Mathematical
Inequalities
& Applications
Volume 22, Number 3 (2019), 837–853 doi:10.7153/mia-2019-22-58

APPROXIMATION BY MARCINKIEWICZ Θ–MEANS OF DOUBLE WALSH–FOURIER SERIES

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(*Communicated by T. Erdélyi*)

Abstract. In this article we discuss the behaviour of Θ-means of quadratical partial sums of double Walsh series of a function in $L^p(G^2)$ $(1 \leqslant p \leqslant \infty)$. In case $p = \infty$ by $L^p(G^2)$ we mean C, the collection of continuous functions on G^2. We present the rate of the approximation by Θ-means, in particular, in $\mathrm{Lip}(\alpha, p)$, where $\alpha > 0$ and $1 \leqslant p \leqslant \infty$.

Our main theorem generalizes two result of Nagy on Nörlund means and weighted means of the cubical partial sums of double Walsh-Fourier series [15, 16]. Specifically, we give the two-dimensional analogue of the two results of Móricz, Siddiqi on Nörlund means [14] and Móricz, Rhoades on weighted means [12].

1. One- and two-dimensional Walsh-Fourier series and summation methods

Now, we give a brief introduction to the Walsh-Fourier analysis [1, 18].

Let $\mathbb{P}$ be the set of positive natural numbers and $\mathbb{N} := \mathbb{P} \cup \{0\}$. Let G denote the Walsh group. The elements of Walsh group G are sequences of numbers 0 and 1, that is $x = (x_0, x_1, \ldots, x_k, \ldots)$ with $x_k \in \{0,1\}$ $(k \in \mathbb{N})$.

The group operation on G is the coordinate-wise addition modulo 2 (denoted by $+$), the normalized Haar measure is denoted by μ. Dyadic intervals are defined in usual way

$$I_0(x) := G, \ \ I_n(x) := \{y \in G : y = (x_0, \ldots, x_{n-1}, y_n, y_{n+1}, \ldots)\}$$

for $x \in G, n \in \mathbb{P}$. They form a base for the neighbourhoods of G. Let $0 = (0 : i \in \mathbf{N}) \in G$ denote the null element of G and $I_n := I_n(0)$ for $n \in \mathbb{N}$. Set $e_i := (0, \ldots, 0, 1, 0, \ldots)$, where the ith coordinate is 1 and the rest are 0 $(i \in \mathbb{N})$.

Let L^p denote the usual Lebesgue spaces on G (with the corresponding norm $\|.\|_p$). In the present paper we follow the notation of Móricz and Siddiqi [14]. For the sake of brevity in notation, we agree to write L^∞ instead of C, as Móricz and Siddiqi did, and set $\|f\|_\infty := \sup\{|f(x)| : x \in G\}$.

Mathematics subject classification (2010): 42C10.

Keywords and phrases: Walsh group, Walsh system, Walsh-Fourier series, Nörlund mean, weighted mean, approximation, modulus of continuity, Lipschitz function, Θ-mean, two-dimensional system, quadratical partial sum.

Research supported by TÁMOP-4.2.2.A-11/1/KONV-2012-0051, GINOP-2.2.1-15-2017-00055 and Shota Rustaveli National Science Foundation grant YS-18-043.

For $x \in G$ we define $|x|$ by

$$|x| := \sum_{i=0}^{\infty} \frac{x_i}{2^{i+1}}.$$

The modulus of continuity in $L^p, 1 \leqslant p \leqslant \infty$, of a function $f \in L^p$ is defined by

$$\omega_p(f,\delta) := \sup_{|t|<\delta} \|f(.+t) - f(.)\|_p, \quad \delta > 0.$$

The Lipschitz classes in L^p for each $\alpha > 0$ are defined by

$$\text{Lip}(\alpha,p) := \{f \in L^p : \omega_p(f,\delta) = O(\delta^\alpha) \text{ as } \delta \to 0\}.$$

For $x = (x^1,x^2) \in G^2$ we define $|x|$ by $|x|^2 := |x^1|^2 + |x^2|^2$. Thus, for $f \in L^p(G^2)$ $(1 \leqslant p \leqslant \infty)$ the modulus of continuity $\omega_p(f,\delta)$ and Lipschitz classes $\text{Lip}(\alpha,p)$ are well defined ($\delta > 0,\ \alpha > 0$). We define the mixed modulus of continuity as follows

$$\omega_{1,2}^p(f,\delta_1,\delta_2) :=$$
$$\sup\{\|f(.+x^1,.+x^2) - f(.+x^1,.) - f(.,.+x^2) + f(.,.)\|_p : |x^1| \leqslant \delta_1, |x^2| \leqslant \delta_2\},$$

where $\delta_1,\delta_2 > 0$.

The Rademacher functions are defined as

$$r_k(x) := (-1)^{x_k} \quad (x \in G, k \in \mathbb{N}).$$

The Walsh-Paley functions are defined by the help of Rademacher functions. That is, $w_0 = 1$ and for $n \geqslant 1$

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k},$$

where the natural number n is expressed in the number system based 2, in the form

$$n = \sum_{i=0}^{\infty} n_i 2^i, \quad n_i \in \{0,1\}\ (i \in \mathbb{N})$$

(in this expression only a finite number of n_i's different from zero). Let the order of $n > 0$ be denoted by $|n| := \max\{j \in \mathbb{N} : n_j \neq 0\}$. The Dirichlet kernels are defined by

$$D_n := \sum_{k=0}^{n-1} w_k,$$

where $n \in \mathbb{P},\ D_0 := 0$. The 2^nth Dirichlet kernels have a closed form (see e.g. [18])

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n, \\ 0, & \text{otherwise } (n \in \mathbb{N}). \end{cases} \tag{1}$$

It is also known that

$$D_{2^A+j}(x) = D_{2^A}(x) + r_A(x)D_j(x), \quad j = 0,1,\ldots,2^A-1. \tag{2}$$

(see [18]). The nth Fejér mean and Fejér kernel of Walsh-Fourier series are defined by

$$\sigma_n(f;x) := \frac{1}{n}\sum_{i=0}^{n-1} S_i(f;x), \quad K_n(x) := \frac{1}{n}\sum_{i=0}^{n-1} D_i(x)$$

In 2018, Toledo [20] improved Yano's [26] basic inequality. He proved that

$$\|K_n\|_1 \leqslant \frac{17}{15} \quad \text{for all } n \in \mathbb{N}. \tag{3}$$

A Sidon type inequality follows in the next lemma [13, Lemma 1], we will apply it, later.

LEMMA 1. (Móricz, Schipp [13]) *For every* $1 < p \leqslant 2$, *sequence* $\{a_k\}$ *of real numbers, and integer* $n \geqslant 1$,

$$\left\| \sum_{k=1}^{n} a_k D_k \right\|_1 \leqslant \frac{2p}{p-1} n^{1-1/p} \left[\sum_{k=1}^{n} |a_k|^p \right]^{1/p}.$$

On G^2 we consider the two-dimensional system as $\{w_{n^1}(x^1) \times w_{n^2}(x^2) : n := (n^1, n^2) \in \mathbb{N}^2\}$. The two-dimensional Fourier coefficients, the rectangular partial sums of the Fourier series and Dirichlet kernels are defined in the usual way. The nth Marcinkiewicz mean and Marcinkiewicz kernel of Walsh-Fourier series are defined by

$$\mathscr{M}_n(f;x^1,x^2) := \frac{1}{n}\sum_{i=0}^{n-1} S_{i,i}(f;x^1,x^2), \quad \mathscr{K}_n(x^1,x^2) := \frac{1}{n}\sum_{i=0}^{n-1} D_i(x^1)D_i(x^2).$$

Next lemma proved by Glukhov [8] is the two-dimensional analogue of Lemma 1 for $p = 2$.

LEMMA 2. (Glukhov [8]) *Let* $\alpha_1, \ldots, \alpha_n$ *be real numbers. Then*

$$\frac{1}{n}\left\| \sum_{k=1}^{n} \alpha_k D_k(.) D_k(..) \right\|_1 \leqslant \frac{c}{\sqrt{n}} \left(\sum_{k=1}^{n} \alpha_k^2 \right)^{1/2},$$

where c *is an absolute constant.*

As a corollary of Lemma 2 there exists a positive constant c such that

$$\|\mathscr{K}_n\|_1 \leqslant c \quad \text{for all } n \in \mathbb{N}. \tag{4}$$

Now, let us set the sequence of matrices Θ_n in the next form

$$\Theta_n := \begin{pmatrix} \theta_{0,1} & 0 & 0 & \dots & 0 \\ \theta_{0,2} & \theta_{1,2} & 0 & \dots & 0 \\ \theta_{0,3} & \theta_{1,3} & \theta_{2,3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \theta_{0,n} & \theta_{1,n} & \theta_{2,n} & \dots & \theta_{n-1,n} \end{pmatrix}$$

We always assume that $\theta_{0,k} = 1$ for all $k \in \{1,\dots,n\}$.

Let the nth (one-dimensional) Θ-mean and kernel be defined by

$$\sigma_n^{\Theta}(f;x) := \sum_{k=0}^{n-1} \theta_{k,n}\hat{f}(k)w_k(x), \quad K_n^{\Theta}(x) := \sum_{k=0}^{n-1} \theta_{k,n}w_k(x) \tag{5}$$

(see [5, 21]). It is easily seen that

$$\sigma_n^{\Theta}(f;x) := \int_G f(t)K_n^{\Theta}(t+x)d\mu(t).$$

Using Abel's transformation we immediately have that

$$\sigma_n^{\Theta}(f;x) = -\sum_{l=1}^{n} \Delta\theta_{l-1,n}S_l(f;x), \tag{6}$$

with the notation $\Delta\theta_{k,n} := \theta_{k+1,n} - \theta_{k,n}$ ($\theta_{n,n} = 0$) for $0 \leqslant k < n$. Let us set $\Delta^2\theta_{k,n} := \Delta\theta_{k+1,n} - \Delta\theta_{k,n}$, where $0 \leqslant k < n$ and $\theta_{n+1,n} := 0$ (it is natural, see the matrix Θ_{n+2}).

Taking into account equality (6) the nth Θ-mean and kernel of quadratical partial sums defined by

$$\begin{aligned} \sigma_n^{\Theta}(f;x^1,x^2) &= -\sum_{l=1}^{n} \Delta\theta_{l-1,n}S_{l,l}(f;x^1,x^2), \\ \mathscr{K}_n^{\Theta}(x^1,x^2) &= -\sum_{l=1}^{n} \Delta\theta_{l-1,n}D_l(x^1)D_l(x^2). \end{aligned} \tag{7}$$

It is also called Marcinkiewicz Θ-summation of double Walsh-Fourier series of a function $f \in L^1(G^2)$ (see [24]).

EXAMPLE 1. *Let $\{q_n : n \geqslant 0\}$ be a sequence of nonnegative numbers. Let us set $Q_n := \sum_{k=0}^{n-1} q_k$ $(n \geqslant 1)$. (It is always assumed that $q_0 > 0$ and $\lim_{n\to\infty} Q_n = \infty$.) If we choose $\theta_{k,n} = \frac{\sum_{i=0}^{n-k-1} q_i}{Q_n}$ $(0 \leqslant k \leqslant n-1)$, taking into account equality (7), we immediately have $\sigma_n^{\Theta}(f) = \sum_{k=1}^{n} \frac{q_{n-k}}{Q_n} S_{k,k}(f)$. It means that Nörlund-mean of quadratical partial sums is a special Θ-mean of quadratical partial sums.*

For the one-dimensional Nörlund means of Walsh-Fourier series of a function f in L^p $(1 \leqslant p \leqslant \infty)$ the rate of the approximation was given in terms of modulus of continuity [14]. In particular, functions in $\mathrm{Lip}(\alpha,p)$, where $\alpha > 0$ and $1 \leqslant p \leqslant \infty$ were considered, also. As special cases Móricz and Siddiqi obtained the earlier results on the rate of the approximation by Cesàro means given by Yano [27], Jastrebova [10] and Skvortsov [19]. The approximation properties of the Cesàro means of negative order was studied by Goginava in 2002 [9]. Recently, Fridli, Manchanda and Siddiqi generalized the result of Móricz and Siddiqi for homogeneous Banach spaces and dyadic Hardy spaces [7]. A few years ago the second author investigated the rate of the approximation by Nörlund means of quadratical partial sums of double Walsh-Fourier series for functions in the space $L^p(G^2)$ $(1 \leqslant p \leqslant \infty)$ [15]. In 2012, the general Nörlund mean method in dimension two was discussed [17], also. Recently, the first author, Baramidze, Memić, Persson, Tephnadze and Wall have some new results with respect to this topic [2, 4, 11].

EXAMPLE 2. *Let $\{p_n : n \geqslant 1\}$ be a sequence of nonnegative numbers. (It is always assumed that $p_1 > 0$ and $\lim_{n\to\infty} P_n = \infty$, which is the condition for regularity.) If we choose $\theta_{k,n} = \frac{\sum_{i=k+1}^{n} p_i}{P_n}$ $(0 \leqslant k \leqslant n-1)$, taking into account equality (7), we get $\sigma_n^{\Theta}(f) = \frac{1}{P_n}\sum_{k=1}^{n} p_k S_{k,k}(f)$. It means that weighted mean of Marcinkiewicz type is a special Θ-mean of Marcinkiewicz type.*

The rate of the approximation by weighted means of one-dimensional Walsh-Fourier series of a function in L^p $(1 \leqslant p \leqslant \infty)$ was presented in terms of modulus of continuity [12]. In particular, functions in $\mathrm{Lip}(\alpha, p)$, where $\alpha > 0$ and $1 \leqslant p \leqslant \infty$ were considered, also. As special cases Móricz and Rhoades obtained the earlier results given by Yano [27], Jastrebova [10] on the rate of the approximation by Cesàro means.

In 2010, the second author discussed the rate of the approximation by weighted means of quadratical partial sums of two-dimensional Walsh-Fourier series for functions in $L^p(G^2)$ $(1 \leqslant p \leqslant \infty)$ [16].

Our work is motivated by the paper of Móricz, Siddiqi [14] on Nörlund mean method and the paper of Móricz, Rhoades [12] on weighted mean method. Both of them present the result for one-dimensional Walsh-Fourier series. Recently, the results in papers [12, 14] were generalized by the authors in paper [3]. Namely the approximation properties of one-dimensional θ-mean was discussed. It is important to note that some ideas are coming from the paper of Chripkó [5]. She studied the order of convergence of Θ-mean with respect to Jacobi-Fourier series.

Our main aim is to investigate the rate of the approximation of Marcinkiewicz Θ-mean in terms of modulus of continuity under some general conditions. Our main theorem (Theorem 1) give a common generalization of two result of the second author [15, 16] (see Example 1 and 2). Specifically, we give the two-dimensional analogue of two results of Móricz, Siddiqi on Nörlund means [14] and Móricz, Rhoades on weighted means [12]. Moreover, we present some new results under general conditions for Marcinkiewicz Θ-summability.

It is important to note that other aspects of Θ-summability methods with respect to Walsh-Fourier series are treated in [21, 22, 23, 24].

2. Auxiliary results

Let $\mathscr{P}_n$ be the collection of one-dimensional Walsh polynomials of order less than n, that is, functions of the form

$$P(x)=\sum_{k=0}^{n-1} a_k w_k(x),$$

where $n\geqslant 1$ and $\{a_k\}$ is a sequence of real numbers. On G^2 we consider the two-dimensional Walsh polynomials of order less than (n,n) as

$$T(x^1,x^2):=\sum_{k=1}^{n}\alpha_k D_k(x^1)D_k(x^2),$$

where $n\geqslant 1$ and $\{\alpha_k\}$ is a sequence of real numbers. We note that not every two-dimensional Walsh-polynomial can be written in this form. The set of this special type two-dimensional polynomials are denoted by $\mathscr{P}_{n,n}$.

The next Lemma can be derived from the method presented in [15, page 313-314].

LEMMA 3. (Nagy [15]) *Let $P\in\mathscr{P}_{2^A,2^A}$, $f\in L^p(G^2)$, where $A,B\in\mathbb{P}$ and $1\leqslant p\leqslant\infty$. Then there exists a positive constant c such that*

$$\left\|\int_{G^2}(f(.+x)-f(.))r_A(x^1)r_A(x^2)P(x)d\mu(x)\right\|_p\leqslant c\|P\|_1\omega^p_{1,2}(f,2^{-A},2^{-A}),$$

with the notation $x=(x^1,x^2)\in G^2$.

As specially it is proved that

$$\left\|\int_{G^2}(f(.+x)-f(.))r_A(x^1)r_A(x^2)\mathscr{K}_j(x)d\mu(x)\right\|_p\leqslant c\omega^p_{1,2}(f,2^{-A},2^{-A}),$$

for $|j|\leqslant A$.

We need the next Lemma proved in [17].

LEMMA 4. (Nagy [17]) *Let $P\in\mathscr{P}_{2^A}$, $f\in L^p(G^2)$ $(1\leqslant p\leqslant\infty)$ and $A\in\mathbb{P}$. Then there exists a positive constant c such that*

$$\left\|\int_{G^2}(f(.+x)-f(.))D_{2^A}(x^2)r_A(x^1)P(x^1)d\mu(x)\right\|_p\leqslant c\|P\|_1\omega_p(f,2^{-A}).$$

For two-dimensional variable $(x^1,x^2)\in G^2$ we use the notations

$$r^1_n(x^1,x^2):=r_n(x^1),\quad D^1_n(x^1,x^2):=D_n(x^1),\quad K^1_n(x^1,x^2):=K_n(x^1),$$
$$r^2_n(x^1,x^2):=r_n(x^2),\quad D^2_n(x^1,x^2):=D_n(x^2),\quad K^2_n(x^1,x^2):=K_n(x^2),$$

for any $n\in\mathbb{N}$.

LEMMA 5. *Let $n > 2$ be a positive number, then we have*

$$\begin{aligned}
\mathscr{K}_n^{\Theta} = &-\sum_{j=0}^{|n|-1}\sum_{k=0}^{2^j-1}\Delta\theta_{2^j+k-1,n}D^1_{2^j}D^2_{2^j} - \sum_{k=0}^{n-2^{|n|}}\Delta\theta_{2^{|n|}+k-1,n}D^1_{2^{|n|}}D^2_{2^{|n|}}\\
&+\sum_{j=0}^{|n|-1}D^2_{2^j}r^1_j\sum_{k=0}^{2^j-2}\Delta^2\theta_{2^j+k-1,n}(k+1)K^1_{k+1} - \sum_{j=0}^{|n|-1}D^2_{2^j}r^1_j\Delta\theta_{2^{j+1}-2,n}2^jK^1_{2^j}\\
&+\sum_{j=0}^{|n|-1}D^1_{2^j}r^2_j\sum_{k=0}^{2^j-2}\Delta^2\theta_{2^j+k-1,n}(k+1)K^2_{k+1} - \sum_{j=0}^{|n|-1}D^1_{2^j}r^2_j\Delta\theta_{2^{j+1}-2,n}2^jK^2_{2^j}\\
&+\sum_{j=0}^{|n|-1}r^1_jr^2_j\sum_{k=0}^{2^j-2}\Delta^2\theta_{2^j+k-1,n}(k+1)\mathscr{K}_{k+1} - \sum_{j=0}^{|n|-1}r^1_jr^2_j\Delta\theta_{2^{j+1}-2,n}2^j\mathscr{K}_{2^j}\\
&-D^2_{2^{|n|}}r^1_{|n|}R^1_n - D^1_{2^{|n|}}r^2_{|n|}R^2_n - r^1_{|n|}r^2_{|n|}\mathscr{R}_n,
\end{aligned}$$

with the notation $R_n = \sum_{k=0}^{n-2^{|n|}}\Delta\theta_{2^{|n|}+k-1,n}D_k$ *and* $\mathscr{R}_n = \sum_{k=0}^{n-2^{|n|}}\Delta\theta_{2^{|n|}+k-1,n}D^1_kD^2_k$.

Proof. First, we use equality (2) for $\mathscr{K}_n^{\Theta}$ (see equality (7), too)

$$\begin{aligned}
\mathscr{K}_n^{\Theta} &= -\sum_{j=0}^{|n|-1}\sum_{l=2^j}^{2^{j+1}-1}\Delta\theta_{l-1,n}D^1_lD^2_l - \sum_{l=2^{|n|}}^{n}\Delta\theta_{l-1,n}D^1_lD^2_l\\
&= -\sum_{j=0}^{|n|-1}\sum_{k=0}^{2^j-1}\Delta\theta_{2^j+k-1,n}D^1_{2^j+k}D^2_{2^j+k} - \sum_{k=0}^{n-2^{|n|}}\Delta\theta_{2^{|n|}+k-1,n}D^1_{2^{|n|}+k}D^2_{2^{|n|}+k}\\
&= -\sum_{j=0}^{|n|-1}\sum_{k=0}^{2^j-1}\Delta\theta_{2^j+k-1,n}D^1_{2^j}D^2_{2^j} - \sum_{j=0}^{|n|-1}\sum_{k=0}^{2^j-1}\Delta\theta_{2^j+k-1,n}r^1_jD^1_kD^2_{2^j}\\
&\quad -\sum_{j=0}^{|n|-1}\sum_{k=0}^{2^j-1}\Delta\theta_{2^j+k-1,n}D^1_{2^j}r^2_jD^2_k - \sum_{j=0}^{|n|-1}\sum_{k=0}^{2^j-1}\Delta\theta_{2^j+k-1,n}r^1_jr^2_jD^1_kD^2_k\\
&\quad -\sum_{k=0}^{n-2^{|n|}}\Delta\theta_{2^{|n|}+k-1,n}D^1_{2^{|n|}}D^2_{2^{|n|}} - D^2_{2^{|n|}}r^1_{|n|}\sum_{k=0}^{n-2^{|n|}}\Delta\theta_{2^{|n|}+k-1,n}D^1_k\\
&\quad -D^1_{2^{|n|}}r^2_{|n|}\sum_{k=0}^{n-2^{|n|}}\Delta\theta_{2^{|n|}+k-1,n}D^2_k - r^1_{|n|}r^2_{|n|}\sum_{k=0}^{n-2^{|n|}}\Delta\theta_{2^{|n|}+k-1,n}D^1_kD^2_k\\
&=: \sum_{l=1}^{8}K_n^{\Theta,l}.
\end{aligned}$$

For the expression $K_n^{\Theta,2}$, $K_n^{\Theta,3}$ and $K_n^{\Theta,4}$ we use Abel's transformation

$$\begin{aligned}
K_n^{\Theta,2} &= -\sum_{j=0}^{|n|-1} D_{2^j}^2 r_j^1 \sum_{k=0}^{2^j-1} \Delta\theta_{2^j+k-1,n} D_k^1 \\
&= -\sum_{j=0}^{|n|-1} D_{2^j}^2 r_j^1 \left(\sum_{k=0}^{2^j-2} \left(\Delta\theta_{2^j+k-1,n} - \Delta\theta_{2^j+k,n}\right) \sum_{i=0}^{k} D_i^1 + \Delta\theta_{2^{j+1}-2,n} \sum_{k=0}^{2^j-1} D_k^1 \right) \\
&= \sum_{j=0}^{|n|-1} D_{2^j}^2 r_j^1 \left(\sum_{k=0}^{2^j-2} \Delta^2\theta_{2^j+k-1,n}(k+1)K_{k+1}^1 - \Delta\theta_{2^{j+1}-2,n} 2^j K_{2^j}^1 \right),
\end{aligned}$$

($K_n^{\Theta,3}$ has got a similar form) and

$$\begin{aligned}
K_n^{\Theta,4} &= -\sum_{j=0}^{|n|-1} r_j^1 r_j^2 \sum_{k=0}^{2^j-1} \Delta\theta_{2^j+k-1,n} D_k^1 D_k^2 \\
&= -\sum_{j=0}^{|n|-1} r_j^1 r_j^2 \left(\sum_{k=0}^{2^j-2} \left(\Delta\theta_{2^j+k-1,n} - \Delta\theta_{2^j+k,n}\right) \sum_{i=0}^{k} D_i^1 D_i^2 + \Delta\theta_{2^{j+1}-2,n} \sum_{k=0}^{2^j-1} D_k^1 D_k^2 \right) \\
&= \sum_{j=0}^{|n|-1} r_j^1 r_j^2 \left(\sum_{k=0}^{2^j-2} \Delta^2\theta_{2^j+k-1,n}(k+1)\mathscr{K}_{k+1} - \Delta\theta_{2^{j+1}-2,n} 2^j \mathscr{K}_{2^j} \right).
\end{aligned}$$

Summarising our results on the expressions $K_n^{\Theta,1},\dots,K_n^{\Theta,8}$, we complete the proof. □

3. The rate of the approximation by Θ-mean of cubical partial sums

In the next theorem the coefficients $\theta_{k,n} \in [0,1]$ for all $k,n \in \mathbb{N}$.

THEOREM 1. *Let $f \in L^p(G^2)$ $(1 \leqslant p \leqslant \infty)$. Let $n > 2$ be a positive integer. Let the finite sequence $\{\theta_{k,n} : 0 \leqslant k \leqslant n-1\}$ of nonnegative numbers be nonincreasing (in sign $\theta_{k,n} \downarrow$).*

a.) Let the finite sequence of differences $\{\Delta\theta_{k,n} : 0 \leqslant k < n\}$ be nonincreasing (in sign $\Delta\theta_{k,n} \downarrow$). We suppose that

$$\theta_{n-1,n} = O\left(\frac{1}{n}\right). \tag{8}$$

Then there exists a positive constant c such that

$$\|\sigma_n^\Theta(f) - f\|_p \leqslant c \sum_{j=0}^{|n|-1} 2^j |\Delta\theta_{2^{j+1}-2,n}| \omega_p\left(f, 2^{-j}\right) + O(\omega_p(f, 2^{-|n|})). \tag{9}$$

b.) Let the finite sequence of differences $\{\Delta\theta_{k,n} : 0 \leqslant k < n\}$ be nondecreasing (in sign $\Delta\theta_{k,n} \uparrow$). Then there exists a positive constant c such that

$$\|\sigma_n^\Theta(f) - f\|_p \leqslant c \sum_{j=0}^{|n|-1} 2^j |\Delta\theta_{2^j-1,n}| \omega_p\left(f, 2^{-j}\right) + O(\omega_p(f, 2^{-|n|})). \tag{10}$$

REMARK 1. *The condition* $0 \leqslant \theta_{k,n} \leqslant 1$ *for all* $k \in \{0,\dots,n-1\}$ *and* $n \in \mathbb{P}$ *is a usual condition, since in Example 1 and 2 it is satisfied.*

For Example 1, easy to see that $\Delta\theta_{2^j-1,n} = -\frac{q_{n-2^j}}{Q_n}$ *and* $\Delta\theta_{2^{j+1}-2,n} = -\frac{q_{n-2^{j+1}+1}}{Q_n}$. *Thus, as a consequence of our main theorem we get back an analogical form of result of second author on Nörlund means of Marcinkiewicz type* [15].

For Example 2, $\Delta\theta_{2^j-1,n} = -\frac{p_{2^j}}{P_n}$ *and* $\Delta\theta_{2^{j+1}-2,n} = -\frac{p_{2^{j+1}-1}}{P_n}$ *hold. Thus, as a consequence of our theorem we have an analogical form of the result of Nagy on weighted means of Marcinkiewicz type* [16].

Proof of Theorem 1. We carry out the proof for $1 \leqslant p < \infty$, for $p = \infty$ the proof is similar (where $L^\infty = C$). During this proof c denotes a positive constant, which may vary at different appearances. Keeping in mind that $\theta_{0,k} = 1$ for all k, we use Lemma 5 and the usual Minkowski's inequality

$$\begin{aligned}
\|\sigma_n^\Theta(f) - f\|_p &= \left(\int_{G^2} |\sigma_n^\Theta(f,x) - f(x)|^p d\mu(x)\right)^{\frac{1}{p}} \\
&= \left(\int_{G^2} \left|\int_{G^2} \mathscr{K}_n^\Theta(t)(f(x+t) - f(x))d\mu(t)\right|^p d\mu(x)\right)^{\frac{1}{p}} \\
&\leqslant \sum_{k=1}^{8} \left(\int_{G^2} \left|\int_{G^2} K_n^{\Theta,k}(t)(f(x+t) - f(x))d\mu(t)\right|^p d\mu(x)\right)^{\frac{1}{p}} \\
&=: \sum_{k=1}^{8} I_{k,n}.
\end{aligned}$$

Using generalized Minkowski's inequality ([28], vol. 1, p. 19) for the expressions $I_{1,n}$ and $I_{5,n}$, we obtain

$$\begin{aligned}
I_{1,n} &\leqslant \sum_{j=0}^{|n|-1} \left|\sum_{k=0}^{2^j-1} \Delta\theta_{2^j+k-1,n}\right| \int_{G^2} D_{2^j}(t^1) D_{2^j}(t^2) \left(\int_{G^2} |f(x+t) - f(x)|^p d\mu(x)\right)^{\frac{1}{p}} d\mu(t) \\
&\leqslant c \sum_{j=0}^{|n|-1} \left|\sum_{k=0}^{2^j-1} \Delta\theta_{2^j+k-1,n}\right| \omega_p\left(f, 2^{-j}\right), \qquad (11)
\end{aligned}$$

and

$$\begin{aligned}
I_{5,n} &\leqslant \left|\sum_{k=0}^{n-2^{|n|}} \Delta\theta_{2^{|n|}+k-1,n}\right| \int_{G^2} D_{2^{|n|}}(t^1) D_{2^{|n|}}(t^2) \left(\int_{G^2} |f(x+t) - f(x)|^p d\mu(x)\right)^{\frac{1}{p}} d\mu(t) \\
&\leqslant c \left|\sum_{k=0}^{n-2^{|n|}} \Delta\theta_{2^{|n|}+k-1,n}\right| \omega_p\left(f, 2^{-|n|}\right). \qquad (12)
\end{aligned}$$

In case a.) (in sign $\Delta\theta_{k,n} \downarrow$) we write $\left|\sum_{k=0}^{2^j-1} \Delta\theta_{2^j+k-1,n}\right| \leqslant -2^j \Delta\theta_{2^{j+1}-2,n}$ and

$$I_{1,n} \leqslant c \sum_{j=0}^{|n|-1} 2^j |\Delta\theta_{2^{j+1}-2,n}| \omega_p\left(f, 2^{-j}\right).$$

In case b.) (in sign $\Delta\theta_{k,n}\uparrow$) we have $\left|\sum_{k=0}^{2^j-1}\Delta\theta_{2^j+k-1,n}\right|\leqslant -2^j\Delta\theta_{2^j-1,n}$ and

$$I_{1,n}\leqslant -\sum_{j=0}^{|n|-1}2^j\Delta\theta_{2^j-1,n}\omega_p\left(f,2^{-j}\right).$$

Since, $\left|\sum_{k=0}^{n-2^{|n|}}\Delta\theta_{2^{|n|}+k-1,n}\right|=\theta_{2^{|n|}-1,n}-\theta_{n,n}\leqslant 1$, in case a.) and b.) we immediately write

$$I_{5,n}\leqslant c\omega_p\left(f,2^{-|n|}\right).$$

For the expression $I_{2,n}$ usual Minkowski's inequality yields

$$\begin{aligned}
I_{2,n} \leqslant\ & \sum_{j=0}^{|n|-1}\sum_{k=0}^{2^j-2}\left|\Delta^2\theta_{2^j+k-1,n}\right|(k+1)\\
&\cdot\left(\int_{G^2}\left|\int_{G^2}D_{2^j}(t^2)r_j(t^1)K_{k+1}(t^1)(f(x+t)-f(x))d\mu(t)\right|^p d\mu(x)\right)^{\frac{1}{p}}\\
&+\sum_{j=0}^{|n|-1}|\Delta\theta_{2^{j+1}-2,n}|2^j\\
&\cdot\left(\int_{G^2}\left|\int_{G^2}D_{2^j}(t^2)r_j(t^1)K_{2^j}(t^1)(f(x+t)-f(x))d\mu(t)\right|^p d\mu(x)\right)^{\frac{1}{p}}\\
=:\ & I_{2,n}^1+I_{2,n}^2.
\end{aligned}$$

From Lemma 4 and inequality (3) we write

$$\begin{aligned}
I_{2,n}^1 &\leqslant c\sum_{j=0}^{|n|-1}\sum_{k=0}^{2^j-2}\left|\Delta^2\theta_{2^j+k-1,n}\right|(k+1)\|K_{k+1}\|_1\omega_p(f,2^{-j})\\
&\leqslant c\sum_{j=0}^{|n|-1}\sum_{k=0}^{2^j-2}\left|\Delta^2\theta_{2^j+k-1,n}\right|(k+1)\omega_p(f,2^{-j}) \qquad (13)
\end{aligned}$$

and

$$\begin{aligned}
I_{2,n}^2 &\leqslant c\sum_{j=0}^{|n|-1}|\Delta\theta_{2^{j+1}-2,n}|2^j\|K_{2^j}\|_1\omega_p(f,2^{-j}) \qquad (14)\\
&\leqslant c\sum_{j=0}^{|n|-1}|\Delta\theta_{2^{j+1}-2,n}|2^j\omega_p(f,2^{-j}). \qquad (15)
\end{aligned}$$

At first, we deal with expression $I^1_{2,n}$. In case a.) (in sign $\Delta\theta_{k,n}\downarrow$),

$$\begin{aligned}\sum_{k=0}^{2^j-2}\left|\Delta^2\theta_{2^j+k-1,n}\right|(k+1) &= \sum_{k=0}^{2^j-2}(\Delta\theta_{2^j+k-1,n}-\Delta\theta_{2^j+k,n})(k+1)\\ &= \sum_{k=0}^{2^j-2}\Delta\theta_{2^j+k-1,n}-(2^j-1)\Delta\theta_{2^{j+1}-2,n}\\ &\leqslant -2^j\Delta\theta_{2^{j+1}-2,n}\end{aligned}$$

and

$$I^1_{2,n}\leqslant c\sum_{j=0}^{|n|-1}2^j|\Delta\theta_{2^{j+1}-2,n}|\omega_p\left(f,2^{-j}\right).$$

In case b.) (in sign $\Delta\theta_{k,n}\uparrow$) we have

$$\begin{aligned}\sum_{k=0}^{2^j-2}\left|\Delta^2\theta_{2^j+k-1,n}\right|(k+1) &= (2^j-1)\Delta\theta_{2^{j+1}-2,n}-\sum_{k=0}^{2^j-2}\Delta\theta_{2^j+k-1,n}\\ &\leqslant -\sum_{k=0}^{2^j-2}\Delta\theta_{2^j+k-1,n}\leqslant -2^j\Delta\theta_{2^j-1,n}\end{aligned}$$

and

$$I^1_{2,n}\leqslant c\sum_{j=0}^{|n|-1}2^j|\Delta\theta_{2^j-1,n}|\omega_p\left(f,2^{-j}\right).$$

Now, we discuss expression $I^2_{2,n}$. In case a.) (in sign $\Delta\theta_{k,n}\downarrow$), we immediately write

$$I^2_{2,n}\leqslant c\sum_{j=0}^{|n|-1}2^j|\Delta\theta_{2^{j+1}-2,n}|\omega_p\left(f,2^{-j}\right).$$

In case b.) (in sign $\Delta\theta_{k,n}\uparrow$) we have

$$I^2_{2,n}\leqslant c\sum_{j=0}^{|n|-1}2^j|\Delta\theta_{2^j-1,n}|\omega_p\left(f,2^{-j}\right).$$

We discuss expression $I_{3,n}$ analogously. For expression $I_{4,n}$ we apply usual Minkowski's inequality

$$\begin{aligned}I_{4,n} \leqslant& \sum_{j=0}^{|n|-1}\sum_{k=0}^{2^j-2}\left|\Delta^2\theta_{2^j+k-1,n}\right|(k+1)\\&\cdot\left(\int_{G^2}\left|\int_{G^2} r_j(t^1)r_j(t^2)\mathscr{K}_{k+1}(t)(f(x+t)-f(x))d\mu(t)\right|^p d\mu(x)\right)^{\frac{1}{p}}\\&+\sum_{j=0}^{|n|-1}|\Delta\theta_{2^{j+1}-2,n}|2^j\\&\cdot\left(\int_{G^2}\left|\int_{G^2} r_j(t^1)r_j(t^2)\mathscr{K}_{2^j}(t)(f(x+t)-f(x))d\mu(t)\right|^p d\mu(x)\right)^{\frac{1}{p}}\\=:&\, I_{4,n}^1+I_{4,n}^2.\end{aligned}$$

By Lemma 3 and inequality (4) we immediately have

$$\begin{aligned}I_{4,n}^1 &\leqslant c\sum_{j=0}^{|n|-1}\sum_{k=0}^{2^j-2}\left|\Delta^2\theta_{2^j+k-1,n}\right|(k+1)\|\mathscr{K}_{k+1}\|_1\omega_{1,2}^p(f,2^{-j},2^{-j})\\&\leqslant c\sum_{j=0}^{|n|-1}\sum_{k=0}^{2^j-2}\left|\Delta^2\theta_{2^j+k-1,n}\right|(k+1)\omega_p(f,2^{-j})\end{aligned}\tag{16}$$

and

$$\begin{aligned}I_{4,n}^2 &\leqslant c\sum_{j=0}^{|n|-1}|\Delta\theta_{2^{j+1}-2,n}|2^j\|\mathscr{K}_{2^j}\|_1\omega_{1,2}^p(f,2^{-j},2^{-j})\\&\leqslant c\sum_{j=0}^{|n|-1}|\Delta\theta_{2^{j+1}-2,n}|2^j\omega_p(f,2^{-j}).\end{aligned}\tag{17}$$

In this point we can apply the same methods for $I_{4,n}^1$ and $I_{4,n}^2$ as we used for the expressions $I_{2,n}^1$ and $I_{2,n}^2$, respectively.

Now, we discuss the expression $I_{6,n}$ (we discuss $I_{7,n}$ analogously). Lemma 4 yields

$$\begin{aligned}I_{6,n} &= \left(\int_{G^2}\left|\int_{G^2} D_{2^{|n|}}(t^1)r_{|n|}(t^2)R_n(t^2)(f(x+t)-f(x))d\mu(t)\right|^p d\mu(x)\right)^{1/p}\\&\leqslant c\|R_n\|_1\omega_p(f,2^{-|n|}).\end{aligned}\tag{18}$$

At last, by Lemma 3 we write

$$\begin{aligned}I_{8,n} &= \left(\int_{G^2}\left|\int_{G^2} r_{|n|}(t^1)r_{|n|}(t^2)\mathscr{R}_n(t)(f(x+t)-f(x))d\mu(t)\right|^p d\mu(x)\right)^{1/p}\\&\leqslant c\|\mathscr{R}_n\|_1\omega_{1,2}^p(f,2^{-|n|},2^{-|n|})\leqslant c\|\mathscr{R}_n\|_1\omega_p(f,2^{-|n|}).\end{aligned}\tag{19}$$

Lemma 1 with $p=2$ implies that

$$\|R_n\|_1 \leqslant c \quad \text{for all } n \in \mathbb{P} \tag{20}$$

and Lemma 2 yields that

$$\|\mathscr{R}_n\|_1 \leqslant c \quad \text{for all } n \in \mathbb{P} \tag{21}$$

in both cases a.) and b.). Namely, denote $\|\mathscr{R}_n\|_1$ or $\|R_n\|_1$ by H_n. From these lemmas we obtain

$$H_n \leqslant c(n-2^{|n|})^{1/2}\left[\sum_{k=0}^{n-2^{|n|}} |\Delta\theta_{2^{|n|}+k-1,n}|^2\right]^{1/2}. \tag{22}$$

Case a.) ($\Delta\theta_{k,n}\downarrow$) then using condition (8)

$$H_n \leqslant c(n-2^{|n|}+1)|\Delta\theta_{n-1,n}| \leqslant cn\theta_{n-1,n} \leqslant c.$$

In case b.) ($\Delta\theta_{k,n}\uparrow$) then

$$\sum_{k=0}^{n-2^{|n|}} |\Delta\theta_{2^{|n|}+k-1,n}|^2 \leqslant (n-2^{|n|}+1)|\Delta\theta_{2^{|n|}-1,n}|^2,$$

and $|\theta_{k,n}| \leqslant c$ (here $c=1$). Since $n-2^{|n|}+1 \leqslant 2^{|n|}$ we write

$$H_n \leqslant c(n-2^{|n|}+1)|\Delta\theta_{2^{|n|}-1,n}| \leqslant c(|\Delta\theta_{0,n}|+\ldots+|\Delta\theta_{2^{|n|}-1,n}|) \leqslant c(\theta_{0,n}-\theta_{2^{|n|},n}) \leqslant c.$$

This yields that the inequalities (20) and (21) are proved for all n. We immediately get

$$I_{6,n} \leqslant c\omega_p\left(f,2^{-|n|}\right) \quad \text{for all } n$$

and

$$I_{8,n} \leqslant c\omega_p\left(f,2^{-|n|}\right) \quad \text{for all } n.$$

This completes the proof. □

In the next Theorem we allow that the finite sequence $\{\theta_{k,n} : 0 \leqslant k \leqslant n-1\}$ has some negative values. Namely, $\theta_{k,n} \in [c_*,1]$ with a negative number c_*.

THEOREM 2. *Let $f \in L^p(G^2)$ $(1 \leqslant p \leqslant \infty)$. Let $n>2$ be a positive natural number. Let the finite sequence $\{\theta_{k,n} : 0 \leqslant k \leqslant n-1\}$ be nonincreasing (in sign $\theta_{k,n}\downarrow$) and $\theta_{n-1,n}<0$.*

a.) Let the finite sequence of differences $\{\Delta\theta_{k,n} : 0 \leqslant k \leqslant n-2\}$ be nonincreasing (in sign $\Delta\theta_{k,n}\downarrow$). Moreover, we suppose that

$$|\theta_{n-1,n}| = O\left(\frac{1}{\sqrt{n}}\right) \quad \text{and} \quad |\Delta\theta_{n-2,n}| = O\left(\frac{1}{n}\right). \tag{23}$$

Then there exists a positive constant c *such that*

$$\|\sigma_n^{\Theta}(f)-f\|_p \leqslant c\sum_{j=0}^{|n|-1} 2^j|\Delta\theta_{2^{j+1}-2,n}|\omega_p\left(f,2^{-j}\right)+O(\omega_p(f,2^{-|n|})). \tag{24}$$

b.) Let the finite sequence of differences $\{\Delta\theta_{k,n}: 0\leqslant k\leqslant n-2\}$ *be nondecreasing (in sign* $\Delta\theta_{k,n}\uparrow$*). Moreover, we suppose that there exists a negative constant* c_**, such that* $\theta_{n-1,n}\geqslant c_*$ *for all* n*. Then there exists a positive constant* c *such that*

$$\|\sigma_n^{\Theta}(f)-f\|_p \leqslant c\sum_{j=0}^{|n|-1} 2^j|\Delta\theta_{2^{j}-1,n}|\omega_p\left(f,2^{-j}\right)+O(\omega_p(f,2^{-|n|})) \tag{25}$$

holds.

Proof of Theorem 2. We make the proof for such a finite sequence $\{\theta_{k,n}: 0\leqslant k\leqslant n-1\}$ for which at least the last member $\theta_{n-1,n}$ is negative.

We use the method and notations of the proof given in Theorem 1.

$$\|\sigma_n^{\Theta}(f)-f\|_p \leqslant \sum_{k=1}^{8} I_{k,n}.$$

Since, the most part of the proof goes in the same way as above written (proofs for $I_{1,n}$, $I_{2,n}$, $I_{3,n}$ and $I_{4,n}$), we give details about the necessary changes.

For the expression $I_{5,n}$ we have inequality (12). Since, $\left|\sum_{k=0}^{n-2^{|n|}}\Delta\theta_{2^{|n|}+k-1,n}\right| = |\theta_{2^{|n|}-1,n}-\theta_{n,n}|\leqslant 1+|c'|$ (where c' is coming from condition (23)), in case a.) and $|\theta_{2^{|n|}-1,n}-\theta_{n,n}|\leqslant 1+|c_*|$ in case b.), we write

$$I_{5,n}\leqslant c\omega_p\left(f,2^{-|n|}\right).$$

For proving the necessary inequality for $I_{6,n}$ (and analogously for $I_{7,n}$) we get

$$I_{6,n}\leqslant c\|R_n\|_1\omega_p(f,2^{-|n|}). \tag{26}$$

from (18).

Lemma 1 with $p=2$ implies that

$$\|R_n\|_1\leqslant c \quad \text{for all } n\in\mathbb{P} \tag{27}$$

and Lemma 2 yields that

$$\|\mathscr{R}_n\|_1\leqslant c \quad \text{for all } n\in\mathbb{P} \tag{28}$$

in both cases a.) and b.). Namely, denote H_n the expressions $\|\mathscr{R}_n\|_1$ or $\|R_n\|_1$. From Lemmas 1. and 2. we obtain

$$H_n\leqslant c(n-2^{|n|})^{1/2}\left[\sum_{k=0}^{n-2^{|n|}}|\Delta\theta_{2^{|n|}+k-1,n}|^2\right]^{1/2}. \tag{29}$$

Case a.) $(\Delta\theta_{k,n}\downarrow)$

$$\sum_{k=0}^{n-2^{|n|}} |\Delta\theta_{2^{|n|}+k-1,n}|^2 \leqslant (n-2^{|n|})|\Delta\theta_{n-2,n}|^2 + |\Delta\theta_{n-1,n}|^2.$$

Using condition (23)

$$\begin{aligned} H_n &\leqslant c(n-2^{|n|})|\Delta\theta_{n-2,n}| + c(n-2^{|n|})^{1/2}|\theta_{n-1,n}| \\ &\leqslant cn|\Delta\theta_{n-2,n}| + cn^{1/2}|\theta_{n-1,n}| \leqslant c. \end{aligned}$$

In case b.) $(\Delta\theta_{k,n}\uparrow)$

$$H_n \leqslant c(n-2^{|n|}+1)|\Delta\theta_{2^{|n|}-1,n}| \leqslant c(\theta_{0,n}-\theta_{2^{|n|},n}) \leqslant c$$

(see the corresponding part in the proof of Theorem 1).

This yields that the inequality (27) and (28) are proved for all n. We immediately get

$$I_{6,n} \leqslant c\omega_p\left(f, 2^{-|n|}\right) \qquad \text{for all } n$$

and

$$I_{8,n} \leqslant c\omega_p\left(f, 2^{-|n|}\right) \qquad \text{for all } n.$$

This completes the proof of our theorem. □

THEOREM 3. *Let $f \in Lip(\alpha, p)$ for some $\alpha > 0$ and $1 \leqslant p \leqslant \infty$. For Θ-mean σ_n^Θ of quadratical partial sums we suppose that the conditions in Theorem 1 hold.*

In case Theorem 1 a.) and Theorem 2 a.) the next equality holds

$$\|\sigma_n^\Theta(f) - f\|_p = \begin{cases} O(n^{-\alpha}), & \text{if } 0 < \alpha < 1, \\ O(\log n/n), & \text{if } \alpha = 1, \\ O(1/n), & \text{if } \alpha > 1. \end{cases}$$

In case Theorem 1 b.), Theorem 2 b.) we have

$$\|\sigma_n^\Theta(f) - f\|_p = O\left(\sum_{j=0}^{|n|-1} |\Delta\theta_{2^j-1,n}|2^{j(1-\alpha)} + 2^{-|n|\alpha}\right).$$

Proof. The proof is similar to the proof of analogical theorem of Móricz and Siddiqi [14] (for more details see [12, 3]). □

REMARK 2. *Let us suppose that the finite sequence of $\{\theta_{k,n} : 0 \leqslant k < n-1\}$ is nondecreasing ($\theta_{k,n}\uparrow$) and bounded by a positive constant. Then Lemma 1 and Lemma 2 do not guarantee the uniform boundedness of the L_1-norm of kernels R_n and $\mathscr{R}_n$, in both cases $\Delta\theta_{k,n}\uparrow$ and $\Delta\theta_{k,n}\downarrow$. So, we do not discuss this case. That is, the situation is the same as in the one-dimensional case.*

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(Received May 14, 2018)

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