Mathematical
Inequalities
& Applications
Volume 22, Number 3 (2019), 855–883 doi:10.7153/mia-2019-22-59

COMMUTATORS OF POTENTIAL TYPE OPERATORS WITH LIPSCHITZ SYMBOLS ON VARIABLE LEBESGUE SPACES WITH DIFFERENT WEIGHTS

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(*Communicated by J. Soria*)

Abstract. We prove that a generalized Fefferman-Phong type condition on a pair of weights u and v is sufficient for the boundedness of the commutators of potential type operators from $L_v^{p(\cdot)}$ into $L_u^{q(\cdot)}$. We also give an improvement of this result in the sense that we not only consider a variable version of power bump conditions, but also weaker norms related to Musielak-Orlicz functions.

We consider a wider class of symbols including Lipschitz symbols and some generalizations.

1. Introduction and main results

In [17], E. Sawyer and R. Wheeden obtained Fefferman-Phong type conditions on a pair of weights in order to prove boundedness results for the fractional integral operator I_α, between Lebesgue spaces with different weights. For the case of one weight, remarkably simple conditions on the weight characterizing the boundedness of I_α were known to hold (see [13]). Motivated by the results above, in [14], C. Pérez considered weaker norms than those involved in the Fefferman-Phong type conditions in [17], and obtained two-weighted boundedness estimates for the potential operator T_K, formally defined by

$$T_K f(x) = \int_{\mathbb{R}^n} K(x-y) f(y)\, dy,$$

whenever this integral is finite where the kernel K is a non-negative and locally integrable function satisfying certain weak growth condition. This article was the motivation for a great variety of subsequent papers related to this kind of operator. For example, in [8] and [9], the authors obtained weighted L^p inequalities of Fefferman-Stein type for T_K and for the higher order commutators with BMO symbols associated

Mathematics subject classification (2010): 42B25.
Keywords and phrases: Potential operators, commutators, variable Lebesgue spaces.
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to this operator, respectively, whenever $1 < p < \infty$. If $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $m \in \mathbb{N}$, the commutator of order m of T_K is formally defined by

$$T_K^{b,m} f(x) = \int_{\mathbb{R}^n} (b(x) - b(y))^m K(x-y) f(y) dy, \tag{1}$$

whenever this integral is finite. In the multilinear context, similar results were proved in [1]. For these commutators two-weighted norm inequalities in the spirit of those given in [14] were proved in [7] in the classical L^p context, and in [11] on the general setting of variable Lebesgue spaces.

The commutators of fractional type operators with Lipschitz symbols were studied by several authors. For instance, in [12] the authors considered unweighted estimates for the mentioned operator acting between different Lebesgue spaces in the context of non-doubling measures.

Characterizations of Lipschitz functions via the boundedness of commutators of fractional integral operators with generalized Lipschitz symbols were given in [15], in the general setting of variable Lebesgue spaces.

In [2] the authors give weighted $L^p - L^q$ estimates for the commutators, with Lipschitz symbols, of a great variety of fractional type operators. Later, in [14] certain extrapolation techniques allow to obtain similar results in variable Lebesgue spaces.

The main aim of this paper is to describe the behavior of the commutators of the potential type operators $T_K^{b,m}$ between variable Lebesgue spaces with different weights, for a wider class of symbols b including Lipschitz symbols and some generalizations. Concretely, we prove that a generalized Fefferman-Phong type condition on a pair of weights u and v is sufficient for the boundedness of the commutator $T_K^{b,m}$, from $L_v^{p(\cdot)}$ into $L_u^{q(\cdot)}$.

When the symbol b belongs to a variable Lipschitz space, we not only consider variable version of power bump conditions, but also we consider weaker norms related to Musielak-Orlicz functions. Thus, in this sense, we are providing an improvement.

In the definition of $T_K^{b,m}$, the function K belongs to a certain class of kernels that satisfy that there exists positive constants δ, c and $0 \leqslant \varepsilon < 1$, with the property that

$$\sup_{2^k < |x| \leqslant 2^{k+1}} K(x) \leqslant \frac{c}{2^{kn}} \int_{\delta(1-\varepsilon)2^k < |y| \leqslant 2\delta(1+\varepsilon)2^k} K(y)\, dy,$$

for all $k \in \mathbb{Z}$. We shall denote this class by $\mathfrak{D}$.

For example, if K is radial an non-increasing, then $K \in \mathfrak{D}$. A basic example of potential operator with radial and non-increasing kernel K is given by the fractional integral operator I_α, which is the convolution with the kernel $K(t) = |t|^{\alpha - n}$, $0 < \alpha < n$. There are other important examples such as the Bessel potential $J_{\beta,\lambda}$, $\beta, \lambda > 0$ with kernels $K_{\beta,\lambda}$ best defined by means of its Fourier transform by

$$\widehat{K_{\beta,\lambda}}(\xi) = (\lambda^2 + |\xi|^2)^{-\beta/2}$$

and $K_{\beta,\lambda}$ is also radial and non-increasing.

Nevertheless, condition $\mathfrak{D}$ involves other type of kernels K such as radial and non-decreasing functions. Moreover, if K is essentially constant on annuli, that is, $K(y) \leqslant CK(x)$ for $|y|/2 \leqslant |x| \leqslant 2|y|$, then $K \in \mathfrak{D}$.

We will be working in a general context that we now introduce.

Let $p(\cdot) : \mathbb{R}^n \to [1,\infty]$ be a measurable function. For $\mathrm{A} \subset \mathbb{R}^n$ we define

$$p_{\mathrm{A}}^- = \inf_{x\in \mathrm{A}} p(x) \qquad\qquad p_{\mathrm{A}}^+ = \sup_{x\in \mathrm{A}} p(x).$$

For simplicity we denote $p^- = p_{\mathbb{R}^n}^-$ and $p^+ = p_{\mathbb{R}^n}^+$.

With $p'(\cdot) = p(\cdot)/(p(\cdot)-1)$ we denote the conjugate exponent of $p(\cdot)$. It is not hard to prove that $(p')^- = (p^+)'$ and $(p')^+ = (p^-)'$.

We say that $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is *globally log-Hölder continuous* on $\mathbb{R}^n$ if it satisfies the following inequalities

$$|\alpha(x) - \alpha(y)| \leqslant \frac{C}{\log(e + 1/|x-y|)},\ x,y \in \mathbb{R}^n$$

and

$$|\alpha(x) - \alpha_\infty| \leqslant \frac{C}{\log(e + |x|)},\ x \in \mathbb{R}^n \tag{2}$$

for some positive constants C and α_∞. It is easy to see that the inequality (2) implies that $\lim_{|x|\to\infty} \alpha(x) = \alpha_\infty$.

We say that $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ if $1 \leqslant p^- \leqslant p^+ \leqslant \infty$ and we denote by $\mathscr{P}^{\log}(\mathbb{R}^n)$ the set of the exponents $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ such that $1/p(\cdot)$ is globally log-Hölder continuous. If $p \in \mathscr{P}(\mathbb{R}^n)$ with $p^+ < \infty$, then $p \in \mathscr{P}^{\log}(\mathbb{R}^n)$ if and only if p is globally log-Hölder continuous.

If $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, we define the function

$$\varphi_{p(\cdot)}(y,t) = \begin{cases} t^{p(y)}, & 1 \leqslant p(y) < \infty \\ \infty \cdot \chi_{(1,\infty)}(t), & p(y) = \infty, \end{cases}$$

for $t \geqslant 0$ and $y \in \mathbb{R}^n$, with the convention $\infty \cdot 0 = 0$, where $\chi_{(1,\infty)}$ denote the characteristic function of $(1,\infty)$. Then the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is the set of the measurable functions f defined on $\mathbb{R}^n$ such that, for some positive λ,

$$\int_{\mathbb{R}^n} \varphi_{p(\cdot)}(x, |f(x)|/\lambda)\, dx < \infty.$$

A Luxemburg norm can be defined in $L^{p(\cdot)}(\mathbb{R}^n)$ by taking

$$\|f\|_{L^{p(\cdot)}} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \varphi_{p(\cdot)}(x, |f(x)|/\lambda)\, dx \leqslant 1\right\}.$$

By $L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$ we denote the space of the functions f such that $f\chi_U \in L^{p(\cdot)}(\mathbb{R}^n)$ for every compact set $U \subset \mathbb{R}^n$.

A locally integrable function w defined in $\mathbb{R}^n$ which is positive almost everywhere is called a weight. For $p(\cdot)\in\mathscr{P}(\mathbb{R}^n)$ we define the weighted variable Lebesgue space $L_w^{p(\cdot)}(\mathbb{R}^n)$ as the set of the measurable functions f defined on $\mathbb{R}^n$ such that $fw\in L^{p(\cdot)}(\mathbb{R}^n)$, and $\|f\|_{L_w^{p(\cdot)}}:=\|fw\|_{L^{p(\cdot)}}$.

By a cube Q in $\mathbb{R}^n$ we shall understand a cube with sides parallel to the coordinate axes. The sidelength of Q is denoted by $\ell(Q)$ and γQ, $\gamma>0$, denotes the cube concentric with Q and with sidelength $\gamma\ell(Q)$.

We shall say that $A\simeq B$ if there exist two positive constants C_1 and C_2 such that $C_1B\leqslant A\leqslant C_2B$.

We define now the functional related with the space where the symbol b belongs. We use $\mathscr{E}$ to denote the class of all cubes Q in $\mathbb{R}^n$ with sides parallel to the axes and consider a functional $a:\mathscr{E}\to[0,\infty)$. We say that a satisfies the T_∞ condition and we denote by $a\in T_\infty$, if there exists a finite positive constant t_∞ such that for every $Q,Q'\in\mathscr{E}$ such that $Q'\subset Q$,

$$a(Q')\leqslant t_\infty a(Q). \tag{3}$$

We denote the least constant t_∞ in (3) by $\|a\|_{t_\infty}$. Clearly, $\|a\|_{t_\infty}\geqslant 1$.

Let $0<\rho<\infty$ and $a\in T_\infty$. We say that a function $b\in L^1_{\text{loc}}(\mathbb{R}^n)$ belongs to the generalized Lipschitz space $\mathscr{L}_a^\rho$ if

$$\sup_Q\frac{1}{a(Q)}\left(\frac{1}{|Q|}\int_Q|b-b_Q|^\rho\,dx\right)^{1/\rho}<\infty \tag{4}$$

where the supremum is taken over all cubes $Q\subset\mathbb{R}^n$ and b_Q denote the average $\frac{1}{|Q|}\int_Q b$ (which sometimes will be denoted by $⨍_Q b$).

We are now in position to state our main results.

The next theorem gives a two weighted boundedness result between variable Lebesgue spaces with different exponents for the commutator $T_K^{b,m}$, when the symbol b belongs to the class $\mathscr{L}_a^\rho$ defined previously. The function $\widetilde{K}$ involved in the condition on the weights is given by

$$\widetilde{K}(t)=\int_{|z|\leqslant t}K(z)\,dz.$$

THEOREM 1.1. *Let $p(\cdot),q(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$ such that $1<p^-\leqslant p(\cdot)\leqslant q(\cdot)\leqslant q^+<\infty$, $K\in\mathfrak{D}$ and $m\in\mathbb{N}\cup\{0\}$. Let $1\leqslant\rho<\infty$, $a\in T_\infty$ and $b\in\mathscr{L}_a^\rho$. Let R,S be two constants such that $R>(p')^+/(p')^-$ and $S>q^+/q^-$. Suppose that (v,w) is any couple of weights such that $v\in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n)$ and, for some positive constant κ and for every cube Q,*

$$a(Q)^m\widetilde{K}(\ell(Q))\frac{\|\chi_Q\|_{L^{q(\cdot)}}}{\|\chi_Q\|_{L^{p(\cdot)}}}\frac{\|\chi_Qv^{-1}\|_{L^{Rp'(\cdot)}}}{\|\chi_Q\|_{L^{Rp'(\cdot)}}}\frac{\|\chi_Qw\|_{L^{Sq(\cdot)}}}{\|\chi_Q\|_{L^{Sq(\cdot)}}}\leqslant\kappa. \tag{5}$$

Then

$$T_K^{b,m}:L_v^{p(\cdot)}(\mathbb{R}^n)\hookrightarrow L_w^{q(\cdot)}(\mathbb{R}^n).$$

More precisely,

$$\left\|T_K^{b,m}f\right\|_{L_w^{q(\cdot)}} \lesssim \kappa \|b\|_{\mathscr{L}_a^1}^m \|f\|_{L_v^{p(\cdot)}}, \forall f \in L_v^{p(\cdot)}(\mathbb{R}^n).$$

In the classical Lebesgue spaces, a proof can be found in [14] for the case $m=0$, that is, $T_K^{b,m} = T_K$; and in [7] for $m \geqslant 1$ and $b \in BMO = \mathscr{L}_a^1$ where $a(Q)=1$. In the variable Lebesgue spaces, when $b \in BMO$ the result above was proved in [11].

Let us observe that, if $a(Q) = |Q|^{\delta/n}$, $0<\delta<1$, then $a \in T_\infty$ and it is known that $\mathscr{L}_a^1 := \mathbb{L}(\delta)$ coincides with the classical Lipschitz spaces Λ_δ define as the set of functions b such that

$$|b(x)-b(y)| \leqslant C|x-y|^\delta$$

for some positive constant C and for every $x,y \in \mathbb{R}^n$.

On the other hand, if $r(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ with $r^+ < \infty$,

$$1 < \gamma \leqslant r^- \leqslant r^+ < \frac{n\gamma}{(n-\gamma)^+} \text{ and } 0 \leqslant \delta(\cdot)/n := 1/\gamma - 1/r(\cdot) < 1/n, \tag{6}$$

in [15, Corollary 3.6] it was proved that the functional $a(Q) = |Q|^{1/\gamma-1}\|\chi_Q\|_{L^{r'(\cdot)}}$ satisfies the T_∞ condition and $\mathscr{L}_a^1 = \mathbb{L}(\delta(\cdot))$ are a variable version of the spaces $\mathbb{L}(\delta)$ defined above. Indeed, let us observe that the functional above can be written as

$$a(Q) \simeq |Q|^{1/\gamma} / \|\chi_Q\|_{L^{r(\cdot)}} \simeq \|\chi_Q\|_{L^{n/\delta(\cdot)}},$$

(see Lemmas 2.5 and 2.6).

In the case that $b \in \mathbb{L}(\delta(\cdot))$, we can improve the theorem above in the sense that we can introduce other type of norms in the conditions on the weights involving generalized Φ-functions (GΦ-functions) (see Section 2 for more information about GΦ-functions). In order to state the results we need some definitions.

The norm associated to a given GΦ-function Ψ is given by

$$\|f\|_{\Psi(\cdot,L)} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \Psi\left(x, \frac{|f(x)|}{\lambda}\right) dx \leqslant 1\right\}$$

and we denote by $L^\Psi(\mathbb{R}^n)$ the space of functions f such that $\|f\|_{\Psi(\cdot,L)} < \infty$.

A corresponding maximal operator associated to Ψ is

$$M_{\Psi(\cdot,L)}f(x) = \sup_{Q \ni x} \frac{\|\chi_Q f\|_{\Psi(\cdot,L)}}{\|\chi_Q\|_{\Psi(\cdot,L)}} \tag{7}$$

and, for $\beta(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, we define the following fractional type version of maximal above as follows

$$M_{\beta(\cdot),\Psi(\cdot,L)}f(x) = \sup_{Q \ni x} \|\chi_Q\|_{L^{\beta(\cdot)}} \frac{\|\chi_Q f\|_{\Psi(\cdot,L)}}{\|\chi_Q\|_{\Psi(\cdot,L)}}. \tag{8}$$

We say that a 3-tuples of GΦ-functions (A,B,D) satisfy condition $\mathscr{F}$ if they verify

9. $\|\chi_Q\|_{A(\cdot,L)}\|\chi_Q\|_{B(\cdot,L)}\lesssim\|\chi_Q\|_{D(\cdot,L)}$ where $\lesssim$ means that there exists a positive constant C such that 9 holds with $\lesssim$ replaced by $\leqslant C$.

10. $A^{-1}(x,t)B^{-1}(x,t)\lesssim D^{-1}(x,t)$ where A^{-1} denote the inverse of A (for the definition of the inverse of a GΦ-function see Section 2).

11. $\|\chi_Q\|_{D(\cdot,L)}\|\chi_Q\|_{D^*(\cdot,L)}\lesssim|Q|$, where D^* is the conjugate function of D (for the definition of the conjugate of a GΦ-function see Section 2).

Necessary conditions on D where given in [4, Remark 4.5.8] and [5, Lemma 4.4.5.] in order to verify 11.

We shall give later some examples of GΦ-functions that satisfy condition $\mathscr{F}$.

We can now give our result.

THEOREM 1.2. *Let $p(\cdot),q(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$ such that $p(\cdot)\leqslant q(\cdot)$, $K\in\mathfrak{D}$ and m a non-negative integer. Let $\beta(\cdot)$ be a function such that $1/\beta(\cdot)=1/p(\cdot)-1/q(\cdot)$. Let $r(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$ and $\delta(\cdot)$ defined as in (6), such that $r_\infty\leqslant r(\cdot)$ and let $b\in\mathbb{L}(\delta(\cdot))$. Let (A,B,D) and (E,H,J) GΦ-functions satisfying condition $\mathscr{F}$,*

$$M_{B(\cdot,L)}:L^{p(\cdot)}(\mathbb{R}^n)\to L^{p(\cdot)}(\mathbb{R}^n) \tag{12}$$

and

$$M_{\beta(\cdot),H(\cdot,L)}:L^{q'(\cdot)}(\mathbb{R}^n)\to L^{p'(\cdot)}(\mathbb{R}^n). \tag{13}$$

Suppose that (v,w) is any couple of weights such that $v\in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n)$ and, for some positive constant κ and for every cube Q,

$$\|\chi_Q\|^m_{L^{n/\delta(\cdot)}}\widetilde{K}(\ell(Q))\frac{\|\chi_Q\|_{L^{q(\cdot)}}}{\|\chi_Q\|_{L^{p(\cdot)}}}\frac{\|\chi_Q v^{-1}\|_{A(\cdot,L)}}{\|\chi_Q\|_{A(\cdot,L)}}\frac{\|\chi_Q w\|_{E(\cdot,L)}}{\|\chi_Q\|_{E(\cdot,L)}}\leqslant\kappa. \tag{14}$$

Then

$$T_K^{b,m}:L_v^{p(\cdot)}(\mathbb{R}^n)\hookrightarrow L_w^{q(\cdot)}(\mathbb{R}^n).$$

More precisely,

$$\left\|T_K^{b,m}f\right\|_{L_w^{q(\cdot)}}\lesssim\kappa\|f\|_{L_v^{p(\cdot)}},\forall f\in L_v^{p(\cdot)}(\mathbb{R}^n).$$

Let us give some examples of GΦ-functions that satisfy the hypothesis of the theorem above.

Notice first that, if we consider $p(\cdot)\in\mathscr{P}(\mathbb{R}^n)$ and $q(\cdot)$ with $q^+<\infty$, then the function $\Psi(x,t)=t^{p(x)}(\log(e+t))^{q(x)}$, $x\in\mathbb{R}^n$, $t\geqslant 0$, is a GΦ-function. In this case, the space $L^\Psi(\mathbb{R}^n)$ will be denoted by $L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)$. In [10, Proposition 2.5] the authors proved that the Hardy-Littlewood maximal operator M is bounded in this space when $p(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$ with $1<p^-\leqslant p^+<\infty$, and $q(\cdot)\in\mathscr{P}^{\log\log}(\mathbb{R}^n)$. We say that

$q(\cdot) \in \mathscr{P}^{\mathrm{loglog}}(\mathbb{R}^n)$ if $q(\cdot) : \mathbb{R}^n \to \mathbb{R}$ with $q^+ < \infty$ such that, for some positive constant C, it satisfies the following inequality

$$|q(x) - q(y)| \leqslant \frac{C}{\log(e + \log(e + 1/|x-y|))}, \text{ for every } x, y \in \mathbb{R}^n.$$

REMARK 1.3. Note that if $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ with $p^+ < \infty$ and $q(\cdot) \in \mathscr{P}^{\mathrm{loglog}}(\mathbb{R}^n)$, then $(pq)(\cdot) \in \mathscr{P}^{\mathrm{loglog}}(\mathbb{R}^n)$ and $(q/p)(\cdot) \in \mathscr{P}^{\mathrm{loglog}}(\mathbb{R}^n)$. Indeed, for every $x, y \in \mathbb{R}^n$,

$$\begin{aligned} |p(x)q(x) - p(y)q(y)| &\leqslant |p(x)||q(x) - q(y)| + |q(y)||p(x) - p(y)| \\ &\lesssim \frac{p^+}{\log(e + \log(e + 1/|x-y|))} + \frac{q^+}{\log(e + 1/|x-y|)} \\ &\lesssim \frac{1}{\log(e + \log(e + 1/|x-y|))}. \end{aligned}$$

This gives $(pq)(\cdot) \in \mathscr{P}^{\mathrm{loglog}}(\mathbb{R}^n)$. Since $(1/p)^+ < \infty$, $(q/p)(\cdot) \in \mathscr{P}^{\mathrm{loglog}}(\mathbb{R}^n)$ follows from the first property.

Examples. Let $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ with $1 < p^- \leqslant p^+ < \infty$ and $\sigma > (p')^+/(p')^-$. The following GΦ-functions satisfy condition $\mathscr{F}$ and the hyphoteses (12) and (13) of the Theorem 1.2.

EXAMPLE 1.1. The 3-tuple (A_1, B_1, D_1) where $A_1(x,t) = t^{\sigma p'(x)}(\log(e+t))^{\sigma p'(x)}$, $B_1(x,t) = t^{(\sigma p')'(x)}$ and $D_1(t) = t\log(e+t)$.

EXAMPLE 1.2. If, in addition, $\mu(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ with $1 < \mu^- \leqslant \mu^+ < \infty$ such that

$$1/\sigma p'(\cdot) - 1/\mu(\cdot) > \varepsilon$$

for some constant $\varepsilon \in (0,1)$ and $\nu(\cdot) \in \mathscr{P}^{\mathrm{loglog}}(\mathbb{R}^n)$ then, the example of 3-tuple is given by (A_2, B_2, D_2) where $A_2(x,t) = t^{\mu(x)}(\log(e+t))^{\nu(x)\mu(x)}$, $B_2(x,t) = t^{(\sigma p')'(x)}$ and $D_2(x,t) = t^{\alpha(x)}(\log(e+t))^{\alpha(x)\nu(x)}$ where $\alpha(\cdot)$ is defined by

$$1/\alpha(\cdot) = 1/\mu(\cdot) + 1/(\sigma p')'(\cdot).$$

In Section 3 we check these examples.

The paper is organized as follows. In Section 2 we introduce basic definitions and known results related to Musielak-Orlicz spaces. We also give some boundedness estimates in this context. In Section 3 we prove a key estimate regarding the $L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)$ norm of χ_Q for $Q \in \mathscr{E}$, using a series of auxiliary lemmas that we prove as well. We also discuss the validity of Examples 1.1 and 1.2. Finally, in Section 4 we prove Theorem 1.1 and Theorem 1.2.

2. Preliminaries

In this section we give some previous definitions and results that we shall be using throughout this paper.

With $\mathscr{M}$ we denote the set of all Lebesgue real valued, measurable functions on $\mathbb{R}^n$.

A convex function $\psi : [0,\infty) \to [0,\infty]$ with $\psi(0) = 0$, $\lim_{t\to 0^+} \psi(t) = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$ is called a Φ-function.

A real function $\Psi : \mathbb{R}^n \times [0,\infty) \to [0,\infty]$ is said to be a generalized Φ-function (GΦ-function), and we denote $\Psi \in G\Phi(\mathbb{R}^n)$, if $\Psi(x,t)$ is Lebesgue-measurable in x for every $t \geqslant 0$ and $\Psi(x,\cdot)$ is a Φ-function for every $x \in \mathbb{R}^n$.

If $\Psi \in G\Phi(\mathbb{R}^n)$, then the set

$$L^{\Psi}(\mathbb{R}^n) := \left\{ f \in \mathscr{M} : \int_{\mathbb{R}^n} \Psi(x, |f(x)|)\, dx < \infty \right\}$$

defines a Banach function space equipped with the Luxemburg-norm given by

$$\|f\|_{\Psi(\cdot,L)} := \inf\left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Psi\left(x, \frac{|f(x)|}{\lambda}\right) dx \leqslant 1 \right\}.$$

The space $L^{\Psi}(\mathbb{R}^n)$ is called a Musielak-Orlicz space.

Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, then $\Psi(x,t) = t^{p(x)} \in G\Phi(\mathbb{R}^n)$. In this case, the space $L^{\Psi}(\mathbb{R}^n)$ is the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ defined in the introduction. If we also consider $r(\cdot)$ with $r^+ < \infty$, then $\Psi(x,t) = t^{p(x)}(\log(e+t))^{r(x)} \in G\Phi(\mathbb{R}^n)$. In this case, the space $L^{\Psi}(\mathbb{R}^n)$ is the space $L^{p(\cdot)}(\log L)^{r(\cdot)}(\mathbb{R}^n)$ introduced before.

Let $\Psi \in G\Phi(\mathbb{R}^n)$, then for any $x \in \mathbb{R}^n$ we denote by $\Psi^*(x,\cdot)$ the conjugate function of $\Psi(x,\cdot)$ which is defined by

$$\Psi^*(x,u) = \sup_{t\geqslant 0}(tu - \Psi(x,t)), \quad u \geqslant 0.$$

For $\Psi \in G\Phi(\mathbb{R}^n)$ that verifies that every simple function belongs to $L^{\Psi^*}(\mathbb{R}^n)$, we have the following norm conjugate formula,

$$\|f\|_{\Psi(\cdot,L)} \simeq \sup_{\|g\|_{\Psi^*(\cdot,L)}\leqslant 1} \int_{\mathbb{R}^n} |f(x)g(x)|\, dx \tag{15}$$

for every function $f \in L^{\Psi}(\mathbb{R}^n)$ (see [4, Corollary 2.7.5]).

The following lemma can be deduced from Lemma 4.4.5 in [5].

LEMMA 2.1. *Let ψ a Φ-function, then the following inequality*

$$\|\chi_Q\|_{\psi} \|\chi_Q\|_{\psi^*} \lesssim |Q|$$

holds for every cube Q in $\mathbb{R}^n$.

Also we can define Ψ^{-1}, the generalized inverse function of Ψ, by

$$\Psi^{-1}(x,t) := \inf\{u \geqslant 0 : \Psi(x,u) \geqslant t\}, \quad x \in \mathbb{R}^n, t \geqslant 0.$$

For example, if $p \in \mathscr{P}(\mathbb{R}^n)$ and $\Psi(x,t) = t^{p(x)}$, we have $\Psi^{-1}(x,t) = t^{1/p(x)}$ and $\Psi^*(x,t) = t^{p'(x)}$.

Note that, by definition of Ψ^*, the following generalization of the Young's inequality holds in this context,

$$vu \leqslant \Psi(x,v) + \Psi^*(x,u), \quad \forall x \in \mathbb{R}^n, \forall v,u \geqslant 0, \tag{16}$$

for any $\Psi \in G\Phi(\mathbb{R}^n)$. If we put $v = \Psi^{-1}(x,t)$ and $u = (\Psi^*)^{-1}(x,t)$ in equation (16) we obtain

$$\Psi^{-1}(x,t)(\Psi^*)^{-1}(x,t) \leqslant \Psi(x,\Psi^{-1}(x,t)) + \Psi^*(x,(\Psi^*)^{-1}(x,t)) \leqslant 2t. \tag{17}$$

Moreover, it can be proved that if $\Psi,\Lambda,\Theta \in G\Phi(\mathbb{R}^n)$ such that $\Psi(x,\cdot),\Lambda(x,\cdot)$ are strictly increasing and $\Psi^{-1}(x,t)\Lambda^{-1}(x,t) \leqslant \Theta^{-1}(x,t)$ for every $x \in \mathbb{R}^n$, and for every $t \geqslant 0$, then

$$\Theta(x,tu) \leqslant \Psi(x,t) + \Lambda(x,u), \quad \forall x \in \mathbb{R}^n, \forall t,u \geqslant 0.$$

The inequality above allows us to prove the following generalized Hölder type inequality in this context. The proof is standard and we omit it.

LEMMA 2.2. *Let $\Psi,\Lambda,\Theta \in G\Phi(\mathbb{R}^n)$ such that $\Psi(x,\cdot),\Lambda(x,\cdot)$ are strictly increasing and*

$$\Psi^{-1}(x,t)\Lambda^{-1}(x,t) \leqslant \Theta^{-1}(x,t), \quad \forall x \in \mathbb{R}^n, \forall t \geqslant 0.$$

Then

$$\|fg\|_{\Theta(\cdot,L)} \lesssim \|f\|_{\Psi(\cdot,L)} \|g\|_{\Lambda(\cdot,L)} \tag{18}$$

for all $f \in L^\Psi(\mathbb{R}^n)$ and $g \in L^\Lambda(\mathbb{R}^n)$.

For example, if $\Theta(x,t) = t^{s(x)}$, $\Psi(x,t) = t^{p(x)}$ and $\Lambda(x,t) = t^{q(x)}$ with exponents $s(\cdot),p(\cdot),q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ and $1/s(\cdot) = 1/p(\cdot) + 1/q(\cdot)$, we obtain that

$$\|fg\|_{L^{s(\cdot)}} \lesssim \|f\|_{L^{p(\cdot)}} \|g\|_{L^{q(\cdot)}}. \tag{19}$$

In the case $s(\cdot) \equiv 1$ inequality (19) becomes

$$\int_{\mathbb{R}^n} |f(y)g(y)|\,dy \lesssim \|f\|_{L^{p(\cdot)}} \|g\|_{L^{p'(\cdot)}} \tag{20}$$

and, for a general $\Psi \in G\Phi(\mathbb{R}^n)$ such that $\Psi(x,\cdot)$ is strictly increasing, from inequality (17) we obtain

$$\int_{\mathbb{R}^n} |f(y)g(y)|\,dy \lesssim \|f\|_{\Psi(\cdot,L)} \|g\|_{\Psi^*(\cdot,L)}, \tag{21}$$

which is an extension of the classical Hölder inequality (see [4]).

Particularly, when we deal with variable Lebesgue spaces, we have the following known results that we shall be using along this paper.

LEMMA 2.3. [4, Lemma 3.4.2] *Let* $p(\cdot)\in\mathscr{P}(\mathbb{R}^n)$ *with* $p^+<\infty$. *Then*

$$\|f\|_{L^{p(\cdot)}}\leqslant C_1 \quad \text{if and only if} \quad \int_{\mathbb{R}^n}|f(x)|^{p(x)}\,dx\leqslant C_2.$$

Moreover, if either constant equals 1 we can take the other equal to 1 as well.

The following lemma describes some properties of the exponent in $\mathscr{P}^{\log}(\mathbb{R}^n)$.

LEMMA 2.4. *Let* $p(\cdot),q(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$ *and* $c\in\mathbb{R}$ *such that* $c\geqslant 1/p^-$, *then the following properties hold:*

(i) $cp(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$.

(ii) $p'(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$.

(iii) *If* $\alpha(\cdot)$ *is the exponent defined by* $1/\alpha(\cdot)=1/p(\cdot)+1/q(\cdot)$, *then* $\alpha(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$.

(iv) *If, in addition,* $p^+,q^+<\infty$, *then* $(pq)(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$.

The next lemma can be deduced from the Corollary 4.5.9 in [4].

LEMMA 2.5. ([4]) *Let* $p(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$. *Then there exists two positive constants* C_p^* *and* C_p^{**} *such that*

$$|Q|\leqslant C_p^*\,\|\chi_Q\|_{L^{p(\cdot)}}\,\|\chi_Q\|_{L^{p'(\cdot)}}\leqslant C_p^{**}|Q|,$$

for every cubes $Q\subset\mathbb{R}^n$. *Note that we can suppose* $C_p^*,C_p^{**}\geqslant 1$.

Moreover, we have the following result.

LEMMA 2.6. [11, Lemma 2.7] *Let* $p(\cdot),q(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$ *such that* $p(\cdot)\leqslant q(\cdot)$. *Suppose that* $1/\beta(\cdot)=1/p(\cdot)-1/q(\cdot)$, *then*

$$\|\chi_Q\|_{L^{p(\cdot)}}\,\|\chi_Q\|_{L^{q(\cdot)}}^{-1}\simeq\|\chi_Q\|_{L^{\beta(\cdot)}},$$

for every cube $Q\subset\mathbb{R}^n$.

Note that Lemma 2.6 implies Lemma 2.5 making the choices $\beta(\cdot):=p(\cdot)$, $q(\cdot):=p'(\cdot)$ and $p(\cdot):=1$.

The following lemma gives a doubling property for the functional $\mathtt{f}(Q):=\|\chi_Q\|_{L^{p(\cdot)}}$ with $p(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$.

LEMMA 2.7. [15, Equation (2.11)] *If* $p(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$ *with* $p^+<\infty$, *then there exists a positive constant* C_p *such that the inequality*

$$\|\chi_{2Q}\|_{L^{p(\cdot)}}\leqslant C_p\,\|\chi_Q\|_{L^{p(\cdot)}} \tag{22}$$

holds for every cube $Q\subset\mathbb{R}^n$.

By iteration of inequality (22) it is not difficult to prove that

$$\|\chi_{\gamma Q}\|_{L^{p(\cdot)}} \lesssim \|\chi_Q\|_{L^{p(\cdot)}} \tag{23}$$

holds for every cube $Q \subset \mathbb{R}^n$, with an appropriate constant depending on γ and C_p.

The next theorem is an useful tool in order to prove Theorem 1.1.

THEOREM 2.8. [4, Theorem 7.3.22] *If* $p \in \mathscr{P}^{\log}(\mathbb{R}^n)$, *then*

$$\sum_{Q\in\mathscr{D}} \|\chi_Q f\|_{L^{p(\cdot)}} \|\chi_Q g\|_{L^{p'(\cdot)}} \leqslant G_p \|f\|_{L^{p(\cdot)}} \|g\|_{L^{p'(\cdot)}}$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$, $g \in L^{p'(\cdot)}(\mathbb{R}^n)$ *and every family* $\mathscr{D}$ *of pairwise disjoint cubes.*

Moreover, a similar result considering overlaping families is the following.

LEMMA 2.9. [11, Lemma 3.5] *Let* $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$, $d \in \mathbb{Z}$ *and* Q_0 *a dyadic cube. If we define*

$$\mathscr{O}_d = \{Q \text{ dyadic cube} : Q \subset Q_0 \text{ and } \ell(Q) = 2^{-d}\},$$

then

$$\sum_{Q\in\mathscr{O}_d} \|f\chi_{3Q}\|_{L^{p(\cdot)}} \|g\chi_{3Q}\|_{L^{p'(\cdot)}} \lesssim \|f\chi_{3Q_0}\|_{L^{p(\cdot)}} \|g\chi_{3Q_0}\|_{L^{p'(\cdot)}} \tag{24}$$

for every $f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$ *and* $g \in L^{p'(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$, *where the implied constant in* $\lesssim$ *does not depend on* d.

In order to prove Lemma 3.7 we state the next result that follows from [3, Lemma 5.5]. Recall that f_Q denotes the average $\frac{1}{|Q|}\int_Q f$.

LEMMA 2.10. ([3]) *Let* $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ *with* $1 < p^- \leqslant p^+ < \infty$. *Then exists a constant* $0 < \nu < 1$ *such that for every cube* Q *and every function* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *with* $f_Q \neq 0$,

$$\||f|^\nu \chi_Q\|_{L^{p(\cdot)}} \lesssim \|\chi_Q\|_{L^{p(\cdot)}} |f_Q|^\nu.$$

The next theorems give boundedness results in Musielak-Orlicz spaces for certain maximal functions.

THEOREM 2.11. [10, Proposition 2.5] *Let* $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ *with* $1 < p^- \leqslant p^+ < \infty$ *and* $q(\cdot) \in \mathscr{P}^{\log\log}(\mathbb{R}^n)$. *Then*

$$M : L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n).$$

THEOREM 2.12. [4, Theorem 7.3.27] *Let* $p(\cdot), s(\cdot), l(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ *such that* $p(\cdot) = s(\cdot)l(\cdot)$ *and* $l^- > 1$. *Then*

$$M_{L^{s(\cdot)}} : L^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p(\cdot)}(\mathbb{R}^n).$$

THEOREM 2.13. [11, Theorem 1.7] *Let* $p(\cdot), q(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ *such that* $p(\cdot) \leqslant q(\cdot)$ *and* $r(\cdot) \in \mathscr{P}^{\log\log}(\mathbb{R}^n)$. *Let* $s(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ *and* $\beta(\cdot)$ *be two functions such that* $1/\beta(\cdot) = 1/p(\cdot) - 1/q(\cdot)$ *and* $1 \leqslant s^- \leqslant s^+ < p^-$. *Then*

$$M_{\beta(\cdot), L^{s(\cdot)}} : L^{p(\cdot)}(\log L)^{r(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{q(\cdot)}(\log L)^{r(\cdot)}(\mathbb{R}^n).$$

REMARK 2.14. Since $1/\beta(\cdot) = 1/q'(\cdot) - 1/p'(\cdot)$, if $1 \leqslant s^- \leqslant s^+ < (q')^-$ we have that

$$M_{\beta(\cdot), L^{s(\cdot)}} : L^{q'(\cdot)}(\log L)^{r(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p'(\cdot)}(\log L)^{r(\cdot)}(\mathbb{R}^n).$$

The following result establishes that the spaces $\mathscr{L}_a^{\rho}$ coincide, for $1 \leqslant \rho < \infty$.

THEOREM 2.15. [6, Corollary 2] *Let* $1 \leqslant \rho < \infty$ *and* $a \in T_\infty$, *then* $\mathscr{L}_a^{\rho} = \mathscr{L}_a^1$ *and*

$$\sup_Q \frac{1}{a(Q)} \left(\fint_Q |b - b_Q|^{\rho}\, dx \right)^{1/\rho} \simeq \sup_Q \frac{1}{a(Q)} \fint_Q |b - b_Q|\, dx.$$

The following lemma can be deduced from the proof of Theorem 2.3 in [15] (see [15, Equation (5.4)], and it will be useful in the proof of Theorem 1.2.

LEMMA 2.16. [15] *Let* $r(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n))$ *with* $r^+ < \infty$ *such that* $r_\infty \leqslant r(\cdot)$,

$$1 < \gamma \leqslant r^- \leqslant r^+ < \frac{n\gamma}{(n-\gamma)^+} \text{ and } \delta(\cdot)/n := 1/\gamma - 1/r(\cdot).$$

Let $b \in \mathbb{L}(\delta(\cdot))$ *then*

$$|b(x) - b(z)| \lesssim |x - z|^{\delta(x)}$$

for every $x, z \in \mathbb{R}^n$.

3. Key auxiliary results

In this section we give some technical lemmas that will be useful in the proof of the main results.

3.1. Estimates of $\|\chi_Q\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}}$

In [4] the authors proved that, if $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$, then $\|\chi_Q\|_{L^{p(\cdot)}} \simeq |Q|^{(1/p)_Q}$ for any cube Q (see [4, Lemma 4.5.3]). Recall that $(1/p)_Q$ denotes the average $|Q|^{-1}\int_Q 1/p(x)\,dx$. We would like to generalize this result to the case of $L^{p(\cdot)}(\log L)^{q(\cdot)}$ norms, that is, estimates of $\|\chi_Q\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}}$ with $p(\cdot), q(\cdot)$ in certain classes of exponents. Concretely, we prove the following result.

PROPOSITION 3.1. *Let $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ such that $1 < p^- \leqslant p^+ < \infty$ and a non--negative function $q(\cdot) \in \mathscr{P}^{\log\log}(\mathbb{R}^n)$. Then*

$$\|\chi_Q\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}} \simeq |Q|^{(1/p)_Q}(\log(e+1/|Q|))^{(q/p)_Q},$$

for every cube Q in $\mathbb{R}^n$.

REMARK 3.2. In particular, when $p(\cdot) = q(\cdot)$ with $1 < p^- \leqslant p^+ < \infty$,

$$\|\chi_Q\|_{L^{p(\cdot)}(\log L)^{p(\cdot)}} \simeq |Q|^{(1/p)_Q}\log(e+1/|Q|) \tag{25}$$

and if, in addition, $q(\cdot) \equiv 0$,

$$\|\chi_Q\|_{L^{p(\cdot)}} \simeq |Q|^{(1/p)_Q}. \tag{26}$$

Since $\psi(t) = t\log(e+t)$ is an invertible Young function, is easy to see that

$$\|\chi_Q\|_{L\log L} \simeq |Q|\log(e+1/|Q|). \tag{27}$$

In order to achieve Proposition 3.1 we need the following lemmas.

LEMMA 3.3. *Let $q(\cdot) \in \mathscr{P}^{\log\log}(\mathbb{R}^n)$ and let Q be a cube in $\mathbb{R}^n$. Then, for every $x, y \in Q$,*

$$(\log(e+1/|Q|))^{q(x)} \simeq (\log(e+1/|Q|))^{q(y)}.$$

Proof. It is enough to show that there exists a positive constant C such that

$$(\log(e+1/|Q|))^{|q(x)-q(y)|} \leqslant C,$$

or equivalenty

$$\exp[|q(x)-q(y)|\log(\log(e+1/|Q|))] \leqslant C. \tag{28}$$

Since $q(\cdot) \in \mathscr{P}^{\log\log}(\mathbb{R}^n)$,

$$\exp(|q(x)-q(y)|\log(\log(e+1/|Q|))) \leqslant \exp\left(C\frac{\log(e+\log(e+1/|Q|))}{\log(e+\log(e+1/|x-y|))}\right). \tag{29}$$

Since $x,y\in Q$, there exists a constant $C_n>1$ such that $|x-y|\leqslant C_n|Q|^{1/n}$. Then

$$\log\left(e+\log\left(e+\frac{1}{C_n|Q|^{1/n}}\right)\right)\leqslant \log\left(e+\log\left(e+\frac{1}{|x-y|}\right)\right).$$

If we prove that

$$\log\left(e+\log\left(e+\frac{1}{|Q|}\right)\right)\leqslant \kappa\log\left(e+\log\left(e+\frac{1}{C_n|Q|^{1/n}}\right)\right) \tag{30}$$

for some positive constant κ then, by (29), we conclude (28).

Let us prove inequality (30). Note that, since $C_n\geqslant 1$,

$$\begin{aligned}\log\left(e+\frac{1}{|Q|}\right)\lesssim \log\left(e+\frac{1}{|Q|^{1/n}}\right)&\leqslant \log\left(C_n e+\frac{C_n}{C_n|Q|^{1/n}}\right)\\ &\leqslant \log(C_n)\log\left(e+\frac{1}{C_n|Q|^{1/n}}\right)+\log\left(e+\frac{1}{C_n|Q|^{1/n}}\right)\\ &\leqslant (1+\log C_n)\log\left(e+\frac{1}{C_n|Q|^{1/n}}\right):=\kappa_1\log\left(e+\frac{1}{C_n|Q|^{1/n}}\right).\end{aligned}$$

Thus, by similar argument, since $\kappa_1\geqslant 1$,

$$\begin{aligned}\log\left(e+\log\left(e+\frac{1}{|Q|}\right)\right)&\leqslant \log\left(e+\kappa_1\log\left(e+\frac{1}{C_n|Q|^{1/n}}\right)\right)\\ &\leqslant (1+\log\kappa_1)\log\left(e+\log\left(e+\frac{1}{C_n|Q|^{1/n}}\right)\right)\\ &:=\kappa\log\left(e+\log\left(e+\frac{1}{C_n|Q|^{1/n}}\right)\right).\quad\square\end{aligned}$$

Let $\alpha(\cdot)$ and $\theta(\cdot)$ be two functions with $0<\alpha^-\leqslant\alpha^+<\infty$ and $0\leqslant\theta^-\leqslant\theta^+<\infty$ and $x\in\mathbb{R}^n$, we denote

$$\phi_{\alpha(x),\theta(x)}(t):=t^{\alpha(x)}(\log(e+t))^{\theta(x)}.$$

Note that, for every fixed $x\in\mathbb{R}^n$, $\phi^{-1}_{\alpha(x),\theta(x)}(\cdot)$ is a Young function, then it is not difficult to prove that

$$\phi^{-1}_{\alpha(x),\theta(x)}(t)\simeq t^{1/\alpha(x)}(\log(e+t))^{-\theta(x)/\alpha(x)} \tag{31}$$

(see, for example, [16]). If, in addition, $\alpha^->1$,

$$\phi^{*}_{\alpha(x),\theta(x)}(t)\simeq t^{\alpha'(x)}(\log(e+t))^{-\theta(x)/(\alpha(x)-1)}. \tag{32}$$

The constants involved in equations (31) and (32) only depend on the extremes of the exponents $\alpha(\cdot)$ and $\theta(\cdot)$.

LEMMA 3.4. *Let* $p(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$ *such that* $1\leqslant p(\cdot)\leqslant p^+<\infty$ *and a non--negative function* $q(\cdot)\in\mathscr{P}^{\log\log}(\mathbb{R}^n)$*. Then for every cube* $Q\subset\mathbb{R}^n$ *we have*

$$\phi^{-1}_{\frac{1}{(1/p)_Q},\frac{(q/p)_Q}{(1/p)_Q}}(1/|Q|)\lesssim \fint_Q \phi^{-1}_{p(x),q(x)}(1/|Q|)\,dx.$$

Proof. Let $Q\subset\mathbb{R}^n$ a cube. Since

$$0<\frac{1}{(1/p)_Q}\leqslant p^+<\infty \quad\text{and}\quad 0\leqslant\frac{(q/p)_Q}{(1/p)_Q}\leqslant q^+p^+<\infty,$$

by equation (31) with $\alpha(\cdot):=1/(1/p)_Q$ and $\theta(\cdot):=(q/p)_Q/(1/p)_Q$, we have

$$\phi^{-1}_{\frac{1}{(1/p)_Q},\frac{(q/p)_Q}{(1/p)_Q}}(1/|Q|)\simeq(1/|Q|)^{(1/p)_Q}(\log(e+(1/|Q|)))^{(q/p)_Q}. \tag{33}$$

Given $x\in Q$, define the mappings

$$h(z):=(1/|Q|)^z(\log(e+(1/|Q|)))^{-(q/p)_Q}$$

and

$$g_x(z):=(1/|Q|)^{1/p(x)}(\log(e+(1/|Q|)))^{-z}$$

for $z\geqslant 0$. Note that, as functions of z, the mappings h and g_x are convex. Thus, by (33) and applying Jensen's inequality twice we have that

$$\begin{aligned}
\phi^{-1}_{\frac{1}{(1/p)_Q},\frac{(q/p)_Q}{(1/p)_Q}}(1/|Q|)\simeq h\left(\left(\frac{1}{p}\right)_Q\right)&\leqslant\fint_Q h\left(\frac{1}{p(x)}\right)dx\\
&=\fint_Q(1/|Q|)^{1/p(x)}(\log(e+(1/|Q|)))^{-(q/p)_Q}\,dx\\
&=\fint_Q g_x\left(\left(\frac{q}{p}\right)_Q\right)dx\leqslant\fint_Q\fint_Q g_x\left(\frac{q(y)}{p(y)}\right)dy\,dx\\
&=\fint_Q\fint_Q\frac{(1/|Q|)^{1/p(x)}}{(\log(e+1/|Q|))^{q(y)/p(y)}}\,dy\,dx.
\end{aligned}$$

From Remark (1.3) we can apply Lemma 3.3 with $q(\cdot):=(q/p)(\cdot)$ to obtain that

$$\phi^{-1}_{\frac{1}{(1/p)_Q},\frac{(q/p)_Q}{(1/p)_Q}}(1/|Q|)\lesssim\fint_Q\frac{(1/|Q|)^{1/p(x)}}{(\log(e+1/|Q|))^{q(x)/p(x)}}\,dx\simeq\fint_Q\phi^{-1}_{p(x),q(x)}(1/|Q|)\,dx,$$

where we have used equation (31) with $\alpha(\cdot):=p(\cdot)$ and $\theta(\cdot):=q(\cdot)$. $\square$

LEMMA 3.5. *Let $p(\cdot),q(\cdot)$ such that $1<p^-\leqslant p^+<\infty$ and $0\leqslant q^-\leqslant q^+<\infty$ and let Q be a cube in $\mathbb{R}^n$. Then for every $t\geqslant 0$,*

$$t\lesssim \fint_Q \phi^{-1}_{p(x),q(x)}(t)\,dx \fint_Q (\log(e+t))^{q(x)}\phi^{-1}_{p'(x),q(x)}(t)\,dx.$$

Proof. It is enough to prove the case $t>0$. Since, by equation (31),

$$\phi^{-1}_{p(x),q(x)}(t)\;\phi^{-1}_{p'(x),q(x)}(t)\simeq \frac{t^{1/p(x)}}{(\log(e+t))^{q(x)/p(x)}}\frac{t^{1/p'(x)}}{(\log(e+t))^{q(x)/p'(x)}}=\frac{t}{(\log(e+t))^{q(x)}},$$

then, by Jensen's inequality, we have

$$\begin{aligned}\fint_Q \phi^{-1}_{p(x),q(x)}(t)\,dx &\simeq t\fint_Q \frac{1}{(\log(e+t))^{q(x)}\phi^{-1}_{p'(x),q(x)}(t)}\,dx\\ &\gtrsim t\frac{1}{\fint_Q(\log(e+t))^{q(x)}\phi^{-1}_{p'(x),q(x)}(t)\,dx}. \qquad \square\end{aligned}$$

Proof of Proposition 3.1. Let Q be a cube in $\mathbb{R}^n$, define

$$f(x):=\chi_Q(x)\phi^{-1}_{p(x),q(x)}(1/|Q|),\quad x\in\mathbb{R}^n$$

and

$$g(x):=\chi_Q(x)(\log(e+1/|Q|))^{q(x)}\phi^{-1}_{p'(x),q(x)}(1/|Q|),\quad x\in\mathbb{R}^n.$$

Note that $\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}}\leqslant 1$ and $\|g\|_{L^{p'(\cdot)}(\log L)^{-q(\cdot)/(p(\cdot)-1)}}\leqslant C_2$ with C_2 a positive constant independent of Q. Indeed, since by (31),

$$\int_{\mathbb{R}^n}\phi_{p(x),q(x)}(f(x))\,dx=\int_Q\phi_{p(x),q(x)}\left(\phi^{-1}_{p(x),q(x)}(1/|Q|)\right)dx\simeq 1,$$

the estimation for f is clear. Note that, for $x\in Q$, by (31),

$$\begin{aligned}\log(e+g(x))&=\log\left[e+(\log(e+1/|Q|))^{q(x)}\phi^{-1}_{p'(x),q(x)}(1/|Q|)\right]\\ &\simeq\log\left[e+(\log(e+1/|Q|))^{q(x)/p(x)}(1/|Q|)^{1/p'(x)}\right]\\ &\geqslant\log\left(e+(1/|Q|)^{1/p'(x)}\right)\geqslant\frac{1}{(p')^+}\log\left(e+1/|Q|\right)\gtrsim\log(e+1/|Q|),\end{aligned}$$

since $(p')^+<\infty$. Thus we have that

$$\begin{aligned}\int_Q\frac{g^{p'(x)}}{(\log(e+g))^{q(x)/(p(x)-1)}}\,dx&\lesssim\int_Q\frac{g^{p'(x)}}{(\log(e+1/|Q|))^{q(x)/(p(x)-1)}}\,dx\\ &\lesssim\fint_Q(\log(e+1/|Q|))^{q(x)\left(p'(x)-\frac{1}{p(x)-1}-1\right)}\,dx\lesssim 1,\end{aligned}$$

since $p'(x)-1/(p(x)-1)-1=0$.

By Lemma 3.5 with $t:=1/|Q|$ we have

$$1 \lesssim |Q| \fint_Q \phi^{-1}_{p(x),q(x)}(1/|Q|)\,dx \fint_Q (\log(e+1/|Q|))^{q(x)}\,\phi^{-1}_{p'(x),q(x)}(1/|Q|)\,dx$$
$$= \fint_Q f\,dx \int_{\mathbb{R}^n} \chi_Q(x) g(x)\,dx. \tag{34}$$

We can apply Hölder's inequality (21) with the functions $\Psi(x,t):=t^{p(x)}(\log(e+t))^{q(x)}$ and $\Psi^*(x,t):=t^{p'(x)}(\log(e+t))^{-q(x)/(p(x)-1)}$ (see equation (32)), to obtain

$$1 \lesssim \fint_Q f(x)\,dx \,\|\chi_Q\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}}\,\|g\|_{L^{p'(\cdot)}(\log L)^{-q(\cdot)/(p(\cdot)-1)}}$$
$$\lesssim \|f_Q\chi_Q\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}} \leqslant \|Mf\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}} \lesssim \|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}} \lesssim 1, \tag{35}$$

where we have used Theorem 2.11.

Since $\fint_Q f(x)\,dx = \fint_Q \phi^{-1}_{p(x),q(x)}(1/|Q|)\,dx > 0$, from equation (35) we obtain that

$$|Q| \fint_Q (\log(e+1/|Q|))^{q(x)}\,\phi^{-1}_{p'(x),q(x)}(1/|Q|)\,dx \lesssim \|\chi_Q\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}}$$
$$\lesssim \frac{1}{\fint_Q \phi^{-1}_{p(x),q(x)}(1/|Q|)\,dx}. \tag{36}$$

By Lemma 3.4 we can estimate the right-hand side of inequality (36) using equation (31) as follow

$$\frac{1}{\fint_Q \phi^{-1}_{p(x),q(x)}(1/|Q|)\,dx} \lesssim \frac{1}{\phi^{-1}_{\frac{1}{(1/p)_Q},\frac{(q/p)_Q}{(1/p)_Q}}(1/|Q|)} \simeq |Q|^{(1/p)_Q}(\log(e+1/|Q|))^{(q/p)_Q}.$$

In order to estimate the left-hand side of inequality (36), if $x\in Q$, by Jensen's inequality and Lemma 3.3,

$$(\log(e+1/|Q|))^{q_Q} \leqslant \fint_Q (\log(e+1/|Q|))^{q(y)}\,dy \simeq (\log(e+1/|Q|))^{q(x)}.$$

Thus by Lemma 3.4 we have

$$|Q| \fint_Q (\log(e+1/|Q|))^{q(x)}\,\phi^{-1}_{p'(x),q(x)}(1/|Q|)\,dx$$
$$\gtrsim |Q|(\log(e+1/|Q|))^{q_Q} \fint_Q \phi^{-1}_{p'(x),q(x)}(1/|Q|)\,dx$$
$$\gtrsim |Q|(\log(e+1/|Q|))^{q_Q}\,\phi^{-1}_{\frac{1}{(1/p')_Q},\frac{(q/p')_Q}{(1/p')_Q}}(1/|Q|)$$
$$\gtrsim |Q|(\log(e+1/|Q|))^{q_Q}\,|Q|^{-(1/p')_Q}\,(\log(e+1/|Q|))^{-(q/p')_Q}$$
$$\simeq |Q|^{(1/p)_Q}\,(\log(e+1/|Q|))^{(q/p)_Q}. \quad \square$$

COROLLARY 3.6. *Let $p(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$ with $p^+<\infty$ and let Q be a cube in $\mathbb{R}^n$. Then*

$$\fint_Q |Q|^{1/p(x)}\,dx \lesssim \|\chi_Q\|_{p(\cdot)}.$$

Proof. From the proof of Proposition 3.1, by using inequality (35) replacing $p(\cdot)$ and $q(\cdot)$ by $p'(\cdot)$ and 0, respectively, we have

$$\|\chi_Q\|_{p'(\cdot)}\fint_Q (1/|Q|)^{1/p'(x)}\,dx \lesssim 1. \tag{37}$$

Since

$$\int_Q (1/|Q|)^{1/p'(x)}\,dx = \int_Q |Q|^{1/p(x)-1}\,dx = \fint_Q |Q|^{1/p(x)}\,dx$$

by (37) we obtain that

$$\frac{\|\chi_Q\|_{p'(\cdot)}}{|Q|}\fint_Q |Q|^{1/p(x)}\,dx \lesssim 1.$$

Thus, by Lemma 2.5,

$$\fint_Q |Q|^{1/p(x)}\,dx \lesssim \|\chi_Q\|_{p(\cdot)}. \quad \square$$

We now show that the Examples 1.1 and 1.2 satisfy the hypotheses of Theorem 1.2.

Let us see 1.1. For $p(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$ with $1<p^-\leqslant p^+<\infty$ and $\sigma>(p')^+/(p')^-$, $A_1(x,t)=t^{\sigma p'(x)}(\log(e+t))^{\sigma p'(x)}$, $B_1(x,t)=t^{(\sigma p')'(x)}$ and $D_1(t)=t\log(e+t)$.

If we define $s(\cdot):=(\sigma p')'(\cdot)$ and $l(\cdot):=p(\cdot)/s(\cdot)$, by Lemma 2.4(ii) and (iv), $s(\cdot),l(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$. Moreover, $l^->1$. In fact, since $\sigma>(p')^+/(p')^-$,

$$(p^-)'=(p')^+<\sigma(p')^-=(\sigma p')^-$$

which implies that

$$p^->[(\sigma p')^-]'=[(\sigma p')']^+$$

and then

$$1<\frac{p^-}{[(\sigma p')']^+}\leqslant l^-.$$

Thus, we can apply Theorem 2.12 and Theorem 2.13 to obtain that

$$M_{L^{(\sigma p')'(\cdot)}}:L^{p(\cdot)}(\mathbb{R}^n)\to L^{p(\cdot)}(\mathbb{R}^n).$$

and

$$M_{\beta(\cdot),L^{(\sigma p')'(\cdot)}}:L^{p(\cdot)}(\mathbb{R}^n)\to L^{p(\cdot)}(\mathbb{R}^n),$$

respectively. Condition 11 it follows from Lemma 2.1. By Remark 3.2,

$$\begin{aligned}\|\chi_Q\|_{A_1(\cdot,L)}\|\chi_Q\|_{B_1(\cdot,L)} &= \|\chi_Q\|_{L^{\sigma p'(\cdot)}(\log L)^{\sigma p'(\cdot)}}\|\chi_Q\|_{L^{(\sigma p')'(\cdot)}}\\ &\simeq |Q|^{(1/\sigma p')_Q}\log(e+1/|Q|)|Q|^{(1/(\sigma p')')_Q} = |Q|\log(e+1/|Q|)\\ &\simeq \|\chi_Q\|_{L\log L} = \|\chi_Q\|_{D_1(L)},\end{aligned}$$

by equation (27), and thus condition 9 is satisfied. Condition 10 follows from the fact that, by equation (31),

$$A_1^{-1}(x,t)B_1^{-1}(x,t) \simeq \frac{t^{1/\sigma p'(x)}}{\log(e+t)}t^{1/(\sigma p')'(x)} = \frac{t}{\log(e+t)} \simeq D_1^{-1}(t).$$

Let us now see 1.2. Recall that for $p(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$ with $1<p^-\leqslant p^+<\infty$ and $\sigma>(p')^+/(p')^-$, $\mu(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$ such that $1<\mu^-\leqslant\mu^+<\infty$ and

$$1/\sigma p'(\cdot)-1/\mu(\cdot)>\varepsilon, \tag{38}$$

for some constant $\varepsilon\in(0,1)$ and $\nu(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$, $A_2(x,t)=t^{\mu(x)}(\log(e+t))^{\nu(x)\mu(x)}$, $B_2(x,t)=t^{(\sigma p')'(x)}$ and $D_2(x,t)=t^{\alpha(x)}(\log(e+t))^{\alpha(x)\nu(x)}$ where $\alpha(\cdot)$ is defined by $1/\alpha(\cdot)=1/\mu(\cdot)+1/(\sigma p')'(\cdot)$.

Note that, by Lemma 2.4(iii), $\alpha(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$. Moreover, $1<\alpha^-\leqslant\alpha^+<\infty$. In fact, by inequality (38),

$$\frac{1}{\alpha(\cdot)}=\frac{1}{\mu(\cdot)}+\frac{1}{(\sigma p')'(\cdot)}<\frac{1}{\sigma p'(\cdot)}+\frac{1}{(\sigma p')'(\cdot)}-\varepsilon=1-\varepsilon.$$

Thus, $\alpha^-\geqslant 1/(1-\varepsilon)>1$. Also,

$$\alpha(\cdot)=\frac{\mu(\cdot)(\sigma p')'(\cdot)}{\mu(\cdot)+(\sigma p')'(\cdot)}\leqslant\mu^+<\infty.$$

Then, by Remark 1.3, $(\alpha\nu)(\cdot),(\mu\nu)(\cdot)\in\mathscr{P}^{\log\log}(\mathbb{R}^n)$. Thus, by Proposition 3.1 and equation (26), we have

$$\begin{aligned}\|\chi_Q\|_{A_2(\cdot,L)}\|\chi_Q\|_{B_2(\cdot,L)} &= \|\chi_Q\|_{L^{\mu(\cdot)}(\log L)^{(\mu\nu)(\cdot)}}\|\chi_Q\|_{L^{(\sigma p')'(\cdot)}}\\ &\simeq |Q|^{(1/\mu)_Q}(\log(e+1/|Q|))^{\nu_Q}|Q|^{(1/(\sigma p')')_Q}\\ &\simeq |Q|^{(1/\alpha)_Q}(\log(e+1/|Q|))^{\nu_Q}\simeq\|\chi_Q\|_{L^{\alpha(\cdot)}(\log L)^{(\alpha\nu)(\cdot)}}\\ &\simeq \|\chi_Q\|_{D_2(\cdot,L)}.\end{aligned}$$

Then 9 holds. On the other hand, by equation (31),

$$A_2^{-1}(x,t)B_2^{-1}(x,t)\simeq\frac{t^{1/\mu(x)}}{(\log(e+t))^{\nu(x)}}t^{1/(\sigma p')'(x)}\simeq\frac{t^{1/\alpha(x)}}{(\log(e+t))^{\nu(x)}}\simeq D_2^{-1}(x,t),$$

thus 10 holds. Note that, by Lemma 2.11 with $p(\cdot) := \alpha(\cdot)$ and $q(\cdot) := (\alpha\nu)(\cdot)$, $M : L^{\alpha(\cdot)}(\log L)^{(\alpha\nu)(\cdot)}(\mathbb{R}^n) \to L^{\alpha(\cdot)}(\log L)^{(\alpha\nu)(\cdot)}(\mathbb{R}^n)$. Thus, by duality (see equation (15)), we have that

$$\begin{aligned}
\|\chi_Q\|_{D_2(\cdot,L)}\|\chi_Q\|_{D_2^*(\cdot,L)} &\lesssim \|\chi_Q\|_{D_2(\cdot,L)} \sup_{\|g\|_{D_2(\cdot,L)}\leqslant 1} \int_Q |g(x)|\,dx \\
&= \sup_{\|g\|_{D_2(\cdot,L)}\leqslant 1} \left\|\chi_Q \int_Q |g(x)|\,dx\right\|_{D_2(\cdot,L)} \\
&= |Q| \sup_{\|g\|_{D_2(\cdot,L)}\leqslant 1} \left\|\chi_Q \frac{1}{|Q|}\int_Q |g(x)|\,dx\right\|_{D_2(\cdot,L)} \\
&\leqslant |Q| \sup_{\|g\|_{D_2(\cdot,L)}\leqslant 1} \|\chi_Q M g\|_{D_2(\cdot,L)} \leqslant |Q|.
\end{aligned}$$

Then condition 11 holds.

3.2. Estimates in $\mathbb{L}(\delta(\cdot))$

We now give some previous estimates for the symbol functions we are interested in.

LEMMA 3.7. *Let k be a positive integer and $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ with $1 < p^- \leqslant p^+ < \infty$. Let $a \in T_\infty$ and $b \in \mathscr{L}_a^1$. Then, for every cube $Q \subset \mathbb{R}^n$,*

$$\frac{\|\chi_Q(b-b_Q)^k\|_{L^{p(\cdot)}}}{\|\chi_Q\|_{L^{p(\cdot)}}} \lesssim \left(a(Q)\,\|b\|_{\mathscr{L}_a^1}\right)^k. \tag{39}$$

Proof. Let Q be a fixed cube. By Lemma 2.10 there exist a constant $0 < \nu < 1$ independent of Q such that for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\|\chi_Q|f|^\nu\|_{L^{p(\cdot)}} \lesssim (|f|_Q)^\nu\,\|\chi_Q\|_{L^{p(\cdot)}}. \tag{40}$$

We now put $f(x) = (b(x)-b_Q)^{k/\nu}$. Noticing that $k/\nu > 1$, by Theorem 2.15, we have

$$\begin{aligned}
(|f|_Q)^\nu &= \left(\frac{1}{|Q|}\int_Q |b(x)-b_Q|^{k/\nu}\,dx\right)^\nu = \left[\frac{a(Q)}{a(Q)}\left(\frac{1}{|Q|}\int_Q |b(x)-b_Q|^{k/\nu}\,dx\right)^{\nu/k}\right]^k \\
&\simeq \left[a(Q)\left(\frac{1}{a(Q)|Q|}\int_Q |b(x)-b_Q|\,dx\right)\right]^k \lesssim \left[a(Q)\,\|b\|_{\mathscr{L}_a^1}\right]^k. \quad \square
\end{aligned}$$

LEMMA 3.8. *Let $a \in T_\infty$ and $b \in \mathscr{L}_a^1$, then the following inequality*

$$|b_{3Q} - b_Q| \lesssim \|a\|_{\mathbf{t}_\infty}\, a(3Q)\,\|b\|_{\mathscr{L}_a^1}$$

holds for every cube $Q \subset \mathbb{R}^n$.

Proof. Let Q be a fixed cube. Then, by T_∞ condition (3), we have that

$$\begin{aligned}|b_{3Q}-b_Q| &\leqslant |b_{3Q}-b_{2Q}|+|b_{2Q}-b_Q|\\ &\leqslant \frac{1}{|2Q|}\int_{2Q}|b(x)-b_{3Q}|\,dx+\frac{1}{|Q|}\int_{Q}|b(x)-b_{2Q}|\,dx\\ &\lesssim \frac{1}{|3Q|}\int_{3Q}|b(x)-b_{3Q}|\,dx+\frac{1}{|2Q|}\int_{2Q}|b(x)-b_{2Q}|\,dx\\ &\lesssim a(3Q)\,\|b\|_{\mathscr{L}_a^1}+a(2Q)\,\|b\|_{\mathscr{L}_a^1}\lesssim \|a\|_{t_\infty}a(3Q)\,\|b\|_{\mathscr{L}_a^1}. \quad\square\end{aligned}$$

In the proof of Theorem 1.2 we shall use the following pointwise estimate for $b\in\mathbb{L}(\delta(\cdot))$.

LEMMA 3.9. *Let $r(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$ with $r_\infty\leqslant r(\cdot)\leqslant r^+<\infty$ and $\delta(\cdot)$ be defined as in (6) and $b\in\mathbb{L}(\delta(\cdot))$. Let Q be a cube in $\mathbb{R}^n$ and $z\in kQ$ for some positive integer k. Then*

$$|b(z)-b_Q|\lesssim\|\chi_Q\|_{n/\delta(\cdot)}.$$

Proof. Note that if $x\in Q$ and $z\in kQ$ for some positive integer k, then we have $|z-x|\lesssim|Q|^{1/n}$. Thus by Lemma 2.16 and Corollary 3.6 we obtain

$$|b(z)-b_Q|\leqslant \fint_Q|b(z)-b(x)|\,dx\lesssim\fint_Q|z-x|^{\delta(x)}\,dx\lesssim\fint_Q|Q|^{\delta(x)/n}\,dx\lesssim\|\chi_Q\|_{n/\delta(\cdot)}. \quad\square$$

4. Proof of main results

In this section we present the proofs of Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. Since $v\in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$ implies that the set of bounded functions with compact support is dense in $L^{p(\cdot)}_v(\mathbb{R}^n)$, it is enough to show that

$$\left\|T_K^{b,m}f\right\|_{L_w^{q(\cdot)}}\lesssim\|f\|_{L_v^{p(\cdot)}}$$

for each non-negative bounded function with compact support f. Moreover, by duality (see equation (15)) this is equivalent to prove that

$$\int_{\mathbb{R}^n}|T_K^{b,m}f(x)|w(x)g(x)\,dx\lesssim\|f\|_{L_v^{p(\cdot)}}$$

for all non-negative bounded functions with compact support f,g with $\|g\|_{L^{q'(\cdot)}}\leqslant 1$.

Let $\overline{K}$ be the function defined by

$$\overline{K}(t)=\sup_{t<|x|\leqslant 2t}K(x),$$

for every $t > 0$. It was proved in [7, Proof of Theorem 2.2] that, if $K \in \mathfrak{D}$, we can estimate the commutator as follows

$$|T_K^{b,m} f(x)| \leqslant \sum_Q \overline{K}\left(\frac{\ell(Q)}{2}\right) \sum_{j=0}^{m} \binom{m}{j} |b(x) - b_Q|^{m-j} \chi_Q(x) \int_{3Q} |b(z) - b_Q|^j f(z)\, dz,$$

where the sum is taken over all dyadic cubes of $\mathbb{R}^n$. Hence

$$\begin{aligned} &\int_{\mathbb{R}^n} |T_K^{b,m} f(x)| w(x) g(x)\, dx \\ &\lesssim \sum_Q \overline{K}\left(\frac{\ell(Q)}{2}\right) \sum_{j=0}^{m} \int_{3Q} |b(z) - b_Q|^j f(z)\, dz \int_Q |b(x) - b_Q|^{m-j} g(x) w(x)\, dx. \end{aligned} \tag{41}$$

Let us denote $s(\cdot) := Rp'(\cdot)$ and $l(\cdot) := Sq(\cdot)$. Since $(p')^+ < R(p')^-$ and $q^+ < Sq^-$ then $(s')^+ < p^-$ and $(l')^+ < (q^+)'$. Let μ, ν be two constants such that

$$(s')^+ < \mu < p^- \qquad \text{and} \qquad (l')^+ < \nu < (q^+)',$$

and $\omega(\cdot), \tau(\cdot)$ defined by

$$\frac{1}{\omega(\cdot)} = \frac{1}{s(\cdot)} + \frac{1}{\mu} \qquad \text{and} \qquad \frac{1}{\tau(\cdot)} = \frac{1}{l(\cdot)} + \frac{1}{\nu}.$$

Observe that, by Lemma 2.4, $\omega(\cdot), \tau(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ since $s(\cdot), l(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$. Using Hölder's inequality (20) twice and Lemma 2.5, we can estimate (41) by a multiple of

$$\begin{aligned} &\sum_Q \overline{K}\left(\frac{\ell(Q)}{2}\right) \sum_{j=0}^{m} |3Q| \frac{\left\|\chi_{3Q}|b - b_Q|^j\right\|_{L^{\omega'(\cdot)}}}{\|\chi_{3Q}\|_{L^{\omega'(\cdot)}}} \frac{\|\chi_{3Q} f\|_{L^{\omega(\cdot)}}}{\|\chi_{3Q}\|_{L^{\omega(\cdot)}}} \\ &\times |Q| \frac{\left\|\chi_Q|b - b_Q|^{m-j}\right\|_{L^{\tau'(\cdot)}}}{\|\chi_Q\|_{L^{\tau'(\cdot)}}} \frac{\|\chi_Q g w\|_{L^{\tau(\cdot)}}}{\|\chi_Q\|_{L^{\tau(\cdot)}}}. \end{aligned} \tag{42}$$

Notice that, by Lemmas 3.7 and 3.8, we have

$$\begin{aligned} \frac{\left\|\chi_{3Q}|b - b_Q|^j\right\|_{L^{\omega'(\cdot)}}}{\|\chi_{3Q}\|_{L^{\omega'(\cdot)}}} &\lesssim \frac{\left\|\chi_{3Q}|b - b_{3Q}|^j\right\|_{L^{\omega'(\cdot)}}}{\|\chi_{3Q}\|_{L^{\omega'(\cdot)}}} + \frac{\left\|\chi_{3Q}|b_{3Q} - b_Q|^j\right\|_{L^{\omega'(\cdot)}}}{\|\chi_{3Q}\|_{L^{\omega'(\cdot)}}} \\ &\lesssim \left(\|a\|_{t_\infty} a(3Q)\, \|b\|_{\mathscr{L}_a^1}\right)^j. \end{aligned}$$

Thus, since $a \in T_\infty$, we can estimate (42) as follows

$$\begin{aligned} &\int_{\mathbb{R}^n} |T_K^{b,m} f(x)| w(x) g(x)\, dx \\ &\lesssim \sum_Q \overline{K}\left(\frac{\ell(Q)}{2}\right) \sum_{j=0}^{m} |3Q| \left(\|a\|_{t_\infty} a(3Q)\, \|b\|_{\mathscr{L}_a^1}\right)^j \frac{\|\chi_{3Q} f\|_{L^{\omega(\cdot)}}}{\|\chi_{3Q}\|_{L^{\omega(\cdot)}}} \\ &\quad \times |Q| \left(\|a\|_{t_\infty} a(Q)\, \|b\|_{\mathscr{L}_a^1}\right)^{m-j} \frac{\|\chi_Q g w\|_{L^{\tau(\cdot)}}}{\|\chi_Q\|_{L^{\tau(\cdot)}}} \\ &\lesssim \|b\|_{\mathscr{L}_a^1}^m \sum_Q a(3Q)^m\, \overline{K}\left(\frac{\ell(Q)}{2}\right) |3Q| \frac{\|\chi_{3Q} f\|_{L^{\omega(\cdot)}}}{\|\chi_{3Q}\|_{L^{\omega(\cdot)}}} |Q| \frac{\|\chi_Q g w\|_{L^{\tau(\cdot)}}}{\|\chi_Q\|_{L^{\tau(\cdot)}}}. \end{aligned} \tag{43}$$

Since g has compact support and $w \in L^{Sq}_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\lim_{\ell(Q)\to\infty} \frac{\|\chi_Q g w\|_{L^{\tau(\cdot)}}}{\|\chi_Q\|_{L^{\tau(\cdot)}}} = 0.$$

Let C_τ, C_τ^*, C_τ^{**} and G_τ be the constants provided by Lemma 2.7, Lemma 2.5 and Theorem 2.8 respectively. If $\alpha > C_\tau C_\tau^* C_\tau^{**} G_\tau$ and $k \in \mathbb{Z}$, it follows that, if for some dyadic cube Q,

$$\alpha^k < \frac{\|\chi_Q g w\|_{L^{\tau(\cdot)}}}{\|\chi_Q\|_{L^{\tau(\cdot)}}}, \tag{44}$$

then Q is contained in dyadic cubes satisfying this condition, which are maximal with respect to the inclusion. Thus, for each integer k there is a family of maximal non-overlapping dyadic cubes $\{Q_{k,j}\}_{j\in\mathbb{Z}}$ satisfying (44). Let $Q'_{k,j}$ be the dyadic cube containing $Q_{k,j}$ with sidelength $2\ell(Q_{k,j})$. Then, by maximality and Lemma 2.7, we have

$$\alpha^k < \frac{\left\|\chi_{Q_{k,j}} g w\right\|_{L^{\tau(\cdot)}}}{\left\|\chi_{Q_{k,j}}\right\|_{L^{\tau(\cdot)}}} \leqslant \frac{\left\|\chi_{Q'_{k,j}}\right\|_{L^{\tau(\cdot)}}}{\left\|\chi_{Q_{k,j}}\right\|_{L^{\tau(\cdot)}}} \frac{\left\|\chi_{Q'_{k,j}} g w\right\|_{L^{\tau(\cdot)}}}{\left\|\chi_{Q'_{k,j}}\right\|_{L^{\tau(\cdot)}}} \leqslant C_\tau \alpha^k \leqslant \alpha^{k+1}.$$

For $k \in \mathbb{Z}$ we define the set

$$\mathscr{C}_k := \left\{ Q \text{ dyadic } : \alpha^k < \frac{\|\chi_Q g w\|_{L^{\tau(\cdot)}}}{\|\chi_Q\|_{L^{\tau(\cdot)}}} \leqslant \alpha^{k+1} \right\}.$$

Then every dyadic cube Q for which $\|\chi_Q g w\|_{L^{\tau(\cdot)}} / \|\chi_Q\|_{L^{\tau(\cdot)}} \neq 0$ belongs to exactly one $\mathscr{C}_k$. Furthermore, if $Q \in \mathscr{C}_k$, it follows that $Q \subset Q_{k,j}$ for some j. Then, from (43) and T_∞ condition (3), we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^n} |T_K^{b,m} f(x)| w(x) g(x)\, dx \\
&\lesssim \|b\|^m_{\mathscr{L}^1_a} \sum_{k\in\mathbb{Z}} \sum_{Q\in\mathscr{C}_k} a(3Q)^m \overline{K}\left(\frac{\ell(Q)}{2}\right) |3Q| \frac{\|\chi_{3Q} f\|_{L^{\omega(\cdot)}}}{\|\chi_{3Q}\|_{L^{\omega(\cdot)}}} |Q| \frac{\|\chi_Q g w\|_{L^{\tau(\cdot)}}}{\|\chi_Q\|_{L^{\tau(\cdot)}}} \\
&\lesssim \|b\|^m_{\mathscr{L}^1_a} \sum_{(k,j)\in\mathbb{Z}\times\mathbb{Z}} \alpha^{k+1} \sum_{Q\in\mathscr{C}_k : Q\subset Q_{k,j}} a(3Q)^m \overline{K}\left(\frac{\ell(Q)}{2}\right) |3Q||Q| \frac{\|\chi_{3Q} f\|_{L^{\omega(\cdot)}}}{\|\chi_{3Q}\|_{L^{\omega(\cdot)}}} \\
&\lesssim \|b\|^m_{\mathscr{L}^1_a} \alpha \sum_{(k,j)\in\mathbb{Z}\times\mathbb{Z}} \frac{\left\|\chi_{Q_{k,j}} g w\right\|_{L^{\tau(\cdot)}}}{\left\|\chi_{Q_{k,j}}\right\|_{L^{\tau(\cdot)}}} a(3Q_{k,j})^m \\
&\quad \times \sum_{Q\in\mathscr{C}_k : Q\subset Q_{k,j}} \overline{K}\left(\frac{\ell(Q)}{2}\right) |3Q||Q| \frac{\|\chi_{3Q} f\|_{L^{\omega(\cdot)}}}{\|\chi_{3Q}\|_{L^{\omega(\cdot)}}}.
\end{aligned} \tag{45}$$

If we show that there is a constant C_K such that, for any dyadic cube Q_0,

$$\sum_{Q : Q\subset Q_0} \overline{K}\left(\frac{\ell(Q)}{2}\right) |3Q||Q| \frac{\|\chi_{3Q} f\|_{L^{\omega(\cdot)}}}{\|\chi_{3Q}\|_{L^{\omega(\cdot)}}} \leqslant C_K \widetilde{K}(\delta(1+\varepsilon)\ell(Q_0)) |3Q_0| \frac{\|\chi_{3Q_0} f\|_{L^{\omega(\cdot)}}}{\|\chi_{3Q_0}\|_{L^{\omega(\cdot)}}}, \tag{46}$$

with ε,δ the numbers provided by condition $\mathfrak{D}$ and $\widetilde{K}(t)=\int_{|z|\leqslant t}K(z)\,dz$, from (45) we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^n}|T_K^{b,m}f(x)|w(x)g(x)\,dx\\
&\lesssim \|b\|_{\mathscr{L}_a^1}^m \sum_{(k,j)\in\mathbb{Z}\times\mathbb{Z}} a(3Q_{k,j})^m C_K\widetilde{K}(\delta(1+\varepsilon)\ell(Q_{k,j}))|3Q_{k,j}|\\
&\quad\times\frac{\left\|\chi_{3Q_{k,j}}f\right\|_{L^{\omega(\cdot)}}}{\left\|\chi_{3Q_{k,j}}\right\|_{L^{\omega(\cdot)}}}\frac{\left\|\chi_{Q_{k,j}}gw\right\|_{L^{\tau(\cdot)}}}{\left\|\chi_{Q_{k,j}}\right\|_{L^{\tau(\cdot)}}}.
\end{aligned}\tag{47}$$

Let $\gamma=\max\{3,\delta(1+\varepsilon)\}$. Note that $\widetilde{K}$ is an increasing function. From (47), by Lemma 2.7, T_∞ condition (3), Hölder's inequality and Lemma 2.6 we have that

$$\begin{aligned}
&\int_{\mathbb{R}^n}|T_K^{b,m}f(x)|w(x)g(x)\,dx\\
&\lesssim \|b\|_{\mathscr{L}_a^1}^m \sum_{(k,j)\in\mathbb{Z}\times\mathbb{Z}} a(\gamma Q_{k,j})^m\widetilde{K}(\gamma\ell(Q_{k,j}))|\gamma Q_{k,j}|\frac{\left\|\chi_{\gamma Q_{k,j}}f\right\|_{L^{\omega(\cdot)}}}{\left\|\chi_{\gamma Q_{k,j}}\right\|_{L^{\omega(\cdot)}}}\frac{\left\|\chi_{\gamma Q_{k,j}}gw\right\|_{L^{\tau(\cdot)}}}{\left\|\chi_{\gamma Q_{k,j}}\right\|_{L^{\tau(\cdot)}}}\\
&\lesssim \|b\|_{\mathscr{L}_a^1}^m \sum_{(k,j)\in\mathbb{Z}\times\mathbb{Z}} a(\gamma Q_{k,j})^m\widetilde{K}(\gamma\ell(Q_{k,j}))|\gamma Q_{k,j}|\frac{\left\|\chi_{\gamma Q_{k,j}}fv\right\|_{\mu}}{\left\|\chi_{\gamma Q_{k,j}}\right\|_{\mu}}\frac{\left\|\chi_{\gamma Q_{k,j}}v^{-1}\right\|_{L^{s(\cdot)}}}{\left\|\chi_{\gamma Q_{k,j}}\right\|_{L^{s(\cdot)}}}\\
&\quad\times\frac{\left\|\chi_{\gamma Q_{k,j}}g\right\|_{\nu}}{\left\|\chi_{\gamma Q_{k,j}}\right\|_{\nu}}\frac{\left\|\chi_{\gamma Q_{k,j}}w\right\|_{L^{l(\cdot)}}}{\left\|\chi_{\gamma Q_{k,j}}\right\|_{L^{l(\cdot)}}}.
\end{aligned}$$

Thus, by Fefferman-Phong type condition (5) on the weights we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n}|T_K^{b,m}f(x)|w(x)g(x)\,dx\\
&\leqslant \kappa\|b\|_{\mathscr{L}_a^1}^m \sum_{(k,j)\in\mathbb{Z}\times\mathbb{Z}}|Q_{k,j}|\frac{\left\|\chi_{\gamma Q_{k,j}}fv\right\|_{\mu}}{\left\|\chi_{\gamma Q_{k,j}}\right\|_{\mu}}\frac{\left\|\chi_{\gamma Q_{k,j}}g\right\|_{\nu}}{\left\|\chi_{\gamma Q_{k,j}}\right\|_{\nu}}\frac{\left\|\chi_{\gamma Q_{k,j}}\right\|_{L^{p(\cdot)}}}{\left\|\chi_{\gamma Q_{k,j}}\right\|_{L^{q(\cdot)}}}.
\end{aligned}$$

Let $\beta(\cdot)$ defined as in Lemma 2.6. Then, by this lemma, the last sum is equivalent to

$$\kappa\|b\|_{\mathscr{L}_a^1}^m \sum_{(k,j)\in\mathbb{Z}\times\mathbb{Z}}|Q_{k,j}|\frac{\left\|\chi_{\gamma Q_{k,j}}fv\right\|_{\mu}}{\left\|\chi_{\gamma Q_{k,j}}\right\|_{\mu}}\left\|\chi_{\gamma Q_{k,j}}\right\|_{L^{\beta(\cdot)}}\frac{\left\|\chi_{\gamma Q_{k,j}}g\right\|_{\nu}}{\left\|\chi_{\gamma Q_{k,j}}\right\|_{\nu}}.\tag{48}$$

For each $k,j\in\mathbb{Z}$ we can consider the sets

$$D_k=\bigcup_{j\in\mathbb{Z}}Q_{k,j}\quad\text{and}\quad F_{k,j}=Q_{k,j}\backslash(Q_{k,j}\cap D_{k+1}).$$

Thus $\{F_{k,j}\}_{(k,j)\in\mathbb{Z}\times\mathbb{Z}}$ is a disjoint family of sets which satisfy

$$|Q_{k,j}\cap D_{k+1}|<\frac{\Pi}{\alpha}|Q_{k,j}| \tag{49}$$

for some positive constant $\Pi<\alpha$, and

$$|Q_{k,j}|<\frac{1}{1-\Pi/\alpha}|F_{k,j}|. \tag{50}$$

Deferring the proof of these inequalities for the moment, we can estimate (48) to obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n}|T_K^{b,m}f(x)|w(x)g(x)\,dx\\
&\lesssim\kappa\,\|b\|_{\mathscr{L}_a^1}^m\sum_{(k,j)\in\mathbb{Z}\times\mathbb{Z}}|F_{k,j}|\frac{\left\|\chi_{\gamma Q_{k,j}}fv\right\|_{\mu}}{\left\|\chi_{\gamma Q_{k,j}}\right\|_{\mu}}\left\|\chi_{\gamma Q_{k,j}}\right\|_{L^{\beta(\cdot)}}\frac{\left\|\chi_{\gamma Q_{k,j}}g\right\|_{\nu}}{\left\|\chi_{\gamma Q_{k,j}}\right\|_{\nu}}\\
&\lesssim\kappa\,\|b\|_{\mathscr{L}_a^1}^m\int_{\mathbb{R}^n}M_{L^\mu}(fv)(y)dyM_{\beta(\cdot),\nu}(g)(y)\lesssim\kappa\,\|b\|_{\mathscr{L}_a^1}^m\|M_{L^\mu}(fv)\|_{L^{p(\cdot)}}\left\|M_{\beta(\cdot),L^\nu}(g)\right\|_{L^{p'(\cdot)}}\\
&\lesssim\kappa\,\|b\|_{\mathscr{L}_a^1}^m\|fv\|_{L^{p(\cdot)}},
\end{aligned}$$

where we have used that by Theorem 2.12, $M_{L^\mu}:L^{p(\cdot)}(\mathbb{R}^n)\hookrightarrow L^{p(\cdot)}(\mathbb{R}^n)$ since $p^->\mu$, and by Remark 2.14, $M_{\beta(\cdot),L^\nu}:L^{q'(\cdot)}(\mathbb{R}^n)\hookrightarrow L^{p'(\cdot)}(\mathbb{R}^n)$ since $(q')^->\nu$ (see (7) and (8) for the definition of this maximal operatos).

To prove (49), note that if for some $k,j,i\in\mathbb{Z}$, $Q_{k,j}\cap Q_{k+1,i}\neq\emptyset$ then, by maximality and the fact that $\alpha>1$, $Q_{k+1,i}\subsetneq Q_{k,j}$. Thus

$$\begin{aligned}
|Q_{k,j}\cap D_{k+1}|&=\left|Q_{k,j}\cap\bigcup_{i\in\mathbb{Z}}Q_{k+1,i}\right|=\left|\bigcup_{i\in\mathbb{Z}}(Q_{k,j}\cap Q_{k+1,i})\right|=\sum_{i:Q_{k+1,i}\subseteq Q_{k,j}}|Q_{k+1,i}|\\
&\leqslant C_\tau^*\sum_{i:Q_{k+1,i}\subseteq Q_{k,j}}\left\|\chi_{Q_{k+1,i}}\right\|_{L^{\tau(\cdot)}}\left\|\chi_{Q_{k+1,i}}\right\|_{L^{\tau'(\cdot)}},
\end{aligned}$$

where the constant C_τ^* is provided by Lemma 2.5. On the other hand, by maximality and the property (44) of the cubes $Q_{k+1,i}$ and $Q_{k,j}$ we have

$$\text{(i) }\alpha^{k+1}<\frac{\left\|\chi_{Q_{k+1,i}}gw\right\|_{L^{\tau(\cdot)}}}{\left\|\chi_{Q_{k+1,i}}\right\|_{L^{\tau(\cdot)}}}\ \text{ and }\ \text{(ii) }\frac{\left\|\chi_{Q_{k,j}}gw\right\|_{L^{\tau(\cdot)}}}{\left\|\chi_{Q_{k,j}}\right\|_{L^{\tau(\cdot)}}}\leqslant C_\tau\alpha^k. \tag{51}$$

Then, by (51)(i) we have

$$\begin{aligned}
|Q_{k,j}\cap D_{k+1}|&\leqslant C_\tau^*\sum_{i:Q_{k+1,i}\subseteq Q_{k,j}}\left\|\chi_{Q_{k+1,i}}\right\|_{L^{\tau(\cdot)}}\left\|\chi_{Q_{k+1,i}}\right\|_{L^{\tau'(\cdot)}}\\
&<C_\tau^*\alpha^{-(k+1)}\sum_{i:Q_{k+1,i}\subseteq Q_{k,j}}\left\|\chi_{Q_{k+1,i}}gw\chi_{Q_{k,j}}\right\|_{L^{\tau(\cdot)}}\left\|\chi_{Q_{k+1,i}}\chi_{Q_{k,j}}\right\|_{L^{\tau'(\cdot)}}.
\end{aligned} \tag{52}$$

Note that, by Theorem 2.8, the following inequality holds

$$\sum_{i\in\mathbb{Z}} \left\|\chi_{Q_{k+1,i}} r\right\|_{L^{\tau(\cdot)}} \left\|\chi_{Q_{k+1,i}} h\right\|_{L^{\tau'(\cdot)}} \leqslant G_\tau \|r\|_{L^{\tau(\cdot)}} \|h\|_{L^{\tau'(\cdot)}},$$

for every $r \in L^{\tau(\cdot)}(\mathbb{R}^n)$ and $h \in L^{\tau'(\cdot)}(\mathbb{R}^n)$. Applying this with $r := gw\chi_{Q_{k,j}}$ and $h := \chi_{Q_{k,j}}$ we can estimate (52) as follows

$$|Q_{k,j} \cap D_{k+1}| < C_\tau^* \alpha^{-(k+1)} G_\tau \left\|gw\chi_{Q_{k,j}}\right\|_{L^{\tau(\cdot)}} \left\|\chi_{Q_{k,j}}\right\|_{L^{\tau'(\cdot)}}.$$

Then, by (51)(ii), we obtain that

$$\begin{aligned} |Q_{k,j} \cap D_{k+1}| &< C_\tau^* \alpha^{-(k+1)} C_\tau \alpha^k G_\tau \left\|\chi_{Q_{k,j}}\right\|_{L^{\tau(\cdot)}} \left\|\chi_{Q_{k,j}}\right\|_{L^{\tau'(\cdot)}} \\ &\leqslant C_\tau^* \alpha^{-(k+1)} C_\tau \alpha^k G_\tau C_\tau^{**} |Q_{k,j}| := \frac{\Pi}{\alpha} |Q_{k,j}| \end{aligned}$$

where the constant C_τ^{**} is provided by Lemma 2.5. This gives (49). Finally,

$$\frac{|F_{k,j}|}{|Q_{k,j}|} = \frac{|Q_{k,j} \setminus (Q_{k,j} \cap D_{k+1})|}{|Q_{k,j}|} = 1 - \frac{|Q_{k,j} \cap D_{k+1}|}{|Q_{k,j}|} > 1 - \frac{\Pi}{\alpha} > 0$$

since, $\alpha > \Pi$, and we obtain (50).

In order to complete the proof we must show that (46) holds. In fact, if $\ell(Q_0) = 2^{-d_0}$ with $d_0 \in \mathbb{Z}$, by Lemma 2.5 we have

$$\begin{aligned} &\sum_{Q:Q\subset Q_0} \overline{K}\left(\frac{\ell(Q)}{2}\right) |3Q||Q| \frac{\|\chi_{3Q} f\|_{L^{\omega(\cdot)}}}{\|\chi_{3Q}\|_{L^{\omega(\cdot)}}} \\ &\lesssim \sum_{d\geqslant d_0} \overline{K}(2^{-d-1}) 2^{-dn} \sum_{Q\subset Q_0:\ell(Q)=2^{-d}} \|f\chi_{3Q}\|_{L^{\omega(\cdot)}} \|\chi_{3Q}\|_{L^{\omega'(\cdot)}}. \end{aligned}$$

Thus, applying Lemma 2.9 with f and $g := \chi_{3Q_0}$, we obtain that

$$\begin{aligned} \sum_{Q:Q\subset Q_0} \overline{K}\left(\frac{\ell(Q)}{2}\right) |3Q||Q| \frac{\|\chi_{3Q} f\|_{L^{\omega(\cdot)}}}{\|\chi_{3Q}\|_{L^{\omega(\cdot)}}} &\lesssim \|f\chi_{3Q_0}\|_{L^{\omega(\cdot)}} \|\chi_{3Q_0}\|_{L^{\omega'(\cdot)}} \sum_{d\geqslant d_0} \overline{K}(2^{-d-1}) 2^{-dn} \\ &\lesssim \|f\chi_{3Q_0}\|_{L^{\omega(\cdot)}} \|\chi_{3Q_0}\|_{L^{\omega'(\cdot)}} \widetilde{K}(\delta(1+\varepsilon)\ell(Q_0)), \end{aligned}$$

where the last estimate follows as in [14]. This proves (46) and concludes the proof of Theorem 1.1. □

Proof of Theorem 1.2. We use the same technique as in the proof of the Theorem 1.1 to obtain that

$$\begin{aligned} &\int_{\mathbb{R}^n} |T_K^{b,m} f(x)| w(x) g(x)\, dx \\ &\lesssim \sum_Q \overline{K}\left(\frac{\ell(Q)}{2}\right) \sum_{j=0}^m \int_{3Q} |b(z)-b_Q|^j f(z)\, dz \int_Q |b(x)-b_Q|^{m-j} g(x) w(x)\, dx. \end{aligned}$$

Hence, by Lemma 3.9,

$$\int_{\mathbb{R}^n} |T_K^{b,m} f(x)| w(x) g(x)\, dx \lesssim \sum_Q \overline{K}\left(\frac{\ell(Q)}{2}\right) \|\chi_Q\|_{n/\delta(\cdot)}^m |Q| \int_{3Q} f(z)\, dz \fint_Q g(x) w(x)\, dx. \tag{53}$$

Thus, given some constant α larger than 2^n and proceeding as in [14, proof of theorem 2.1], for each $k \in \mathbb{Z}$ there exists a family of maximal non-overlaping dyadic cubes $\{Q_{k,j}\}_{j\in\mathbb{Z}}$, the Calderón-Zygmund cubes, such that we can estimate (53) by a multiple of

$$\sum_{(k,j)\in\mathbb{Z}\times\mathbb{Z}} \widetilde{K}(\ell(\gamma Q_{k,j})) \left\|\chi_{Q_{k,j}}\right\|_{n/\delta(\cdot)}^m |Q_{k,j}| \fint_{\gamma Q_{k,j}} f(z) dz \fint_{\gamma Q_{k,j}} g(z) w(z) dz, \tag{54}$$

where $\gamma = \max\{3, \delta(1+\varepsilon)\}$ with ε, δ the numbers provided by condition $\mathfrak{D}$. By condition $\mathscr{F}$ and Hölder's inequality we have

$$\fint_{\gamma Q_{k,j}} f(z)\, dz \lesssim \frac{\left\|\chi_{\gamma Q_{k,j}} f\right\|_{D(\cdot,L)}}{\left\|\chi_{\gamma Q_{k,j}}\right\|_{D(\cdot,L)}} \frac{\left\|\chi_{\gamma Q_{k,j}}\right\|_{D^*(\cdot,L)}}{\left\|\chi_{\gamma Q_{k,j}}\right\|_{D^*(\cdot,L)}} \lesssim \frac{\left\|\chi_{\gamma Q_{k,j}} f v\right\|_{B(\cdot,L)}}{\left\|\chi_{\gamma Q_{k,j}}\right\|_{B(\cdot,L)}} \frac{\left\|\chi_{\gamma Q_{k,j}} v^{-1}\right\|_{A(\cdot,L)}}{\left\|\chi_{\gamma Q_{k,j}}\right\|_{A(\cdot,L)}}$$

and

$$\fint_{\gamma Q_{k,j}} g(z) w(z)\, dz \lesssim \frac{\left\|\chi_{\gamma Q_{k,j}} g w\right\|_{J(\cdot,L)}}{\left\|\chi_{\gamma Q_{k,j}}\right\|_{J(\cdot,L)}} \frac{\left\|\chi_{\gamma Q_{k,j}}\right\|_{J^*(\cdot,L)}}{\left\|\chi_{\gamma Q_{k,j}}\right\|_{J^*(\cdot,L)}} \lesssim \frac{\left\|\chi_{\gamma Q_{k,j}} g\right\|_{H(\cdot,L)}}{\left\|\chi_{\gamma Q_{k,j}}\right\|_{H(\cdot,L)}} \frac{\left\|\chi_{\gamma Q_{k,j}} w\right\|_{E(\cdot,L)}}{\left\|\chi_{\gamma Q_{k,j}}\right\|_{E(\cdot,L)}}.$$

Then from (54) and by Fefferman-Phong type condition (14) on the weights we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} |T_K^{b,m} f(x)| w(x) g(x)\, dx \\
&\lesssim \sum_{(k,j)\in\mathbb{Z}\times\mathbb{Z}} \widetilde{K}(l(\gamma Q_{k,j})) \left\|\chi_{Q_{k,j}}\right\|_{n/\delta(\cdot)}^m |Q_{k,j}| \frac{\left\|\chi_{\gamma Q_{k,j}} f v\right\|_{B(\cdot,L)}}{\left\|\chi_{\gamma Q_{k,j}}\right\|_{B(\cdot,L)}} \frac{\left\|\chi_{\gamma Q_{k,j}} v^{-1}\right\|_{A(\cdot,L)}}{\left\|\chi_{\gamma Q_{k,j}}\right\|_{A(\cdot,L)}} \\
&\quad \times \frac{\left\|\chi_{\gamma Q_{k,j}} g\right\|_{H(\cdot,L)}}{\left\|\chi_{\gamma Q_{k,j}}\right\|_{H(\cdot,L)}} \frac{\left\|\chi_{\gamma Q_{k,j}} w\right\|_{E(\cdot,L)}}{\left\|\chi_{\gamma Q_{k,j}}\right\|_{E(\cdot,L)}} \\
&\leqslant \kappa \sum_{(k,j)\in\mathbb{Z}\times\mathbb{Z}} |Q_{k,j}| \frac{\left\|\chi_{\gamma Q_{k,j}} f v\right\|_{B(\cdot,L)}}{\left\|\chi_{\gamma Q_{k,j}}\right\|_{B(\cdot,L)}} \frac{\left\|\chi_{\gamma Q_{k,j}} g\right\|_{H(\cdot,L)}}{\left\|\chi_{\gamma Q_{k,j}}\right\|_{H(\cdot,L)}} \frac{\left\|\chi_{\gamma Q_{k,j}}\right\|_{L^{p(\cdot)}}}{\left\|\chi_{\gamma Q_{k,j}}\right\|_{L^{q(\cdot)}}}.
\end{aligned}$$

Let $\beta(\cdot)$ be defined as in Lemma 2.6. Then, by this lemma, the last sum is equivalent to

$$\kappa \sum_{(k,j)\in\mathbb{Z}\times\mathbb{Z}} |Q_{k,j}| \frac{\left\|\chi_{\gamma Q_{k,j}} f v\right\|_{B(\cdot,L)}}{\left\|\chi_{\gamma Q_{k,j}}\right\|_{B(\cdot,L)}} \left\|\chi_{\gamma Q_{k,j}}\right\|_{L^{\beta(\cdot)}} \frac{\left\|\chi_{\gamma Q_{k,j}} g\right\|_{H(\cdot,L)}}{\left\|\chi_{\gamma Q_{k,j}}\right\|_{H(\cdot,L)}}.$$

We shall use the following properties of Calderón-Zygmund cubes. For each integers k and j we can consider the sets $D_k = \bigcup_{j\in\mathbb{Z}} Q_{k,j}$ and $F_{k,j} = Q_{k,j} \setminus (Q_{k,j} \cap D_{k+1})$. Thus $\{F_{k,j}\}_{(k,j)\in\mathbb{Z}\times\mathbb{Z}}$ is a disjoint family of sets which satisfy

$$|Q_{k,j}| < \frac{1}{1-\frac{2^n}{\alpha}}|F_{k,j}|.$$

Then

$$\begin{aligned}\int_{\mathbb{R}^n} T_K f(x)g(x)w(x)\,dx &\lesssim \kappa \sum_{(k,j)\in\mathbb{Z}\times\mathbb{Z}} |F_{k,j}| \frac{\left\|\chi_{\gamma Q_{k,j}} f v\right\|_{B(\cdot,L)}}{\left\|\chi_{\gamma Q_{k,j}}\right\|_{B(\cdot,L)}} \left\|\chi_{\gamma Q_{k,j}}\right\|_{L^{\beta(\cdot)}} \frac{\left\|\chi_{\gamma Q_{k,j}} g\right\|_{H(\cdot,L)}}{\left\|\chi_{\gamma Q_{k,j}}\right\|_{H(\cdot,L)}} \\ &\leqslant \kappa \int_{\mathbb{R}^n} M_{B(L,\cdot)}(fv)(y) M_{\beta(\cdot),H(L,\cdot)}(g)(y)dy \\ &\lesssim \kappa \left\|M_{B(L,\cdot)}(fv)\right\|_{L^{p(\cdot)}} \left\|M_{\beta(\cdot),H(L,\cdot)}(g)\right\|_{L^{p'(\cdot)}} \lesssim \kappa \|fv\|_{L^{p(\cdot)}}\end{aligned}$$

where we have used the hyphotesis (12) and (13). □

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(Received May 15, 2018)

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