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QUANTITATIVE WEIGHTED L^p BOUNDS FOR THE MARCINKIEWICZ INTEGRAL

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Abstract. Let Ω be homogeneous of degree zero, have mean value zero and integrable on the unit sphere, and μ_Ω be the higher-dimensional Marcinkiewicz integral associated with Ω. In this paper, the authors proved that if $\Omega \in L^q(S^{n-1})$ for some $q \in (1,\infty]$, then for $p \in (q',\infty)$ and $w \in A_p(\mathbb{R}^n)$, the bound of μ_Ω on $L^p(\mathbb{R}^n,w)$ is less than $C[w]_{A_{p/q'}}^{\max\{\frac{1}{2},\frac{1}{p-q'}\}+\max\{1,\frac{q'}{p-q'}\}}$.

1. Introduction

We will work on $\mathbb{R}^n$, $n \geqslant 2$. Let M be the Hardy-Littlewood maximal operator, and $A_p(\mathbb{R}^n)$ $(p\in(1,\infty))$ be the weight function class of Muckenhoupt, that is,

$$A_p(\mathbb{R}^n) = \{w \text{ is nonnegative and locally integrable in } \mathbb{R}^n : [w]_{A_p} < \infty\}$$

(see [12, Chapter 9] for the properties of $A_p(\mathbb{R}^n)$), where and in what follows,

$$[w]_{A_p} := \sup_Q \Big(\frac{1}{|Q|}\int_Q w(x)dx\Big)\Big(\frac{1}{|Q|}\int_Q w^{-\frac{1}{p-1}}(x)dx\Big)^{p-1},$$

which is called the A_p constant of w. In the remarkable work, Buckley [4] proved that if $p\in(1,\infty)$ and $w\in A_p(\mathbb{R}^n)$, then

$$\|Mf\|_{L^p(\mathbb{R}^n,w)} \lesssim_{n,p} [w]_{A_p}^{\frac{1}{p-1}}\|f\|_{L^p(\mathbb{R}^n,w)}. \tag{1.1}$$

Moreover, the estimate (1.1) is sharp since the exponent $1/(p-1)$ can not be replaced by a smaller one. Since then, the sharp dependence of the weighted estimates of singular integral operators in terms of the $A_p(\mathbb{R}^n)$ constant has been considered by many authors. Petermichl [22, 23] solved this question for Hilbert transform and Riesz transform. Hytönen [13] proved that for a Calderón-Zygmund operator T and $w\in A_2(\mathbb{R}^n)$,

$$\|Tf\|_{L^2(\mathbb{R}^n,w)} \lesssim_n [w]_{A_2}\|f\|_{L^2(\mathbb{R}^n,w)}. \tag{1.2}$$

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This solved the so-called A_2 conjecture. Lerner [17, 18] gave two simple proofs of the A_2 conjecture by controlling the Calderón-Zygmund operator using sparse operators.

Recently, considerable attention has been paid to the weighted bounds for rough singular integral operators. Hytönen, Roncal and Tapiola [16] considered the weighted bounds of rough homogeneous singular integral operators defined by

$$T_{\Omega}f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(y')}{|y|^n} f(x-y)dy = \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \int_{\varepsilon<|x-y|<R} \frac{\Omega(y')}{|y|^n} f(x-y)dy,$$

where Ω is homogeneous of degree zero, integrable on the unit sphere S^{n-1} and has mean value zero. For $w \in \cup_{p>1} A_p(\mathbb{R}^n)$, $[u]_{A_\infty}$ is the A_∞ constant of u, defined by

$$[u]_{A_\infty} = \sup_{Q\subset\mathbb{R}^d} \frac{1}{u(Q)} \int_Q M(u\chi_Q)(x)dx,$$

see [28]. By a quantitative weighted estimate for the Calderón-Zygmund operators satisfying a Dini-condition, approximation to the identity and interpolation with change of measures, Hytönen, Roncal and Tapiola (see Theorem 1.4 in [16]) proved that

THEOREM 1.1. *Let Ω be homogeneous of degree zero, have mean value zero on S^{n-1} and $\Omega \in L^\infty(S^{n-1})$. Then for $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^n)$,*

$$\|T_\Omega f\|_{L^p(\mathbb{R}^n,w)} \lesssim \|\Omega\|_{L^\infty(S^{n-1})} \{w\}_{A_p} (w)_{A_p} \|f\|_{L^p(\mathbb{R}^n,w)}, \tag{1.3}$$

where and in the following, for $p \in (1,\infty)$,

$$\{w\}_{A_p} = [w]_{A_p}^{\frac{1}{p}} \max\{[w]_{A_\infty}^{\frac{1}{p'}}, [w^{1-p'}]_{A_\infty}^{\frac{1}{p}}\},$$

and

$$(w)_{A_p} = \max\{[w]_{A_\infty}, [w^{1-p'}]_{A_\infty}\}.$$

Conde-Alonso, Culiuc, Di Plinio and Ou [6] proved that for bounded function f and g, and $p \in (1,\infty)$,

$$\left| T_\Omega f(x) g(x)dx \right| \lesssim p' \sup_{\mathscr{S}} \sum_{Q\in\mathscr{S}} \langle |f|\rangle_Q \langle |g|\rangle_{Q,p} |Q|, \tag{1.4}$$

where the supremum is taken over all sparse family of cubes (see definition in Section 2), $\langle |f|\rangle_Q$ denotes the mean value of $|f|$ on Q, and for $r \in (0,\infty)$, $\langle |f|\rangle_{Q,r} = \big(\langle |f|^r\rangle_Q\big)^{1/r}$. By (1.4) Conde-Alonso et al recovered the conclusion in Theorem 1.1. By some new estimates for sparse operators, Li, Pérez, Rivera-Rios and Roncal [21] improved the estimate (1.3) proved that for $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^n)$,

$$\|T_\Omega f\|_{L^p(\mathbb{R}^n,w)} \lesssim \|\Omega\|_{L^\infty(S^{n-1})} \{w\}_{A_p} \min\{[w]_{A_\infty}, [w^{-\frac{1}{1-p}}]_{A_\infty}\} \|f\|_{L^p(\mathbb{R}^n,w)}.$$

Now we consider the Marcinkiewicz integral operator. For $n \geqslant 2$, let Ω be homogeneous of degree zero, integrable and have mean value zero on the unit sphere S^{n-1}. Define the Marcinkiewicz integral operator μ_Ω by

$$\mu_\Omega(f)(x) = \Big(\int_0^\infty |F_{\Omega,t}f(x)|^2 \frac{dt}{t^3}\Big)^{1/2},$$

where

$$F_{\Omega,t}f(x) = \int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy$$

for $f \in \mathscr{S}(\mathbb{R}^n)$. Stein [24] proved that if $\Omega \in \mathrm{Lip}_\gamma(S^{n-1})$ with $\gamma \in (0,1]$, then μ_Ω is bounded on $L^p(\mathbb{R}^n)$ for $p \in (1,2]$. Benedek, Calderón and Panzone [3] showed that the $L^p(\mathbb{R}^n)$ boundedness ($p \in (1,\infty)$) of μ_Ω holds true under the condition that $\Omega \in C^1(S^{n-1})$. Walsh [26] proved that for each $p \in (1,\infty)$, $\Omega \in L(\ln L)^{1/r}(\ln\ln L)^{2(1-2/r')}(S^{n-1})$ is a sufficient condition such that μ_Ω is bounded on $L^p(\mathbb{R}^n)$, where $r = \min\{p, p'\}$ and $p' = p/(p-1)$. Ding, Fan and Pan [7] proved that if $\Omega \in H^1(S^{n-1})$ (the Hardy space on S^{n-1}), then μ_Ω is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1,\infty)$; Al-Salman et al. [2] proved that $\Omega \in L(\ln L)^{1/2}(S^{n-1})$ is a sufficient condition such that μ_Ω is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1,\infty)$. Ding, Fan and Pan [8] considered the boundedness on weighted $L^p(\mathbb{R}^n)$ with $A_p(\mathbb{R}^n)$ when $\Omega \in L^q(S^{n-1})$ for some $q \in (1,\infty]$. For more details about the operator μ_Ω, one can see [1, 5, 7, 9] and the related references therein.

The purpose of this paper is to establish an analogue of (1.3) for the Marcinkiewicz integral operator with kernel $\Omega \in L^q(S^{n-1})$ for some $q \in (1,\infty]$. We remark that in this paper, we are very much motivated by [16] and some ideas from Lerner's recent paper [18]. For $p, r \in (1,\infty)$ and $w \in A_p(\mathbb{R}^n)$, set

$$\{w\}_{A_p,r;s} = [w]_{A_p}^{\frac{1}{r}} \max\{[w]_{A_\infty}^{(\frac{1}{s}-\frac{1}{r})_+}, [w^{1-p'}]_{A_\infty}^{\frac{1}{r}}\},$$

where and in what follows, $(\frac{1}{r}-\frac{1}{p})_+ = \max\{\frac{1}{r}-\frac{1}{p}, 0\}$. It is obvious that $\{w\}_{A_p,p;1} = \{w\}_{A_p}$. Moreover, by the fact that

$$[w]_{A_\infty} \leqslant [w]_{A_p}, \ [w^{1-p'}]_{A_\infty} \leqslant [w^{1-p'}]_{A_{p'}} = [w]_{A_p}^{\frac{1}{p-1}},$$

we know that

$$(w)_{A_p} \leqslant [w]_{A_p}^{\max\{1, \frac{1}{p-1}\}}, \tag{1.5}$$

and

$$\{w\}_{A_p,r;s} \leqslant [w]_{A_p}^{\max\{\frac{1}{s}, \frac{p}{p-1}\frac{1}{r}\}}. \tag{1.6}$$

Our main result can be stated as follows.

THEOREM 1.2. *Let Ω be homogeneous of degree zero, have mean value zero on S^{n-1}, and $\Omega \in L^q(S^{n-1})$ for some $q \in (1,\infty]$. Let $p \in (q',\infty)$ and $w \in A_{p/q'}(\mathbb{R}^n)$. Then*

$$\|\mu_\Omega(f)\|_{L^p(\mathbb{R}^n,w)} \lesssim \|\Omega\|_{L^q(S^{n-1})} \{w\}_{A_{p/q'},p;2} (w)_{A_{p/q'}} \|f\|_{L^p(\mathbb{R}^n,w)}.$$

In particular (by (1.5) and (1.6)),

$$\|\mu_\Omega(f)\|_{L^p(\mathbb{R}^n,w)} \lesssim \|\Omega\|_{L^q(S^{n-1})}[w]_{A_{p/q'}}^{\max\{\frac{1}{2},\frac{1}{p-q'}\}+\max\{1,\frac{q'}{p-q'}\}}\|f\|_{L^p(\mathbb{R}^n,w)}.$$

REMARK 1.3. For $t\in[1,2]$ and $j\in\mathbb{Z}$, set

$$K_t^j(x)=\frac{1}{2^j}\frac{\Omega(x)}{|x|^{n-1}}\chi_{\{2^{j-1}t<|x|\leqslant 2^jt\}}(x). \tag{1.7}$$

Let

$$\widetilde{\mu}_\Omega(f)(x)=\Big(\int_1^2\sum_{j\in\mathbb{Z}}\big|F_jf(x,t)\big|^2dt\Big)^{1/2}, \tag{1.8}$$

with

$$F_jf(x,t)=\int_{\mathbb{R}^n}K_t^j(x-y)f(y)dy.$$

A trivial computation shows that

$$\mu_\Omega(f)(x)\approx\widetilde{\mu}_\Omega(f)(x). \tag{1.9}$$

REMARK 1.4. To prove Theorem 1.2, we will employ the scheme used in [16], that is, approximating the operator $\widetilde{\mu}_\Omega$ defined in (1.8) by certain operators $\{\widetilde{\mu}_\Omega^l\}_l$ with smooth kernels, establishing the quantitative weighted bounds for $\{\widetilde{\mu}_\Omega^l\}_l$ and then using interpolation with change of measures. An ingredient in the procedure of establishing the refined weighted bounds for $\{\widetilde{\mu}_\Omega^l\}_l$ is a new grand maximal operator, which is a variant of the grand maximal operator introduced by Lerner [18] and that is suitable for square functions.

We make some conventions. In what follows, C always denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line. We use the symbol $A\lesssim B$ to denote that there exists a positive constant C such that $A\leqslant CB$. For a set $E\subset\mathbb{R}^n$, χ_E denotes its characteristic function.

2. Proof of Theorem 1.2

Recall that the standard dyadic grid in $\mathbb{R}^n$ consists of all cubes of the form

$$2^{-k}([0,1)^n+j),\, k\in\mathbb{Z},\ j\in\mathbb{Z}^n.$$

Denote the standard grid by $\mathscr{D}$.

As usual, by a general dyadic grid $\mathscr{D}$, we mean a collection of cubes with the following properties: (i) for any cube $Q\in\mathscr{D}$, its side length $\ell(Q)$ is of the form 2^k for some $k\in\mathbb{Z}$; (ii) for any cubes $Q_1,Q_2\in\mathscr{D}$, $Q_1\cap Q_2\in\{Q_1,Q_2,\emptyset\}$; (iii) for each $k\in\mathbb{Z}$, the cubes of side length 2^k form a partition of $\mathbb{R}^n$.

Let $\eta \in (0,1)$ and $\mathscr{S}$ be a family of cubes. We say that $\mathscr{S}$ is η-sparse, if for each fixed $Q \in \mathscr{S}$, there exists a measurable subset $E_Q \subset Q$, such that $|E_Q| \geqslant \eta|Q|$ and $\{E_Q\}$ are pairwise disjoint. Associated with the sparse family $\mathscr{S}$ and $r \in (0,\infty)$, we define the sparse operator $\mathscr{A}^r_{\mathscr{S}}$ by

$$\mathscr{A}^r_{\mathscr{S}} f(x) = \Big\{ \sum_{Q \in \mathscr{S}} \big(\langle |f| \rangle_Q\big)^r \chi_Q(x) \Big\}^{1/r}.$$

We use $\mathscr{A}_{\mathscr{S}}$ to denote $\mathscr{A}^1_{\mathscr{S}}$.

The following result was proved by Hytönen and Lacey [14], see also Hytönen and Li [15].

LEMMA 2.1. *Let $p \in (1,\infty)$ and $r \in (0,\infty)$, $w \in A_p(\mathbb{R}^n)$. Then for a sparse family $\mathscr{S} \subset \mathscr{D}$ with $\mathscr{D}$ a dyadic grid,*

$$\|\mathscr{A}^r_{\mathscr{S}} f\|_{L^p(\mathbb{R}^n,w)} \lesssim [w]_{A_p}^{\frac{1}{p}} \big([w]_{A_\infty}^{(\frac{1}{r}-\frac{1}{p})_+} + [w^{-\frac{1}{p-1}}]_{A_\infty}^{\frac{1}{p}}\big) \|f\|_{L^p(\mathbb{R}^n,w)}.$$

Let Ω be homogeneous of degree zero, integrable on S^{n-1} and K_t^j be defined as in (1.7). It was proved in [11], if $\Omega \in L^q(S^{n-1})$ for some $q \in (1,\infty]$, then there exists a constant $\alpha \in (0,1)$ such that for $t \in [1,2]$ and $\xi \in \mathbb{R}^n \backslash \{0\}$,

$$|\widehat{K_t^j}(\xi)| \lesssim \|\Omega\|_{L^q(S^{n-1})} \min\{1, |2^j \xi|^{-\alpha}\}. \tag{2.1}$$

Here and in what follows, for $h \in \mathscr{S}'(\mathbb{R}^n)$, $\widehat{h}$ denotes the Fourier transform of h. Moreover, if $\int_{S^{n-1}} \Omega(x')dx' = 0$, then

$$|\widehat{K_t^j}(\xi)| \lesssim \|\Omega\|_{L^1(S^{n-1})} \min\{1, |2^j \xi|\}. \tag{2.2}$$

In what follows, we assume that $\|\Omega\|_{L^q(S^{n-1})} = 1$.

Let $\phi \in C_0^\infty(\mathbb{R}^n)$ be a nonnegative function such that $\int_{\mathbb{R}^n} \phi(x)dx = 1$, $\mathrm{supp}\,\phi \subset \{x : |x| \leqslant 1/4\}$. For $l \in \mathbb{Z}$, let $\phi_l(y) = 2^{-nl}\phi(2^{-l}y)$. It is easy to verify that for any $\varsigma \in (0,1)$,

$$|\widehat{\phi_l}(\xi) - 1| \lesssim \min\{1, |2^l \xi|^\varsigma\}. \tag{2.3}$$

Let

$$F_j^l f(x,t) = \int_{\mathbb{R}^n} K_t^j * \phi_{j-l}(x-y) f(y)\,dy.$$

Define the operator $\widetilde{\mu}^l_\Omega$ by

$$\widetilde{\mu}^l_\Omega(f)(x) = \Big(\int_1^2 \sum_{j \in \mathbb{Z}} |F_j^l f(x,t)|^2 dt \Big)^{1/2}.$$

By Fourier transform estimates (2.1), (2.2) and (2.3), and Plancherel's theorem, we have that for some positive constant θ depending only on n,

$$\begin{aligned}\|\widetilde{\mu}_\Omega(f)-\widetilde{\mu}^l_\Omega(f)\|^2_{L^2(\mathbb{R}^n)} &\leqslant \int_1^2\Big\|\Big(\sum_{j\in\mathbb{Z}}\big|F_lf(\cdot,t)-F^l_jf(\cdot,t)\big|^2\Big)^{\frac12}\Big\|^2_{L^2(\mathbb{R}^n)}dt \qquad (2.4)\\ &=\int_1^2\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n}|\widehat{K^j_t}(\xi)|^2|1-\widehat{\phi_{j-l}}(\xi)|^2|\widehat{f}(\xi)|^2d\xi dt\\ &\lesssim 2^{-2\theta l}\|f\|^2_{L^2(\mathbb{R}^n)}.\end{aligned}$$

LEMMA 2.2. *Let Ω be homogeneous of degree zero and belong to $L^q(S^{n-1})$ for some $q\in(1,\infty]$, K^j_t be defined as in (1.7). Then for $l\in\mathbb{N}$, $R>0$ and $y\in\mathbb{R}^n$ with $|y|<R/4$,*

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}\Big(\int_{2^kR<|x|\leqslant 2^{k+1}R}\sup_{t\in[1,2]}\Big|K^j_t*\phi_{j-l}(x+y)-K^j_t*\phi_{j-l}(x)\Big|^qdx\Big)^{\frac1q}\\ &\lesssim \frac{1}{(2^kR)^{n/q'}}\min\{1,2^l\frac{|y|}{2^kR}\}.\end{aligned}$$

Proof. We will employ the idea from [27]. It is obvious that for $r\in[1,\infty)$,

$$\|\phi_{j-l}(\cdot+y)-\phi_{j-l}(\cdot)\|_{L^{r'}(\mathbb{R}^n)}\lesssim 2^{(l-j)n/r}\min\{1,2^{l-j}|y|\}.$$

Observe that

$$\sup_{t\in[1,2]}\Big|K^j_t*\phi_{j-l}(x+y)-K^j_t*\phi_{j-l}(x)\Big|\lesssim\int_{\mathbb{R}^n}\widetilde{K}^j(z)\big|\phi_{j-l}(x+y-z)-\phi_{j-l}(x-z)\big|dz,$$

with $\widetilde{K}^j(z)=|z|^{-n}|\Omega(z)|\chi_{\{2^{j-2}\leqslant|z|\leqslant 2^{j+2}\}}(z)$. Thus, by the fact $\mathrm{supp}K^j_t*\phi_{j-l}\subset\{x\in\mathbb{R}^n:2^{j-2}\leqslant|x|\leqslant 2^{j+2}\}$, we deduce that

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}\Big(\int_{2^kR<|x|\leqslant 2^{k+1}R}\sup_{t\in[1,2]}\Big|K^j_t*\phi_{j-l}(x+y)-K^j_t*\phi_{j-l}(x)\Big|^qdx\Big)^{\frac1q}\\ &\lesssim\sum_{j\in\mathbb{Z}:2^j\approx 2^kR}\|\widetilde{K}^j\|_{L^q(\mathbb{R}^n)}\|\phi_{j-l}(\cdot+y)-\phi_{j-l}(\cdot)\|_{L^1(\mathbb{R}^n)}\lesssim(2^kR)^{-n/q'}\min\{1,2^l\frac{|y|}{2^kR}\}.\end{aligned}$$

This completes the proof of Lemma 2.2. □

LEMMA 2.3. *Let Ω be homogeneous of degree zero and have mean value zero. Suppose that $\Omega\in L^q(S^{n-1})$ for some $q\in(1,\infty]$. Then for any $l\in\mathbb{N}$, $\widetilde{\mu}^l_\Omega$ is bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$ with bound Cl.*

Proof. The proof is fairly standard. For the sake of self-containedness, we present the proof here. Our goal is to prove that for any $\lambda > 0$,

$$\big|\{x \in \mathbb{R}^n : \widetilde{\mu}_\Omega^l(f)(x) > \lambda\}\big| \lesssim l\lambda^{-1}\|f\|_{L^1(\mathbb{R}^n)}. \tag{2.5}$$

For each fixed $\lambda > 0$, applying the Calderón-Zygmund decomposition to $|f|$ at level λ, we obtain a sequence of cubes $\{Q_i\}$ with disjoint interiors, such that

$$\lambda < \frac{1}{|Q_i|}\int_{Q_i}|f(y)|dy \leqslant 2^n\lambda,$$

and $|f(y)| \lesssim \lambda$ for a. e. $y \in \mathbb{R}^n\backslash(\cup_i Q_i)$. Set

$$g(y) = f(y)\chi_{\mathbb{R}^n\backslash\cup_i Q_i}(y) + \sum_i \langle f\rangle_{Q_i}\chi_{Q_i}(y),$$

$$b(y) = \sum_i b_i(y), \text{ with } b_i(y) = \big(f(y) - \langle f\rangle_{Q_i}\big)\chi_{Q_i}(y).$$

By (2.4) and the $L^2(\mathbb{R}^n)$ boundedness of $\widetilde{\mu}_\Omega$, we know that $\widetilde{\mu}_\Omega^l$ is also bounded on $L^2(\mathbb{R}^n)$ with bound independent of l. Therefore,

$$|\{x \in \mathbb{R}^n : \widetilde{\mu}_\Omega^l(g)(x) > \lambda/2\}| \lesssim \lambda^{-2}\|\widetilde{\mu}_\Omega^l g\|_{L^2(\mathbb{R}^n)}^2 \lesssim \lambda^{-1}\|f\|_{L^1(\mathbb{R}^n)}.$$

Let $E_\lambda = \cup_i 4nQ_i$. It is obvious that $|E_\lambda| \lesssim \lambda^{-1}\|f\|_{L^1(\mathbb{R}^n)}$. The proof of (2.5) is now reduced to prove that

$$|\{x \in \mathbb{R}^n\backslash E_\lambda : \widetilde{\mu}_\Omega^l(b)(x) > \lambda/2\}| \lesssim l\lambda^{-1}\|f\|_{L^1(\mathbb{R}^n)}. \tag{2.6}$$

We now prove (2.6). For each fixed cube Q_i, let y_i be the center of Q_i. For $x, y, z \in \mathbb{R}^n$, set

$$S_t^{j,l}(x; y, z) = |K_t^j * \phi_{j-l}(x-y) - K_t^j * \phi_{j-l}(x-z)|.$$

A trivial computation involving Minkowski's inequality and vanishing moment of b_i gives us that for $x \in \mathbb{R}^n$,

$$\begin{aligned}\widetilde{\mu}_\Omega^l(b)(x) &\leqslant \sum_i \Big(\int_1^2 \sum_{j\in\mathbb{Z}} \Big(\int_{\mathbb{R}^n} S_t^{j,l}(x; y, y_i)|b_i(y)|dy\Big)^2 dt\Big)^{\frac{1}{2}} \\ &\leqslant \sum_i \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} \Big(\int_1^2 \{S_t^{j,l}(x; y, y_i)\}^2 dt\Big)^{\frac{1}{2}} |b_i(y)|dy \\ &\leqslant \sum_i \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} \sup_{t\in[1,2]} S_t^{j,l}(x; y, y_i)|b_i(y)|dy.\end{aligned}$$

On the other hand, we get from Lemma 2.2 that

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n\backslash E_\lambda} \sup_{t\in[1,2]} S_t^{j,l}(x; y, y_i)dx = &\sum_{k=1}^{l} \sum_{j\in\mathbb{Z}} \int_{2^{k+2}nQ_i\backslash 2^{k+1}nQ_i} \sup_{t\in[1,2]} S_t^{j,l}(x; y, y_i)dx \\ &+ \sum_{k=l+1}^{\infty} \sum_{j\in\mathbb{Z}} \int_{2^{k+2}nQ_i\backslash 2^{k+1}nQ_i} \sup_{t\in[1,2]} S_t^{j,l}(x; y, y_i)dx \lesssim l.\end{aligned}$$

This in turn yields to that

$$\int_{\mathbb{R}^n\setminus E_\lambda}\widetilde{\mu}^l_{\Omega}(b)(x)dx \leqslant \sum_i\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n\setminus E_\lambda}\sup_{t\in[1,2]} S^{j,l}_t(x;y,y_i)dx|b_i(y)|dy \lesssim l\int_{\mathbb{R}^n}|f(y)|dy.$$

Inequality (2.6) now follows directly. □

LEMMA 2.4. *Let Ω be homogeneous of degree zero and have mean value zero. Suppose that $\Omega\in L^q(S^{n-1})$ for some $q\in(1,\infty]$. Let $\mathscr{M}_{\widetilde{\mu}^l_{\Omega}}$ be the grand maximal operator defined by*

$$\mathscr{M}_{\widetilde{\mu}^l_{\Omega}}f(x)=\sup_{Q\ni x}\sup_{\xi\in Q}|\widetilde{\mu}^l_{\Omega}(f\chi_{\mathbb{R}^n\setminus 3Q})(\xi)|.$$

Then $\mathscr{M}_{\widetilde{\mu}^l_{\Omega}}$ is bounded from $L^{q'}(\mathbb{R}^n)$ to $L^{q',\infty}(\mathbb{R}^n)$ with bound Cl.

Proof. Let $x\in\mathbb{R}^n$ and $Q\subset\mathbb{R}^n$ be a cube containing x. Denote by B_x the closed ball centered at x with radius $2\operatorname{diam}Q$. Then $3Q\subset B_x$. For each $\xi\in Q$, we can write

$$\begin{aligned}|\widetilde{\mu}^l_{\Omega}(f\chi_{R^n\setminus 3Q})(\xi)| &\leqslant |\widetilde{\mu}^l_{\Omega}(f\chi_{\mathbb{R}^n\setminus B_x})(\xi)-\widetilde{\mu}^l_{\Omega}(f\chi_{\mathbb{R}^n\setminus B_x})(x)|\\ &\quad+|\widetilde{\mu}^l_{\Omega}(f\chi_{B_x\setminus 3Q})(\xi)|+|\widetilde{\mu}^l_{\Omega}(f\chi_{\mathbb{R}^n\setminus B_x})(x)|.\end{aligned}$$

It is obvious that

$$\begin{aligned}&|\widetilde{\mu}^l_{\Omega}(f\chi_{\mathbb{R}^n\setminus B_x})(\xi)-\widetilde{\mu}^l_{\Omega}(f\chi_{\mathbb{R}^n\setminus B_x})(x)|\\ &\leqslant \Big(\int_1^2\sum_{j\in\mathbb{Z}}\Big|\int_{\mathbb{R}^n}R^{j,l}_t(x;y,\xi)f(y)\chi_{\mathbb{R}^n\setminus B_x}(y)dy\Big|^2dt\Big)^{\frac12},\end{aligned}$$

where

$$R^{j,l}_t(x;y,\xi)=|K^j_t*\phi_{l-j}(x-y)-K^j_t*\phi_{l-j}(\xi-y)|.$$

A trivial computation involving Hölder's inequality gives us that

$$\begin{aligned}&\sup_{t\in[1,2]}\sum_{j\in\mathbb{Z}}\Big|\int_{\mathbb{R}^n}R^{j,l}_t(x;y,\xi)f(y)\chi_{\mathbb{R}^n\setminus B_x}(y)dy\Big|\\ &\leqslant \sum_j\sum_{k=1}^{\infty}\Big(\int_{2^kB_x\setminus 2^{k-1}B_x}\sup_{t\in[1,2]}|R^{j,l}_t(x;y,\xi)|^qdy\Big)^{\frac1q}\Big(\int_{2^kB_x}|f(y)|^{q'}dy\Big)^{\frac1{q'}}\\ &\lesssim lM_{q'}f(x), \qquad (2.7)\end{aligned}$$

where $M_{q'}f(x)=\{M(|f|^{q'})(x)\}^{1/q'}$. For each fixed $t\in[1,2]$ and $j\in\mathbb{Z}$ with $2^j\approx\operatorname{diam}Q$,

$$|F^l_j(f\chi_{B_x\setminus 3Q})(x,t)|\leqslant\|K^j_t*\phi_{l-j}\|_{L^q(\mathbb{R}^n)}\|f\chi_{B_x}\|_{L^{q'}(\mathbb{R}^n)}\lesssim M_{q'}f(x).$$

Recall that $\operatorname{supp} K_t^j * \phi_{j-l} \subset \{x \in \mathbb{R}^n : 2^{j-2} \leqslant |x| \leqslant 2^{j+2}\}$. It then follows that

$$|\widetilde{\mu}_\Omega^l(f\chi_{B_x\setminus 3Q})(\xi)| = \Big(\int_1^2 \sum_j |F_j^l(f\chi_{B_x\setminus 3Q})(x,t)|^2 dt\Big)^{\frac{1}{2}}$$
$$\leqslant \sum_{j:2^j \approx \operatorname{diam} Q} \Big(\int_1^2 |F_j^l(f\chi_{B_x\setminus 3Q})(x,t)|^2 dt\Big)^{\frac{1}{2}} \lesssim M_{q'}f(x). \quad (2.8)$$

To estimate $\widetilde{\mu}_\Omega^l(f\chi_{\mathbb{R}^n\setminus B_x})(x)$, write

$$\begin{aligned}
\widetilde{\mu}_\Omega^l(f\chi_{\mathbb{R}^n\setminus B_x})(x) &\leqslant \widetilde{\mu}_\Omega^l(f)(x) + \Big(\int_1^2 \sum_{j\in\mathbb{Z}} |F_j^l(f\chi_{B_x})(x,t)|^2 dt\Big)^{\frac{1}{2}} \\
&= \widetilde{\mu}_\Omega^l(f)(x) + \Big(\int_1^2 \sum_{j:2^j\leqslant 4\operatorname{diam} Q} |F_j^l(f\chi_{B_x})(x,t)|^2 dt\Big)^{\frac{1}{2}} \\
&\leqslant 2\widetilde{\mu}_\Omega^l(f)(x) + \Big(\int_1^2 \sum_{j:2^j\leqslant 4\operatorname{diam} Q} |F_j^l(f\chi_{\mathbb{R}^n\setminus B_x})(x,t)|^2 dt\Big)^{\frac{1}{2}} \\
&=: 2\widetilde{\mu}_\Omega^l(f)(x) + \mathrm{D}f(x).
\end{aligned}$$

For the case of $q = \infty$,

$$|K_t^j * \phi_{l-j}(x)| \lesssim |x|^{-n}\chi_{2^{j-2}\leqslant|x|\leqslant 2^{j+2}}(x),\ t \in [1,2]. \quad (2.9)$$

On the other hand, if $q \in (1,\infty)$, then we have that

$$\sup_{t\in[1,2]} \int_{\mathbb{R}^n} |K_t^j * \phi_{l-j}(x-y)||f(y)|dy \lesssim M_{q'}Mf(x) \lesssim M_{q'}f(x). \quad (2.10)$$

Therefore,

$$\begin{aligned}
\mathrm{D}f(x) &\leqslant \sum_{j\in\mathbb{Z}:\operatorname{diam} Q/4\leqslant 2^j\leqslant 4\operatorname{diam} Q} \sup_{t\in[1,2]} \int_{\mathbb{R}^n} |K_t^j * \phi_{l-j}(x-y)||f(y)|dy \\
&\lesssim M_{q'}f(x). \quad (2.11)
\end{aligned}$$

Combining estimates (2.7), (2.8) and (2.11) yields that

$$\mathscr{M}_{\widetilde{\mu}_\Omega^l} f(x) \lesssim lM_{q'}f(x) + \widetilde{\mu}_\Omega^l f(x).$$

The desired boundedness for $\mathscr{M}_{\widetilde{\mu}_\Omega^l}$, follows from the last inequality and Lemma 2.3. □

LEMMA 2.5. *Let Ω be homogeneous of degree zero and have mean value zero. Suppose that $\Omega \in L^q(S^{n-1})$ for some $q \in (1,\infty]$. Then for $p \in (q',\infty)$ and $w \in A_p(\mathbb{R}^n)$,*

$$\|\widetilde{\mu}_\Omega^l f\|_{L^p(\mathbb{R}^n,w)} \lesssim l\{w\}_{A_{p/q'},p;2}\|f\|_{L^p(\mathbb{R}^n,w)}.$$

Proof. First, we claim that for each bounded function f with compact support, there exists a sparse family of cubes $\mathscr{S}$, such that for almost everywhere $x\in\mathbb{R}^n$,

$$\big[\widetilde{\mu}_{\Omega}^{l}(f)(x)\big]^2\lesssim l^2\sum_{Q\in\mathscr{S}}\langle|f|\rangle_{Q,q'}^2\chi_Q(x). \tag{2.12}$$

If we can prove this estimate, then for $p\in(q',\infty)$ and $w\in A_{p/q'}(\mathbb{R}^n)$, we deduce from Lemma 2.1 that .

$$\|\widetilde{\mu}_{\Omega}^{l}(f)\|_{L^p(\mathbb{R}^n,w)}\lesssim l\big\|\mathscr{A}_{\mathscr{S}}^{\frac{2}{q'}}(|f|^{q'})\big\|_{L^{p/q'}(\mathbb{R}^n,w)}^{\frac{1}{q'}}\lesssim l\{w\}_{A_{p/q'},p;2}\|f\|_{L^p(\mathbb{R}^n,w)}.$$

We now prove the estimate (2.12). We will employ the ideas of Lerner [17], via a variant of the grand maximal operator $\mathscr{M}_{\widetilde{\mu}_{\Omega}^{l}}$. Let $Q_0\subset\mathbb{R}^n$ be a cube. We define the operator $\mathscr{M}^{*}_{\widetilde{\mu}_{\Omega}^{l},Q_0}$ as

$$\mathscr{M}^{*}_{\widetilde{\mu}_{\Omega}^{l},Q_0}f(x)=\sup_{Q\ni x,Q\subset Q_0}\Big\|\Big(\int_1^2\sum_{j=J_Q}^{\infty}|F_l^j(f\chi_{3Q_0})(\cdot,t)|^2\frac{dt}{t}\Big)^{\frac12}\Big\|_{L^\infty(Q)},$$

where and in what follows, for a cube $Q\subset\mathbb{R}^n$, J_Q is the integer such that $2^{J_Q-1}\leqslant 4\ell(Q)<2^{J_Q}$, and $J_Q^*\in\mathbb{Z}$ such that $2^{J_Q^*-1}\leqslant 16n\ell(Q)<2^{J_Q^*}$. Let $x\in\mathbb{R}^n$, $Q\subset Q_0$ such that $x\in Q$. For each $\xi\in Q$, write

$$\begin{aligned}\int_1^2\sum_{j=j_Q}^{\infty}|F_l^j(f\chi_{3Q_0})(\xi,t)|^2\frac{dt}{t}&=\int_1^2\sum_{j=J_Q}^{J_Q^*}|F_l^j(f\chi_{3Q_0})(\xi,t)|^2\frac{dt}{t}\\&\quad+\int_1^2\sum_{j=J_Q^*}^{\infty}|F_l^j(f\chi_{3Q_0})(\xi,t)|^2\frac{dt}{t}.\end{aligned}$$

Applying estimates (2.9) and (2.10), we have that

$$\int_1^2\sum_{j=J_Q}^{J_Q^*}|F_l^j(f\chi_{3Q_0})(\xi,t)|^2\frac{dt}{t}\lesssim\big(M_{q'}(f\chi_{3Q_0})(x)\big)^2.$$

Note that for each $t\in[1,2]$,

$$F_l^j(f\chi_{3Q_0})(\xi,t)=F_l^j(f\chi_{3Q_0\backslash 3Q})(\xi,t).$$

Therefore,

$$\mathscr{M}^{*}_{\widetilde{\mu}_{\Omega}^{l},Q_0}f(x)\lesssim M_{q'}(f\chi_{3Q_0})(x)+\mathscr{M}_{\widetilde{\mu}_{\Omega}^{l}}(f\chi_{3Q_0})(x). \tag{2.13}$$

Let

$$\begin{aligned}E=&\big\{x\in Q_0:\widetilde{\mu}_{\Omega}^{l}(f\chi_{3Q_0})(x)>Dl\langle|f|\rangle_{3Q_0,q'}\big\}\\&\cup\big\{x\in Q_0:\mathscr{M}^{*}_{\widetilde{\mu}_{\Omega}^{l},Q_0}f(x)>Dl\langle|f|\rangle_{3Q_0,q'}\big\},\end{aligned}$$

where D is a positive constant. By Lemma 2.3, Lemma 2.4 and (2.13), we have that $|E| \leqslant \frac{1}{2^{n+2}}|Q_0|$ if we choose D large enough. Now on the cube Q_0, we apply the Calderón-Zygmund decomposition to χ_E at level $\frac{1}{2^{n+1}}$, and obtain pairwise disjoint cubes $\{P_j\} \subset \mathscr{D}(Q_0)$, such that

$$\frac{1}{2^{n+1}}|P_j| \leqslant |P_j \cap E| \leqslant \frac{1}{2}|P_j|$$

and $|E \backslash \cup_j P_j| = 0$. Observe that $\sum_j |P_j| \leqslant \frac{1}{2}|Q_0|$. Write

$$\begin{aligned}\widetilde{\mu}_\Omega^l(f\chi_{3Q_0})(x)^2\chi_{Q_0}(x) &= \widetilde{\mu}_\Omega^l(f\chi_{3Q_0})(x)^2\chi_{Q_0\backslash\cup_j P_j}(x)\\ &\quad+\sum_j\Big(\int_1^2\sum_{m=J_{P_j}}^{\infty}|F_l^m(f\chi_{3Q_0})(x,t)|^2\frac{dt}{t}\Big)\chi_{P_j}(x)\\ &\quad+\sum_j\Big(\int_1^2\sum_{m=-\infty}^{J_{P_j}-1}|F_l^m(f\chi_{3Q_0})(x,t)|^2\frac{dt}{t}\Big)\chi_{P_j}(x).\end{aligned}$$

The facts that $|E \backslash \cup_j P_j| = 0$ implies that

$$\widetilde{\mu}_\Omega^l(f\chi_{3Q_0})(x)^2\chi_{Q_0\backslash\cup_j P_j}(x) \lesssim l^2\langle|f|\rangle_{3Q_0,q'}^2\chi_{Q_0}(x).$$

Since $P_l \cap E^c \neq \emptyset$, we deduce that

$$\begin{aligned}\sum_j\Big(\int_1^2\sum_{m=J_{P_j}}^{\infty}|F_l^m(f\chi_{3Q_0})(x,t)|^2\frac{dt}{t}\Big)\chi_{P_j}(x) &\lesssim \sum_j\inf_{y\in P_j}\big(\mathscr{M}^*_{\widetilde{\mu}_\Omega^l,Q_0}f(y)\big)^2\chi_{P_j}(x)\\ &\lesssim l^2\langle|f|\rangle_{3Q_0,q'}^2\chi_{Q_0}(x).\end{aligned}$$

On the other hand, it is easy to verify that when $t \in [1,2]$, $x \in P_j$ and $m \leqslant J_{P_j}-1$,

$$F_l^m(f\chi_{3Q_0\backslash 3P_j})(x,t) = 0,$$

and

$$\Big(\int_1^2\sum_{m=-\infty}^{J_{P_j}-1}|F_l^m(f\chi_{3Q_0})(x,t)|^2\frac{dt}{t}\Big)\chi_{P_j}(x) \leqslant l^2\big(\widetilde{\mu}_\Omega^l(f\chi_{3P_j})(x)\big)^2\chi_{P_j}(x).$$

Thus, for almost everywhere $x \in Q_0$,

$$\big(\widetilde{\mu}_\Omega^l(f\chi_{3Q_0})(x)\big)^2 \leqslant C\langle|f|\rangle_{3Q_0,q'}^2\chi_{Q_0}(x)+\sum_j\big\{\widetilde{\mu}_\Omega^l(f\chi_{3P_j})(x)\big\}^2\chi_{P_j}(x). \quad (2.14)$$

By iterating (2.14), we immediately get that there exists a $\frac{1}{2}$- sparse family of cubes $\mathscr{F} \subset \mathscr{D}(Q_0)$ such that for almost everywhere $x \in Q_0$,

$$\big(\widetilde{\mu}_\Omega^l(f\chi_{3Q_0})(x)\big)^2\chi_{Q_0}(x) \lesssim l^2\sum_{Q\in\mathscr{F}}\langle|f|\rangle_{3Q,q'}^2\chi_Q(x). \quad (2.15)$$

We can now conclude the proof of Lemma 2.5. In fact, as in [17], we decompose $\mathbb{R}^n$ by cubes $\{Q_l\}$, such that $\mathrm{supp} f \subset 3Q_l$ for each l, and Q_l's have disjoint interiors. Then for each l, we have a $\frac{1}{2}$-sparse family of cubes $\mathscr{F}_l \subset \mathscr{D}(Q_l)$, such that for almost everywhere $x \in \mathbb{R}^n$,

$$\big(\widetilde{\mu}_{\Omega}^{l}(f\chi_{3Q_l})(x)\big)^2 \chi_{Q_l}(x) \lesssim l^2 \sum_{Q\in\mathscr{F}_l} \langle |f|\rangle_{3Q,q'}^2 \chi_Q(x).$$

Let $\mathscr{S} = \cup_l \{3Q : Q \in \mathscr{F}_l\}$. Summing over the last inequality yields (2.12). □

REMARK 2.6. Lerner [19] established the sharp weighted bounds for square functions. Let ψ be an integrable function, have integral zero, and for some constant $\varepsilon \in (0,1)$,

$$|\psi(x)| \lesssim \frac{1}{(1+|x|)^{n+\varepsilon}}, \quad \int_{\mathbb{R}^n} |\psi(x+h)-\psi(x)|dx \lesssim |h|^{\varepsilon}.$$

Let $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times \mathbb{R}_+$ and $\Gamma_\alpha(x) = \{(y,t) \in \mathbb{R}_+^{n+1} : |y-x| \leqslant \alpha t\}$. Set $\psi_t(x) = t^{-n}\psi(x/t)$. Define the square function $S_{\alpha,\psi}$ by

$$S_{\alpha,\psi}(f)(x) = \Big(\int_{\Gamma_\alpha(x)} |f*\psi_t(x)|^2 \frac{dtdy}{t^{n+1}}\Big)^{\frac{1}{2}}.$$

Lerner [19, Section 4] proved that for $p \in (1,\infty)$, $w \in A_p(\mathbb{R}^n)$ and $\alpha \in [1,\infty)$,

$$\|S_{\alpha,\psi}(f)\|_{L^p(\mathbb{R}^n,w)} \leqslant \alpha^n \sup_{\mathscr{S}} \|\mathscr{A}_{\mathscr{S}}^2 f\|_{L^p(\mathbb{R}^n,w)},$$

where the supremum is taken over all sparse family of cubes. Thus,

$$\|S_{\alpha,\psi}(f)\|_{L^p(\mathbb{R}^n,w)} \leqslant C_{n,\psi,p}\alpha^n \{w\}_{A_p,p;2} \|f\|_{L^p(\mathbb{R}^n,w)}. \tag{2.16}$$

Moreover, this estimate is sharp. Repeating the proof of Lemma 2.5, we can prove the following result, which is new for the Marcinkiewicz integral.

THEOREM 2.7. *Let Ω be homogeneous of degree zero and have mean value zero. Suppose that $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$ for some $\alpha \in (0,1]$. Then*

(1) *for bounded function f with compact support, there exists a sparse family of cubes $\mathscr{S}$, such that for almost everywhere $x \in \mathbb{R}^n$,*

$$\mu_\Omega(f)(x) \lesssim \mathscr{A}_{\mathscr{S}}^2 f(x);$$

(2) *for $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^n)$,*

$$\|\mu_\Omega(f)\|_{L^p(\mathbb{R}^n,w)} \lesssim \{w\}_{A_p,p;2} \|f\|_{L^p(\mathbb{R}^n,w)}. \tag{2.17}$$

Note that (2.17) is analogue to (2.16).

Armed with the preceding results we are in the position to prove Theorem 1.2.

Proof of Theorem 1.2. Without loss of generality, we may assume that $\|\Omega\|_{L^q(S^{n-1})} = 1$. By (2.4), we know that

$$\|\widetilde{\mu}_\Omega^{2^l}(f) - \widetilde{\mu}_\Omega^{2^{l+1}}(f)\|_{L^2(\mathbb{R}^n)} \lesssim 2^{-\theta 2^l}\|f\|_{L^2(\mathbb{R}^n)}, \tag{2.18}$$

and the series

$$\widetilde{\mu}_\Omega = \sum_{l=1}^{\infty}(\widetilde{\mu}_\Omega^{2^{l+1}} - \widetilde{\mu}_\Omega^{2^l}) + \widetilde{\mu}_\Omega^2$$

converges in the $L^2(\mathbb{R}^n)$ operator norm. Let $p \in (q', \infty)$ and $w \in A_{p/q'}(\mathbb{R}^n)$, by [**?**, Corollary 3.16 and Corollary 3.17], we know that for $\varepsilon = c_n/(w)_{A_{p/q'}}$ with c_n a constant depending only on n, $w^{1+\varepsilon} \in A_{p/q'}(\mathbb{R}^n)$,

$$[w^{1+\varepsilon}]_{A_{p/q'}} \lesssim [w]_{A_{p/q'}}^{1+\varepsilon},$$

and

$$[w^{1+\varepsilon}]_{A_\infty} \lesssim [w]_{A_\infty}^{1+\varepsilon},\ [w^{(1-(\frac{p}{q'})')(1+\varepsilon)}]_{A_\infty} \lesssim [w^{1-(\frac{p}{q'})'}]_{A_\infty}^{1+\varepsilon}.$$

Therefore,

$$\{w^{1+\varepsilon}\}_{A_{p/q'},p;2} \lesssim \{w\}_{A_{p/q'},p;2}^{1+\varepsilon}.$$

Lemma 2.4 tells us that

$$\|\widetilde{\mu}_\Omega^{2^l}(f) - \widetilde{\mu}_\Omega^{2^{l+1}}(f)\|_{L^p(\mathbb{R}^n, w^{1+\varepsilon})} \lesssim 2^l\{w^{1+\varepsilon}\}_{A_{p/q'},p;2}\|f\|_{L^p(\mathbb{R}^n, w^{1+\varepsilon})}. \tag{2.19}$$

On the other hand, by interpolating the estimates (2.18) and (2.19) with $w = 1$, we know that for some $\rho = \rho_p \in (0, 1)$,

$$\|\widetilde{\mu}_\Omega^{2^l}(f) - \widetilde{\mu}_\Omega^{2^{l+1}}(f)\|_{L^p(\mathbb{R}^n)} \lesssim 2^{-\rho 2^l}\|f\|_{L^p(\mathbb{R}^n)}. \tag{2.20}$$

By interpolation with change of measures (see [25]), we deduce from (2.19) and (2.20) that

$$\|\widetilde{\mu}_\Omega^{2^l}(f) - \widetilde{\mu}_\Omega^{2^{l+1}}(f)\|_{L^p(\mathbb{R}^n, w)} \lesssim 2^l 2^{-\rho\frac{\varepsilon}{1+\varepsilon}2^l}\{w\}_{A_{p/q'},p;2}\|f\|_{L^p(\mathbb{R}^n, w)}.$$

As in [16], a trivial computation involving the inequality $\mathrm{e}^x \geqslant x^2/2$, now shows that

$$\sum_{l=1}^{\infty} 2^l 2^{-\rho 2^l\frac{\varepsilon}{1+\varepsilon}} \lesssim \sum_{l:2^l \leqslant \varepsilon^{-1}} 2^l + \sum_{l:2^l > \varepsilon^{-1}} 2^l\Big(\frac{2^l\varepsilon}{1+\varepsilon}\Big)^{-2} \lesssim (w)_{A_{p/q'}}.$$

We finally get that

$$\begin{aligned}\|\widetilde{\mu}_\Omega(f)\|_{L^p(\mathbb{R}^n, w)} &\leqslant \|\widetilde{\mu}_\Omega^2(f)\|_{L^p(\mathbb{R}^n, w)} + \sum_{l=1}^{\infty}\|\widetilde{\mu}_\Omega^{2^{l+1}}(f) - \widetilde{\mu}_\Omega^{2^l}(f)\|_{L^p(\mathbb{R}^n, w)} \\ &\lesssim \{w\}_{A_{p/q'},p;2}(w)_{A_{p/q'}}\|f\|_{L^p(\mathbb{R}^n, w)}.\end{aligned}$$

This via (1.9) completes the proof of Theorem 1.2. □

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