{\hat{\mathbb{K}}} f(\lambda,m)\,\varphi_{\lambda,m}(x,t)\,d\hat{\mu}_\alpha(\lambda,m), \tag{10}$$

where

$$d\hat{\mu}_\alpha(\lambda,m)=L_m^\alpha(0)\delta_m\otimes\frac{\lambda^{3\alpha+1}}{2^{2\alpha-1}\Gamma(\alpha+\frac{1}{2})}d\lambda. \tag{11}$$

Denote, for $1\leqslant p<+\infty$, $L^p(\mathbb{K})=L^p(\mathbb{K},d\mu_\alpha)$ the space of measurable functions $f:\mathbb{K}\longrightarrow\mathbb{C}$ such that

$$\|f\|_{p,\mu_\alpha}=\left(\int_{\mathbb{K}}|f(x,t)|^p d\mu_\alpha(x,t)\right)^{\frac{1}{p}}<+\infty.$$

and

$$\|f\|_{\infty,\mu_\alpha}=\sup_{\mathbb{K}}|f(x,t)|$$

We introduce $L^p(\hat{\mathbb{K}})$ the space of measurable function $g:\hat{\mathbb{K}}\longrightarrow\mathbb{C}$ which checks

$$\|g\|_{p,\hat{\mu}_\alpha}=\left(\int_{\hat{\mathbb{K}}}|g(\lambda,m)|^p d\hat{\mu}_\alpha(\lambda,m)\right)^{\frac{1}{p}}<+\infty.$$

and

$$\|g\|_{\infty,\hat{\mu}_\alpha}=\sup_{\hat{\mathbb{K}}}|g(\lambda,m)|.$$

The Laguerre-Bessel transform is well defined on $L^1(\mathbb{K})$ and verifies

$$\|\mathscr{F}_{LB}f\|_{\infty,\hat{\mu}_\alpha}\leqslant\|f\|_{1,\mu_\alpha}. \tag{12}$$

Furthermore it can be extended to an isometric isomorphism checking the following Plancherel formula (cf. [12])

$$\|\mathscr{F}_{LB}f\|_{2,\hat{\mu}_\alpha}=\|f\|_{2,\mu_\alpha}. \tag{13}$$

Throughout all this paper, we consider for $T\subset\mathbb{K}$ and $W\subset\hat{\mathbb{K}}$ two measurable subsets. We define the orthogonal projections on $L^2(\mathbb{K})$ P_T and P_W by

$$P_Tf=\chi_Tf \qquad \text{and} \qquad P_Wf=\mathscr{F}_{LB}^{-1}(\chi_W\mathscr{F}_{LB}f),$$

where χ_T and χ_W are the characteristic functions of T and W.

3. Weak and strong annihilating pairs

In this paragraph, we proceed in a similar way as in [1] and [7] to prove that if $T \subset \mathbb{K}$ and $W \subset \hat{\mathbb{K}}$ are two measurable subsets such that $\mu_\alpha(T) < +\infty$ and $\hat{\mu}_\alpha(W) < +\infty$ then the pair (T,W) is weakly annihilating. This means that we can not find a nonzero function supported on T and its Fourier Laguerre Bessel transform supported on W. As a consequence, we deduce that (T,W) is strong annihilating pair. In other words there exists a constant C as follows :

$$\|f\|^2_{2,\mu_\alpha} \leqslant C(T,W)\left(\|f\chi_{T^c}\|^2_{2,\mu_\alpha} + \|\mathscr{F}_{LB}f\chi_{W^c}\|^2_{2,\hat{\mu}_\alpha}\right).$$

With respect to the homogenous measure on $\mathbb{K}$, we introduce the dilated of (x,t) in $\mathbb{K}$ by $\delta_r(x,t) = (rx, r^2t)$ and the dilated of (λ,m) in $\hat{\mathbb{K}}$ by $\delta'_r(\lambda,m) = (r^2\lambda, m)$.
If we denote $f_r(x,t) = r^{-(6\alpha+4)}f(\delta_{\frac{1}{r}}(x,t))$ then we have

$$\int_{\mathbb{K}} f_r(x,t)d\mu_\alpha(x,t) = \int_{\mathbb{K}} f(x,t)d\mu_\alpha(x,t) \tag{14}$$

Consider $D_rf = r^{-(3\alpha+2)}f(\delta_{\frac{1}{r}}(x,t))$. By a change of variables, we get

$$\mathscr{F}_{LB}D_rf = \hat{D}_{\frac{1}{r}}\mathscr{F}_{LB}f, \tag{15}$$

where

$$\hat{D}_rf(\lambda,m) = r^{-(3\alpha+2)}f(\delta'_{\frac{1}{r}}(\lambda,m)). \tag{16}$$

LEMMA 1. *Let $T \subset \mathbb{K}$ and $W \subset \hat{\mathbb{K}}$ be two measurable subsets with finite measure i.e. $\mu_\alpha(T) < +\infty$ and $\hat{\mu}_\alpha(W) < +\infty$. Then the Hilbert-Schmidt norm of P_WP_T is finite and we have*

$$\|P_WP_T\|^2_{HS} \leqslant \mu_\alpha(T)\,\hat{\mu}_\alpha(W). \tag{17}$$

Proof. From relation (10),

$$P_W\,P_Tf(x,t) = \int_{\hat{\mathbb{K}}} \chi_W(\lambda,m)\mathscr{F}_{LB}P_Tf(\lambda,m)\varphi_{\lambda,m}(x,t)\,d\hat{\mu}_\alpha(\lambda,m).$$

Denote

$$g_{(x',t')}(\lambda,m) = \chi_W(\lambda,m)\,\varphi_{\lambda,m}(x',t')$$

and

$$\Psi(x',t',x,t) = \chi_T(x',t')\,\mathscr{F}^{-1}_{LB}g_{(x',t')}(x,t).$$

By Fubini's theorem, we have

$$P_W\,P_Tf(x,t) = \int_{\mathbb{K}} f(x',t')\Psi(x',t',x,t)\,d\mu_\alpha(x',t').$$

Ψ is called the kernel of the integral operator $P_W P_T$ and the Hilbert Schmidt norm of this operator is given by

$$\|P_W P_T\|_{HS}^2 = \|\Psi\|_{L^2(\mathbb{K})\otimes L^2(\mathbb{K})}^2.$$

Furthermore,

$$\|\Psi\|_{L^2(\mathbb{K})\otimes L^2(\mathbb{K})}^2 = \int_{\mathbb{K}} |\chi_T(x',t')|^2 \left(\int_{\mathbb{K}} |\mathscr{F}_{LB}^{-1} g_{(x',t')}(x,t)|^2 \, d\mu_\alpha(x,t) \right) d\mu_\alpha(x',t').$$

From Plancherel formula, we obtain

$$\begin{aligned}\|\Psi\|_{L^2(\mathbb{K})\otimes L^2(\mathbb{K})}^2 &= \int_T \left(\int_{\hat{\mathbb{K}}} |\chi_W(\lambda,m)\varphi_{\lambda,m}(x',t')|^2 \, d\hat{\mu}_\alpha(\lambda,m) \right) d\mu_\alpha(x',t') \\ &\leqslant \mu_\alpha(T)\,\hat{\mu}_\alpha(W).\end{aligned}$$

Which gives the wanted result. □

LEMMA 2. *Let f be a function in $L^2(\mathbb{K})$ with finite support. Then $\{D_\lambda f\}_{\lambda>0}$, the dilates of f, are linearly independent.*

REMARK 1. Lemma 3.4 in [7] does'nt allow us to deduce Lemma 2 since the delated of (x,t) in $\mathbb{K}$ has not the same form as in the Euclidean case. Nevertheless, we give here an interesting proof which could be used while considering the case of several variables.

Proof. Assume that we have a vanishing linear combinations of dilates of f

$$\sum_{finite} \alpha_i f_i(x,t) = 0$$

We denote $\beta_i = \alpha_i r_i^{-(\alpha+2)}$ and $g(\frac{x}{r_i}) = f_0(\frac{x}{r_i},0)$ then

$$\sum_{finite} \beta_i g(\frac{x}{r_i}) = 0.$$

Applying the Euclidean Fourier we get

$$\sum_{finite} \beta_i r_i \mathscr{F} g(r_i x) = 0.$$

Since $g \in L^1(\mathbb{R})$ then $\mathscr{F} g \in C_0$. Invoking [9, lemma 2.1], one can see that $\mathscr{F} g$ has linearly independent dilates. Therefore $\beta_i = 0$ so that $\alpha_i = 0$, which proves that f_i are linearly independent. □

Now, we are able to announce analogous of Benedicks-Amrein-Berthier theorem for the Laguerre-Bessel transform.

THEOREM 1. *Let $f \in L^2(\mathbb{K})$, $T \subset \mathbb{K}$ and $W \subset \hat{\mathbb{K}}$ be two measurable subsets. If $supp(f) \subset T$, $supp(\mathscr{F}_L f) \subset W$ and $\mu_\alpha(T), \hat{\mu}_\alpha(W) < +\infty$ then $f = 0$.*

Proof. Assume that there exists a function $f_0 \neq 0$ such as $supp(f_0) = T_0$ and $W_0 = supp(\mathscr{F}_{LB} f_0)$ have both finite measure i.e $0 < \mu_\alpha(T_0), \hat{\mu}_\alpha(W_0) < +\infty$.
Let T_1 (resp W_1) be a measurable subset of $\mathbb{K}$ (resp $\hat{\mathbb{K}}$) of finite measure $0 < \mu_\alpha(T_1) < +\infty$ (resp $0 < \hat{\mu}_\alpha(W_1) < +\infty$) such that $T_0 \subset T_1$ (resp $W_0 \subset W_1$).
We have, for $r > 0$,

$$\mu_\alpha(T_1 \cup \delta_r T_0) = \|\chi_{\delta_r T_0} - \chi_{T_1}\|^2_{2,\mu_\alpha} + <\chi_{\delta_r T_0}, \chi_{T_1}>_{\mu_\alpha} .$$

The function $: r \longmapsto \mu_\alpha(T_1 \cup \delta_r T_0)$ is continuous on $(0, +\infty)$. The same holds for $: r \longmapsto \hat{\mu}_\alpha(W_1 \cup \delta'_r W_0)$. This allows us to build an infinite sequence of distinct numbers $(r_i)_{i=0}^{\infty} \subset (0, +\infty)$ with $r_0 = 1$, as follows

$$\mu_\alpha(T) < 2\mu_\alpha(T_0) \text{ and } \hat{\mu}_\alpha(W) < 2\hat{\mu}_\alpha(W_0),$$

where $T = \bigcup_{i=0}^{+\infty} \delta_{r_i} T_0$ and $W = \bigcup_{i=0}^{+\infty} \delta'_{\frac{1}{r_i}} W_0$.

Let $f_i = D_{r_i} f_0$, then $supp(f_i) = \delta_{r_i} T_0 \subset T$. Relation (15) allows us to see that $supp(\mathscr{F}_{LB} f_i) = \delta'_{\frac{1}{r_i}} W_0 \subset W$. Consequently, Lemma 2 implies that $dim(Im(P_T) \cap Im(P_W))$ $= +\infty$ which contradicts Lemma 1 since

$$dim(Im(P_T) \cap Im(P_W)) = \|P_T \cap P_W\|^2_{HS} \leqslant \|P_T\, P_W\|^2_{HS}.$$

This proves Theorem 1. □

THEOREM 2. *Let $T \subset \mathbb{K}$, $W \subset \hat{\mathbb{K}}$ be a pair of measurable subsets of finite measures $\mu_\alpha(T)$, $\hat{\mu}_\alpha(W) < +\infty$. Then there exists a constant $C(T,W)$ such that, for all $f \in L^2(\mathbb{K})$,*

$$\|f\|^2_{2,\mu_\alpha} \leqslant C(T,W) \left(\|f \chi_{T^c}\|^2_{2,\mu_\alpha} + \|\mathscr{F}_{LB} f \chi_{W^c}\|^2_{2,\hat{\mu}_\alpha} \right). \tag{18}$$

Proof. Since $|\varphi_{\lambda,m}(x,t)| \leqslant 1$ then the $\mathscr{F}_{LB}$ of a function with finite support is bounded. Therefore, we find similarly as in [7, Corollary 3.7] the wanted result. □

Now consider the homogeneous norm on $\mathbb{K}$ defined, for all $(x,t) \in \mathbb{K}$, by

$$|(x,t)|_{\mathbb{K}} = (x^4 + 4t^2)^{\frac{1}{4}} \tag{19}$$

and the quasinorm, defined on $\hat{\mathbb{K}}$ by

$$|(\lambda, m)|_{\hat{\mathbb{K}}} = 4|\lambda|(m + \frac{\alpha+1}{2}). \tag{20}$$

We can deduce the following Heisenberg type enequality :

COROLLARY 1. *Let s, $\beta > 0$. Then*

$$\forall f \in L^2(\mathbb{K}), \qquad \|\, |(x,t)|^s_{\mathbb{K}} f\|^{\frac{2\beta}{s+\beta}}_{2,\mu_\alpha} . \|\, |(\lambda,m)|^{\frac{\beta}{2}}_{\hat{\mathbb{K}}} \mathscr{F}_{LB} f\|^{\frac{2s}{s+\beta}}_{2,\hat{\mu}_\alpha} \geqslant C\|f\|^2_{2,\mu_\alpha}, \tag{21}$$

where C is a constant which depends on s, β and α.

Proof. Applying Theorem 2 on unitary balls in $\mathbb{K}$ and $\hat{\mathbb{K}}$ for the function $D_r f$, we get

$$\|D_r f\|^2_{2,\mu_\alpha} \leqslant C\left(\|D_r f\chi_{B_1^c}\|^2_{2,\mu_\alpha} + \|\mathscr{F}_{LB} D_r f\chi_{\mathbb{B}_1^c}\|^2_{2,\hat{\mu}_\alpha}\right).$$

Therefore, from relation (15), we have

$$\begin{aligned}\|D_r f\|^2_{2,\mu_\alpha} &\leqslant C\left(\||(x,t)|^s_{\mathbb{K}} D_r f\chi_{B_1^c}\|^2_{2,\mu_\alpha} + \||(\lambda,m)|^{\frac{\beta}{2}}_{\hat{\mathbb{K}}} \hat{D_{\frac{1}{r}}}\mathscr{F}_{LB} f\,\chi_{\mathbb{B}_1^c}\|^2_{2,\hat{\mu}_\alpha}\right)\\ &\leqslant C\left(\||(x,t)|^s_{\mathbb{K}} D_r f\|^2_{2,\mu_\alpha} + \||(\lambda,m)|^{\frac{\beta}{2}}_{\hat{\mathbb{K}}} \hat{D_{\frac{1}{r}}}\mathscr{F}_{LB} f\|^2_{2,\hat{\mu}_\alpha}\right).\end{aligned}$$

Which implies

$$\|f\|^2_{2,\mu_\alpha} \leqslant C\left(r^{2s}\|\,|(x,t)|^s_{\mathbb{K}} f\|^2_{2,\mu_\alpha} + r^{-\beta}\|\,|(\lambda,m)|^{\frac{\beta}{2}}_{\hat{\mathbb{K}}}\mathscr{F}_{LB} f\|^2_{2,\hat{\mu}_\alpha}\right).$$

By optimizing in $r > 0$, we obtain the desired result. □

Remark that, for $s = \beta$, we get the same inequality established by S. Hamam and L. Kamoun using heat functions in [10].

4. Donoho-Stark theorem for Laguerre-Bessel transform

4.1. L^2-version of Donoho-Stark theorem

In this section we will extend the Donoho-Stark uncertainty principle to the Laguerre-Bessel transform in the space $L^2(\mathbb{K})$. We say that f is ε_T-concentrated on a set T if and only if

$$\|f - P_T f\|_{2,\mu_\alpha} \leqslant \varepsilon_T. \tag{22}$$

We say also that f is ε_W-bandlimited or $\mathscr{F}_{LB} f$ is ε_W-concentrated on a set W if and only if

$$\|f - P_W f\|_{2,\mu_\alpha} \leqslant \varepsilon_W. \tag{23}$$

The operator P_T is bounded from $L^p(\mathbb{K})$, $1 \leqslant p \leqslant \infty$ into it self and we have

$$\|P_T f\|_{p,\mu_\alpha} \leqslant \|f\|_{p,\mu_\alpha},\ f \in L^p(\mathbb{K}). \tag{24}$$

The same result holds for P_W and we have

$$\|P_W f\|_{p,\mu_\alpha} \leqslant \|f\|_{p,\mu_\alpha},\ f \in L^p(\mathbb{K}) \tag{25}$$

THEOREM 3. *Let $T \subset \mathbb{K}$, $W \subset \hat{\mathbb{K}}$ be measurable sets and suppose that $\|f\|_{2,\mu_\alpha} = \|\mathscr{F}_{LB} f\|_{2,\hat{\mu}_\alpha} = 1$. Assume that $\varepsilon_T^2 + \varepsilon_W^2 < 1$, f is ε_T-concentrated on T and $\mathscr{F}_{LB} f$ is ε_W concentrated on W. Then*

$$\mu_\alpha(T)\hat{\mu}_\alpha(W) \geqslant \left(1 - \sqrt{\varepsilon_T^2 + \varepsilon_W^2}\right)^2. \tag{26}$$

Proof. Since $\|f\|_{2,\mu_\alpha} = \|\mathscr{F}_{LB}f\|_{2,\hat{\mu}_\alpha} = 1$ and $\varepsilon_T^2 + \varepsilon_W^2 < 1$, the measures of T and W must both be non-zero. Indeed, if not, then the ε_T-concentration of f implies that $\|f - P_T f\|_{2,\mu_\alpha} = \|f\|_{2,\hat{\mu}_\alpha} = 1 \leqslant \varepsilon_T$, which contradicts with $\varepsilon_T < 1$, likewise for $\mathscr{F}_{LB}(f)$.
We have two cases : if $\mu_\alpha(T)\hat{\mu}_\alpha(W) \geqslant 1$ then, in this case, relation (26) is obvious.
If $\mu_\alpha(T)\hat{\mu}_\alpha(W) < 1$ then relations (24) and (25) allow to conclude that

$$\|P_W P_T\| = \|P_W P_T\|_{L^2(\mathbb{K}) \to L^2(\mathbb{K})} \leqslant 1.$$

Furthermore, from Lemma 1, since $\|P_W P_T\| \leqslant \|P_W P_T\|_{HS}$, we get in this case that

$$\|P_W P_T\| = \|P_W P_T\|_{L^2(\mathbb{K}) \to L^2(\mathbb{K})} < 1.$$

According to Lemma 3.1 in [7], we have

$$\forall f \in L^2(\mathbb{K}), \qquad \|f\|_{2,\mu_\alpha}^2 \leqslant (1 - \|P_W P_T\|)^{-2}(\Big(\|f\chi_{T^c}\|_{2,\mu_\alpha}^2 + \|\mathscr{F}_{LB}f\chi_{W^c}\|_{2,\hat{\mu}_\alpha}^2\Big).$$

Then by Lemma 1, we obtain

$$\forall f \in L^2(\mathbb{K}), \qquad \|f\|_{2,\mu_\alpha}^2 \leqslant (1 - \sqrt{\mu_\alpha(T)\hat{\mu}_\alpha(W)})^{-2}(\Big(\|f\chi_{T^c}\|_{2,\mu_\alpha}^2 + \|\mathscr{F}_{LB}f\chi_{W^c}\|_{2,\hat{\mu}_\alpha}^2\Big).$$

Therefore if f is ε_T-concentrated on T and $\mathscr{F}_{LB}f$ is ε_W-concentrated on W then we get easily relation (26). $\square$

REMARK 2. Theorem 3 improves the analog of Donoho-Stark theorem which could be stated as

THEOREM 4. *Let $T \subset \mathbb{K}, W \subset \hat{\mathbb{K}}$ be measurable sets and suppose that $\|f\|_{2,\mu_\alpha} = \|\mathscr{F}_{LB}f\|_{2,\hat{\mu}_\alpha} = 1$. Assume that $\varepsilon_T + \varepsilon_W < 1$, f is ε_T-concentrated on T and $\mathscr{F}_{LB}f$ is ε_W concentrated on W. Then*

$$\mu_\alpha(T)\hat{\mu}_\alpha(W) \geqslant (1 - \varepsilon_T - \varepsilon_W)^2.$$

4.2. L^1-version of Donoho-Stark theorem

In the following we shall consider the case of $f \in L^1(\mathbb{K})$.
As in the $L^2(\mathbb{K})$ case, we say that $f \in L^1(\mathbb{K})$ is ε_T-concentrated to T if

$$\|f - P_T f\|_{1,\mu_\alpha} \leqslant \varepsilon_T \|f\|_{1,\mu_\alpha}. \tag{27}$$

Let $B_{\mu_\alpha,1}(W)$ denote the subspace of $L^1(\mathbb{K})$ which consists of all $g \in L^1(\mathbb{K})$ where $P_W g = g$. We say that f is ε_W-bandlimited on W if there is a $g \in B_{\mu_\alpha,1}(W)$ with

$$\|f - g\|_{1,\mu_\alpha} \leqslant \varepsilon_W \|f\|_{1,\mu_\alpha}. \tag{28}$$

Here we denote by $\|P_T\|_{1,W}$ the operator norm of

$$P_T : B_{\mu_\alpha,1}(W) \longrightarrow L_{1,\mu_\alpha}.$$

We have the following lemma :

LEMMA 3.

$$\|P_T\|_{1,W} \leqslant \mu_\alpha(T)\hat{\mu}_\alpha(W). \tag{29}$$

Proof. Let $f \in B_{\mu_\alpha,1}(W)$. We can notice that

$$f = \mathscr{F}_{LB}^{-1}(\chi_W \mathscr{F}_{LB} f)$$

Then by Fubini's theorem, we get

$$f(x,t) = \int_{\mathbb{K}} f(y,s)\left(\int_W \varphi_{-\lambda,m}(y,s)\varphi_{\lambda,m}(x,t)d\hat{\mu}_\alpha(\lambda,m)\right)d\mu_\alpha(x,t).$$

According to relation (8), we have

$$\|f\|_{\infty,\mu_\alpha} \leqslant \mu_\alpha(W)\|f\|_{1,\mu_\alpha}.$$

Therefore

$$\begin{aligned}\|P_T f\|_{1,\mu_\alpha} &= \int_T |f(x,t)|d\mu_\alpha(x,t)\\ &\leqslant \mu_\alpha(T)\|f\|_{\infty,\mu_\alpha}\\ &\leqslant \mu_\alpha(T)\hat{\mu}_\alpha(W)\|f\|_{1,\mu_\alpha}.\end{aligned}$$

Thus

$$\|P_T\|_{1,W} = \sup_{f\in B_{\mu_\alpha,1}(W)} \frac{\|P_T f\|_{1,\mu_\alpha}}{\|f\|_{1,\mu_\alpha}} \leqslant \mu_\alpha(T)\hat{\mu}_\alpha(W). \quad \square$$

THEOREM 5. *Let $T \subset \mathbb{K}, W \subset \hat{\mathbb{K}}$ be measurable sets and $f \in L^1(\mathbb{K})$. If f is ε_T-concentrated to T and ε_W-bandlimited to W in $L^1(\mathbb{K})$ then*

$$\mu_\alpha(T)\hat{\mu}_\alpha(W) \geqslant \frac{1-\varepsilon_T-\varepsilon_W}{1+\varepsilon_W}. \tag{30}$$

Proof. Let $f \in L^1(\mathbb{K})$, we have

$$\|P_T f\|_{1,\mu_\alpha} \geqslant \|f\|_{1,\mu_\alpha} - \|f - P_T f\|_{1,\mu_\alpha}.$$

Since f is ε_T-concentrated on T then

$$\|P_T f\|_{1,\mu_\alpha} \geqslant (1-\varepsilon_T)\|f\|_{1,\mu_\alpha}.$$

Moreover, f is ε_W-bandlimited, there is a $g \in B_{\mu_\alpha,1}(W)$ with

$$\|g-f\|_{1,\mu_\alpha} \leqslant \varepsilon_W\|f\|_{1,\mu_\alpha}.$$

Therefore, it follows that

$$\begin{aligned}\|P_T g\|_{1,\mu_\alpha} &\geqslant \|P_T f\|_{1,\mu_\alpha} - \|P_T(g-f)\|_{1,\mu_\alpha}\\ &\geqslant (1-\varepsilon_T-\varepsilon_W)\|f\|_{1,\mu_\alpha}\end{aligned}$$

and

$$\begin{aligned}\|g\|_{1,\mu_\alpha} &\leqslant \|f\|_{1,\mu_\alpha} + \|g-f\|_{1,\mu_\alpha}\\ &= (1+\varepsilon_W)\|f\|_{1,\mu_\alpha}.\end{aligned}$$

Consequently,

$$\frac{\|P_T g\|_{1,\mu_\alpha}}{\|g\|_{1,\mu_\alpha}} \geqslant \frac{1-\varepsilon_T-\varepsilon_W}{1+\varepsilon_W}$$

which implies that

$$\|P_T\|_{1,W} \geqslant \frac{1-\varepsilon_T-\varepsilon_W}{1+\varepsilon_W}.$$

Using Lemma 3 we obtain the L^1-version of Donoho-Stark's theorem for Laguerre-Bessel transform. □

4.3. An uncertainty principle for $L^1(\mathbb{K})\cap L^2(\mathbb{K})$

Notice that the $L^1\cap L^2$-version of Donoho-Stark uncertainty principle was first proved in [8]. In this paper we give similarly the result : $L^1(\mathbb{K})\cap L^2(\mathbb{K})$-version of Donoho-Stark theorem for Laguerre-Bessel transform.

THEOREM 6. *Let $f\in L^1(\mathbb{K})\cap L^2(\mathbb{K})$. If f is ε_T-concentrated to T in $L^1(\mathbb{K})$ and $\mathscr{F}_{LB}(f)$ is ε_W-concentrated to W in $L^2(\hat{\mathbb{K}})$ then*

$$\mu_\alpha(T)\hat{\mu}_\alpha(W) \geqslant (1-\varepsilon_T)^2(1-\varepsilon_W)^2. \tag{31}$$

Proof. Assume that $\mu_\alpha(T)<\infty$ and $\widehat{\mu}_\alpha(W)<\infty$.
Let $f\in L^1(\mathbb{K})\cap L^2(\mathbb{K})$. Since $\mathscr{F}_{LB}(f)$ is ε_W-concentrated to W in $L^2(\hat{\mathbb{K}})$ then

$$\begin{aligned}\|f\|_{2,\mu_\alpha} &\leqslant \varepsilon_W\|f\|_{2,\mu_\alpha} + \Big(\int_{\mathbb{W}} |\mathscr{F}_{LB}(f)(\lambda)|^2 d\widehat{\mu}_\alpha(\lambda)\Big)^{1/2}\\ &\leqslant \varepsilon_W\|f\|_{2,\mu_\alpha} + \sqrt{\widehat{\mu}_\alpha(W)}\,\|\mathscr{F}_{LB}f\|_{\infty,\widehat{\mu}_\alpha}.\end{aligned}$$

Using relation (12), we obtain

$$(1-\varepsilon_W)\|f\|_{2,\mu_\alpha} \leqslant \sqrt{\widehat{\mu}_\alpha(W)}\|f\|_{1,\mu_\alpha}. \tag{32}$$

On the other hand, since f is ε_T-concentrated to T in $L^1(\mathbb{K})$ we get

$$\|f\|_{1,\mu_\alpha} \leqslant \varepsilon_T\|f\|_{1,\mu_\alpha} + \int_{\mathbb{T}} |f(\lambda)|d\mu_\alpha(\lambda)$$

Then, from Cauchy-Schwartz inequality, we get

$$(1-\varepsilon_T)\|f\|_{1,\mu_\alpha} \leqslant \sqrt{\mu_\alpha(T)}\|f\|_{2,\mu_\alpha}. \tag{33}$$

Combining (32) and (33) we obtain the result of this theorem. □

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