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MONOTONICITY PATTERNS AND FUNCTIONAL INEQUALITIES FOR CLASSICAL AND GENERALIZED WRIGHT FUNCTIONS

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Abstract. Our aim in this paper is to present the completely monotonicity and convexity properties for the Wright function. As a consequences of these results, we present some functional inequalities. Moreover, we derive the monotonicity and log-convexity results for the generalized Wright functions. As applications, we present several new inequalities (like Turán type inequalities) and we prove some geometric properties for the four–parametric Mittag–Leffler functions.

1. Introduction

Special functions like Mittag–Leffler functions and Wright functions $E_{\alpha,\beta}(z)$ and $W_{\alpha,\beta}(z)$ are frequently used in the solution of linear partial fractional differential equations, the number theory regarding the asymptotic of the number of some special partitions of the natural numbers and in the boundary–value problems for the fractional diffusion-wave equation, that is, the linear partial integro-differential equation obtained from the classical diffusion or wave equation by replacing the first- or second order time derivative by a fractional derivative of order α with $0<\alpha<2$. It was found that the corresponding Green functions can be represented in terms of the Wright function. This special function is related to modified Bessel functions of the first kind, and thus their properties can be useful in problems of mathematical physics.

The special case of Fox-Wright function which we consider in this paper, is the Wright function which is defined by the series representation, valid in the whole complex plane

$$W_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{k!\Gamma(\alpha k+\beta)},\ \alpha>-1,\beta\in\mathbb{C}. \tag{1}$$

It is an entire function of order $1/(1+\alpha)$, which has been known also as generalized Bessel function.

Our aim in this paper is twofold: on one hand it is to prove the complete monotonicity properties for the Wright function $W_{\alpha,\beta}(-z)$ for $\alpha,\beta>,0$ and $0<z<1$. As a consequence, we derive some functional inequalities as well as lower and upper

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bounds for the Wright function. On the other hand, by using the complete monotonicity property for the classical Wright function we obtain the complete monotonicity for the generalized Wright function, and consequently we get also the monotonicity property for the four–parametric Mittag–Leffler function. This paper is a continuation of the author results on Turán type and related inequalities for different class of special functions, cf. [4, 6, 7, 8, 9, 10, 11, 13, 15].

The paper is organized as follows: In section 2, we present new integral representation for the Wright function. Moreover, we derive some monotonicity and convexity results for the function $z \mapsto W_{\alpha,\beta}(-z)$. As a consequence, we establish a number of functional inequalities. In section 3, the monotonicity property for generalized Wright function is proved. As applications, we prove several new inequalities for this functions. In particular, we gave some Turán type inequalities for the generalized Wright function. Finally, in section 4, we apply some of our main results of Section 3 with a view to deriving some new inequalities for the four–parametric Mittag–Leffler function.

Each of the following definitions will be used in our investigation.

DEFINITION 1. A function $f:(0,\infty)\subseteq\mathbb{R}\to\mathbb{R}$ is said to be completely monotonic if f has derivatives of all orders and satisfies the following inequalities:

$$(-1)^n f^{(n)}(x) \geqslant 0, \ \ (x>0, \ \text{and} \ n\in\mathbb{N}=\{1,2,3,...\}).$$

DEFINITION 2. A function $f:[a,b]\subseteq\mathbb{R}\to\mathbb{R}$ is said to be log-convex if its natural logarithm $\log f$ is convex, that is, for all $x,y\in[a,b]$ and $\alpha\in[0,1]$ we have

$$f(\alpha x+(1-\alpha)y)\leqslant [f(x)]^{\alpha}[f(y)]^{1-\alpha}.$$

If the above inequality is reversed then f is called a log-concave function. It is also known that if g is differentiable, then f is log-convex (log-concave) if and only if f'/f is increasing (decreasing).

2. The Wright functions: monotonicity patterns and functional inequalities

In the next Lemma we present new integral representation for the Wright function $W_{\alpha,\beta}(z)$.

LEMMA 1. *Let $\beta>\alpha>0$. Then the the Wright function $W_{\alpha,\beta}(z)$ has the following integral representation*

$$W_{\alpha,\beta}(z)=c_{\alpha,\beta}\int_0^1 (1-t^{1/\alpha})^{\beta-\alpha-1}W_{\alpha,\alpha}(zt)dt, \ z\in\mathbb{R}, \tag{2}$$

where $c_{\alpha,\beta}=\frac{1}{\alpha\Gamma(\beta-\alpha)}$. In particular,

$$W_{\alpha,\alpha+1}(z)=\frac{1}{\alpha}\int_0^1 W_{\alpha,\alpha}(zt)dt.$$

Proof. By using the definition of the Wright function $W_{\alpha,\beta}(z)$, we get

$$\begin{aligned}\int_0^1 (1-t^{1/\alpha})^{\beta-\alpha-1} W_{\alpha,\alpha}(z)dt &= \int_0^1 (1-t^{1/\alpha})^{\beta-\alpha-1} \sum_{k=0}^{\infty} \frac{(zt)^k}{k!\Gamma(\alpha+k\alpha)} dt \\ &= \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(\alpha+k\alpha)} \left(\int_0^1 (1-t^{1/\alpha})^{\beta-\alpha-1} t^k dt\right) z^k \\ &= \alpha \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(\alpha+k\alpha)} \left(\int_0^1 (1-t)^{\beta-\alpha-1} t^{\alpha k+\alpha-1} dt\right) z^k \\ &= \alpha \sum_{k=0}^{\infty} \frac{B(\beta-\alpha,\alpha k+\alpha)}{k!\Gamma(\alpha+k\alpha)} z^k \\ &= \frac{W_{\alpha,\beta}(z)}{c_{\alpha,\beta}},\end{aligned}$$

where $B(x,y)$ is the Beta function defined by $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$. Finally, letting in (2) the value $\beta = \alpha+1$ we obtain the integral representation for the Wright function $W_{\alpha,\alpha+1}(z)$. □

LEMMA 2. *Let $\alpha > 0$ and $\beta > x^\star$, where $x^\star \simeq 1.461632144...$ is the abscissa of the minimum of the Gamma function, then the function $W_{\alpha,\beta}(-z)$ is non–negative for all $z \in (0,1)$.*

Proof. Let $u_k(z) = \frac{z^k}{k!\Gamma(\alpha k+\beta)}$, we get

$$W_{\alpha,\beta}(-z) = u_0(z) - u_1(z) + \sum_{k=2}^{\infty} (-1)^k u_k(z). \tag{3}$$

Elementary calculations reveal that for $0 < z < 1$, and $k \geqslant 2$

$$\frac{u_{k+1}(z)}{u_k(z)} = \frac{\Gamma(\alpha k+\beta)z}{(k+1)\Gamma(\alpha k+\beta+\alpha)} \leqslant \frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha k+\beta+\alpha)}. \tag{4}$$

From the previous inequality and using the fact that $z \mapsto \Gamma(z)$ is increasing on $(x^\star,\infty)$ we deduce that $\frac{u_{k+1}(z)}{u_k(z)} \leqslant 1$. Therefore, for fixed $0 < z < 1$, the sequence $k \mapsto u_k(z)$ is decreasing with regards $k \geqslant 2$ and u_k tends to 0 as $k \longrightarrow \infty$. From (3) and since the Gamma function is increasing on $(x^\star,\infty)$, we have

$$\begin{aligned}W_{\alpha,\beta}(-z) \geqslant u_0(z) - u_1(z) &= \frac{1}{\Gamma(\beta)} - \frac{z}{\Gamma(\beta+\alpha)} \\ &\geqslant \frac{1}{\Gamma(\beta)} - \frac{1}{\Gamma(\beta+\alpha)} \geqslant 0.\end{aligned}$$

The proof of Lemma 2 is complete. □

THEOREM 1. *Let $\beta > \alpha > x^{\star}$. Then, the function $z \mapsto \check{W}_{\alpha,\beta}(z) = W_{\alpha,\beta}(-z)$ is completely monotonic and log-convex on $(0,1)$. Furthermore, the following inequalities*

$$\check{W}_{\alpha,\beta+2\alpha}(z)\check{W}_{\alpha,\beta}(z) - \left(\check{W}_{\alpha,\beta+\alpha}(z)\right)^2 \geqslant 0,\ 0<z<1, \tag{5}$$

$$\check{W}_{\alpha,\beta}(z) \geqslant \frac{e^{-\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}z}}{\Gamma(\beta)},\ 0<z<1, \tag{6}$$

are valid.

Proof. By using the differentiation formula

$$\frac{d}{dz}W_{\alpha,\beta}(z) = W_{\alpha,\beta+\alpha}(z), \tag{7}$$

Lemma 1 and Lemma 2, we have for $n \in \mathbb{N}$ and $\beta > \alpha > 0$,

$$(-1)^n\left(\check{W}_{\alpha,\beta}(z)\right)^{(n)} = c_{\alpha,\beta}\int_0^1 (1-t^{1/\alpha})^{\beta-\alpha-1}t^n\check{W}_{\alpha,\alpha+n\alpha}(zt)dt \geqslant 0,$$

for all $z \in (0,1)$. Thus, the function $z \mapsto \check{W}_{\alpha,\beta}(z)$ is completely monotonic and consequently is log-convex, since every completely monotonic function is log–convex, see [16, p.167]. Now, focus on the Turán type inequality (1). Since the function $z \mapsto \check{W}_{\alpha,\beta}(z)$ is log-convex on $(0,1)$, it follows that the function $z \mapsto \check{W}'_{\alpha,\beta}(z)/\check{W}_{\alpha,\beta}(z)$ is increasing on $(0,1)$. Thus

$$\left(\frac{\check{W}'_{\alpha,\beta}(z)}{\check{W}_{\alpha,\beta}(z)}\right)' = \frac{\check{W}_{\alpha,\beta+2\alpha}(z)\check{W}_{\alpha,\beta}(z) - \left(\check{W}_{\alpha,\beta+\alpha}(z)\right)^2}{\check{W}^2_{\alpha,\beta}(z)} \geqslant 0.$$

Next, to prove the inequality (6), we set

$$F(z) = \log\left(\Gamma(\beta)\check{W}_{\alpha,\beta}(z)\right) \text{ and } G(z) = z.$$

By usnig the fact that $z \mapsto \check{W}'_{\alpha,\beta}(z)/\check{W}_{\alpha,\beta}(z)$ is increasing on $(0,\infty)$ and monotone form of l'Hospital's rule [1], we deduce that the function $z \mapsto F(z)/G(z) = (F(z) - F(0))/(G(z)-G(0))$ is increasing on $(0,1)$, and consequently

$$\frac{F(z)}{G(z)} \geqslant \lim_{x\longrightarrow 0} F'(z) = -\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

This completes the proof of the Theorem 1. □

THEOREM 2. *The following inequalities hold true:*
a. *For $\beta-\alpha \geqslant 1$ and $z>0$, we have:*

$$W_{\alpha,\beta}(z) \leqslant \left(\frac{\Gamma(2\alpha)}{\Gamma^2(\beta)}\right)\cdot\left(\frac{e^{\frac{\Gamma(\alpha)z}{\Gamma(2\alpha)}}-1}{z}\right). \tag{8}$$

b. *For $\beta-\alpha \geqslant 2$ and $z>0$, we have:*

$$W_{\alpha,\beta+1}(z)W_{\alpha,\beta-1}(z) \leqslant \frac{\Gamma(\beta-\alpha)}{\Gamma(\beta-\alpha-1)\Gamma(\beta-\alpha+1)}W_{\alpha,\alpha+1}(z)W_{\alpha,\beta}(z). \tag{9}$$

In particular, we get

$$2W_{\alpha,\alpha+3}(z) \leqslant W_{\alpha,\alpha+2}(z). \tag{10}$$

Proof. **a.** In [5, Theorem 6.1], the author it was proved that

$$W_{\alpha,\beta}(z) \leqslant \frac{e^{\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}z}}{\Gamma(\beta)},\ z>0. \tag{11}$$

In view of (2) and (11), we obtain

$$W_{\alpha,\beta}(z) \leqslant \frac{c_{\alpha,\beta}}{\Gamma(\beta)}\int_0^1 (1-t^{1/\alpha})^{\beta-\alpha-1}e^{\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}zt}dt. \tag{12}$$

Now, recall the Chebyshev integral inequality [12, p. 40]: if $f,g:[a,b]\longrightarrow\mathbb{R}$ are synchoronous (both increasing or decreasing) integrable functions, and $p:[a,b]\longrightarrow\mathbb{R}$ is a positive integrable function, then

$$\int_a^b p(t)f(t)dt\int_a^b p(t)g(t)dt \leqslant \int_a^b p(t)dt\int_a^b p(t)f(t)g(t)dt. \tag{13}$$

Note that if f and g are asynchronous (one is decreasing and the other is increasing), then (13) is reversed. For this consider the functions $p,f,g:[0,1]\longrightarrow\mathbb{R}$ defined by:

$$p(t)=1,\ f(t)=\frac{c_{\alpha,\beta}}{\Gamma(\beta)}(1-t^{1/\alpha})^{\beta-\alpha-1} \text{ and } g(t)=e^{\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}zt}.$$

Since the function f is decreasing and g increasing if $\beta-\alpha\geqslant 1$. On the other hand, we have

$$\int_0^1 f(t)dt=\frac{\alpha c_{\alpha,\beta}}{\Gamma(\beta)}B(\beta-\alpha,\alpha)=\frac{\Gamma(\alpha)}{\Gamma^2(\beta)},\ \text{and}\ \int_0^1 g(t)dt=\frac{\Gamma(2\alpha)(e^{\frac{\Gamma(2\alpha)z}{\Gamma(\alpha)}}-1)}{\Gamma(\alpha)z}.$$

So, using the Chebyshev inequality (13) we get inequality (8).
b. Another use of the Chebyshev integral inequality (13), that is $p,f,g:[0,1]\longrightarrow\mathbb{R}$ defined by:

$$p(t)=W_{\alpha,\alpha}(zt),\ f(t)=c_{\alpha,\beta+1}(1-t^{1/\alpha})^{\beta-\alpha} \text{ and } g(t)=c_{\alpha,\beta-1}(1-t^{1/\alpha})^{\beta-\alpha-2}.$$

Observe that the functions f and g are decreasing on $(0,\infty)$ for all $\beta-\alpha\geqslant 2$. Furthermore, by using the Chebyshev inequality (13) and the integral representation (2) we have

$$\begin{aligned} & W_{\alpha,\beta+1}(z)W_{\alpha,\beta-1}(z) \\ \leqslant & \left(\int_0^1 W_{\alpha,\alpha}(zt)dt\right).\left(c_{\alpha,\beta+1}c_{\alpha,\beta-1}\int_0^1 (1-t^{1/\alpha})^{2\beta-2\alpha-2}W_{\alpha,\alpha}(zt)dt\right) \\ \leqslant & \left(\int_0^1 W_{\alpha,\alpha}(zt)dt\right).\left(c_{\alpha,\beta+1}c_{\alpha,\beta-1}\int_0^1 (1-t^{1/\alpha})^{\beta-\alpha-1}W_{\alpha,\alpha}(zt)dt\right) \\ = & \frac{\alpha c_{\alpha,\beta+1}c_{\alpha,\beta-1}}{c_{\alpha,\beta}}W_{\alpha,\alpha+1}(z)W_{\alpha,\beta}(z), \end{aligned} \tag{14}$$

and consequently (9) as well. Finally, setting in (9) the value $\beta=\alpha+2$ we deduce that the inequality (10) is hold true. □

In order to establish a bilateral functional inequalities for $W_{\alpha,\beta}(z)$, we need the Fox–Wright function ${}_p\Psi_q(z)$ defined by

$${}_p\Psi_q\left[{}^{(a_1,\alpha_1),\dots,(a_p,\alpha_p)}_{(b_1,\beta_1),\dots,(b_q,\beta_q)}\Big|z\right]=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^p\Gamma(a_j+\alpha_jk)}{\prod_{j=1}^q\Gamma(b_j+\beta_jk)}\frac{z^k}{k!}, \tag{15}$$

where $z,a_i,b_j\in\mathbb{C}$, $\alpha_i,\beta_j\in\mathbb{R}$ for $i\in\{1,...,p\}$ and $j\in\{1,...,q\}$. The series (15) converges absolutely and uniformly for all bounded $|z|$, $z\in\mathbb{C}$ when

$$1+\sum_{j=1}^q\beta_j-\sum_{i=1}^p\alpha_i>0.$$

We note that the inequality (18) in the next Theorem complements and improve the inequality (6).

THEOREM 3. *Let $\beta>\alpha>0$. The following inequalities hold true:*

$$\left(\frac{\Gamma(\alpha)}{\Gamma(\beta)}\right).e^{\frac{\Gamma(2\alpha)\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\alpha+\beta)}|z|}\leqslant {}_1\Psi_1\left[{}^{(\alpha,\alpha)}_{(\beta,\alpha)}\Big|z\right]\leqslant\left(\frac{\Gamma(\alpha)}{\Gamma(\beta)}\right)-\left(\frac{\Gamma(2\alpha)(1-e^{|z|})}{\Gamma(\beta+\alpha)}\right),\ z\in\mathbb{R}, \tag{16}$$

$$W_{\alpha,\beta}(z)\leqslant\left(\frac{1}{\Gamma(\beta)}\right)-\left(\frac{\Gamma(2\alpha)(1-e^{\frac{\Gamma(\alpha)z}{\Gamma(2\alpha)}})}{\Gamma(\alpha)\Gamma(\beta+\alpha)}\right),\ z>0, \tag{17}$$

$$\check{W}_{\alpha,\beta}(z)\geqslant\frac{e^{\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}z}}{\Gamma(\beta)},\ 0<z<1. \tag{18}$$

Proof. We recall that Pogány and Srivastava it was proved in [14, Theorem 4] and [14, eq. (22)] that for all ${}_p\Psi_q$ satisfying

$$\psi_1>\psi_2\ \text{ and }\ \psi_1^2<\psi_0\psi_2, \tag{19}$$

the two–sided inequality

$$\psi_0 e^{\psi_1\psi_0^{-1}|x|} \leqslant {}_p\Psi_q\left[\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\middle| x\right] \leqslant \psi_0-(1-e^{|x|})\psi_1, \tag{20}$$

holds true for all $x\in\mathbb{R}$. Here

$$\psi_m=\frac{\prod_{j=1}^{p}\Gamma(a_j+\alpha_j m)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j m)},\ j\in\{1,2,3\}.$$

In our case, we have

$$\psi_0=\frac{\Gamma(\alpha)}{\Gamma(\beta)},\ \ \psi_1=\frac{\Gamma(2\alpha)}{\Gamma(\beta+\alpha)}\text{ and }\psi_2=\frac{\Gamma(3\alpha)}{\Gamma(\beta+2\alpha)}.$$

On the other hand, due to log–convexity property of the Gamma function $\Gamma(z)$, the ratios $z\mapsto\Gamma(z+a)/\Gamma(z)$ is increasing on $(0,\infty)$ when $a>0$. Thus implies that the following inequality:

$$\frac{\Gamma(z+a)}{\Gamma(z)}\leqslant\frac{\Gamma(z+a+b)}{\Gamma(z+b)}, \tag{21}$$

holds for all $a,b,z>0$. Letting $z=2\alpha$, $a=\alpha$ and $b=\beta-\alpha>0$ in (12) we get $\psi_1>\psi_2$. This proves the left–hand side of inequality (19). Now, we consider the function $f:(0,\infty)\longrightarrow\mathbb{R}$ defined by:

$$f_\alpha(z)=\frac{\Gamma(z)\Gamma(z+2\alpha)}{\Gamma^2(\alpha+z)}.$$

Thus,

$$\frac{f'_\alpha(z)}{f_\alpha(z)}=\psi(z)+\psi(z+2\alpha)-2\psi(z+\alpha), \tag{22}$$

where $\psi(z)=\Gamma'(z)/\Gamma(z)$ is the Euler digamma function. By using the Legendre's formula

$$\psi(z)=-\gamma+\int_0^1\frac{t^{z-1}-1}{t-1}dt,$$

where γ is the Euler–Mascheroni constant, we have

$$\frac{f'_\alpha(z)}{f_\alpha(z)}=\int_0^1\frac{t^{z-1}}{t-1}g_\alpha(t)dt, \tag{23}$$

where $g_\alpha(t)=1+t^{2\alpha}-2t^\alpha$, $t\in[0,1]$. Thus $g'_\alpha(t)=2\alpha t^{\alpha-1}(t^\alpha-1)\leqslant 0$, for all $t\in[0,1]$, consequently the function $t\mapsto g_\alpha(t)$ is decreasing on $[0,1]$ and satisfies $g_\alpha(0)=1$ and $g_\alpha(1)=0$. So, the function $z\mapsto f_\alpha(z)$ is decreasing on $(0,\infty)$. In particular $f_\alpha(\beta)\leqslant f_\alpha(\alpha)$, which implies the right hand side of (19). Then,

$$\left(\frac{\Gamma(\alpha)}{\Gamma(\beta)}\right).e^{\frac{\Gamma(2\alpha)\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\alpha+\beta)}z}\leqslant{}_1\Psi_1\left[\begin{matrix}(\alpha,\alpha)\\(\beta,\alpha)\end{matrix}\middle| z\right]\leqslant\left(\frac{\Gamma(\alpha)}{\Gamma(\beta)}\right)-\left(\frac{\Gamma(2\alpha)(1-e^{|z|}}{\Gamma(\beta+\alpha)}\right) \tag{24}$$

for all $z \in \mathbb{R}$. Now, we prove the inequality (17) From the integral representation (2) and (11), we have

$$\begin{aligned} W_{\alpha,\beta}(z) &\leqslant \frac{c_{\alpha,\beta}}{\Gamma(\alpha)}\int_0^1 (1-t^{1/\alpha})^{\beta-\alpha-1} e^{\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}zt}dt \\ &= \frac{c_{\alpha,\beta}}{\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{(\Gamma(\alpha)/\Gamma(2\alpha)z)^n}{n!}\int_0^1 (1-t^{1/\alpha})^{\beta-\alpha-1}t^n dt \\ &= \frac{\alpha c_{\alpha,\beta}}{\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{(\Gamma(\alpha)/\Gamma(2\alpha)z)^n}{n!}\int_0^1 (1-t)^{\beta-\alpha-1}t^{\alpha n+\alpha-1}dt \\ &= \frac{\alpha c_{\alpha,\beta}}{\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{B(\beta-\alpha,\alpha n+\alpha)(\Gamma(\alpha)/\Gamma(2\alpha)z)^n}{n!} \\ &= \frac{1}{\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha n+\alpha)(\Gamma(\alpha)/\Gamma(2\alpha)z)^n}{n!\Gamma(\alpha n+\beta)} \\ &= \frac{1}{\Gamma(\alpha)}{}_1\Psi_1\left[{}^{(\alpha,\alpha)}_{(\beta,\alpha)}\Big|\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}z\right]. \end{aligned} \tag{25}$$

So, by the right hand side of inequality (16) and (25) we deduce that the inequality (17) holds true for all $z > 0$. Similar arguments would lead us to prove the inequality (18). By means of the integral representation (2) and the inequality (6) we have

$$\begin{aligned} W_{\alpha,\beta}(z) &\geqslant \frac{c_{\alpha,\beta}}{\Gamma(\alpha)}\int_0^1 (1-t^{1/\alpha})^{\beta-\alpha-1} e^{-\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}zt}dt \\ &= \frac{c_{\alpha,\beta}}{\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{(-\Gamma(\alpha)/\Gamma(2\alpha)z)^n}{n!}\int_0^1 (1-t^{1/\alpha})^{\beta-\alpha-1}t^n dt \\ &= \frac{1}{\Gamma(\alpha)}{}_1\Psi_1\left[{}^{(\alpha,\alpha)}_{(\beta,\alpha)}\Big|-\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}z\right]. \end{aligned} \tag{26}$$

Combining the left hand side of inequality (16) and (26) we obtain the inequality (18). This evidently completes the proof of Theorem 3. □

3. The generalized Wright functions: monotonicity patterns and functional inequalities

In [2], the authors introduced the definition of the generalized Wright function $W_{\alpha,\beta}^{\gamma,\sigma}(z)$:

$$W_{\alpha,\beta}^{\gamma,\sigma}(z) = \sum_{n=0}^{\infty}\frac{(\gamma)_n}{(\sigma)_n\Gamma(\alpha n+\beta)}\frac{z^n}{n!},\ \alpha\in\mathbb{R}, \beta,\gamma,\sigma,z\in\mathbb{C}, \tag{27}$$

where

$$(\tau)_n = \frac{\Gamma(\tau+n)}{\Gamma(\tau)} = \tau(\tau+1)...(\tau+n-1),$$

is a Pochhammer symbol. The function $W_{\alpha,\beta}^{\gamma,\sigma}(z)$ is an entire function of order $1/(1+\alpha)$ and has the following integral representation [2, Theorem 2, eq. (34)]

$$W_{\alpha,\beta}^{\gamma,\sigma}(z) = \frac{\Gamma(\sigma)}{\Gamma(\gamma)\Gamma(\sigma-\gamma)} \int_0^1 t^{\gamma-1}(1-t)^{\sigma-\gamma-1} W_{\alpha,\beta}(zt)dt, \tag{28}$$

where $\alpha > -1, \beta, \gamma, \sigma, z \in \mathbb{C}$ and $\Re(\sigma) > \Re(\gamma) > 0$.

THEOREM 4. *The following assertions are true:*
a. *The function* $z \mapsto \hat{W}_{\alpha,\beta}^{\gamma,\sigma}(z)$ *is completely monotonic and log–convex on* $(0,1)$, *for all* $\alpha,\gamma,\sigma > 0$ *such that* $\beta > \alpha > x^\star$ *and* $\sigma > \gamma$. *Moreover, the following inequalities hold true:*

$$\check{W}_{\alpha,\beta}^{\gamma,\sigma}(x+y) \geqslant \frac{\check{W}_{\alpha,\beta}^{\gamma,\sigma}(x)\check{W}_{\alpha,\beta}^{\gamma,\sigma}(y)}{\Gamma(\beta)}, \ 0 < x+y < 1, \tag{29}$$

$$\frac{\gamma+1}{\sigma+1}\hat{W}_{\alpha,\beta+2\alpha}^{\gamma+2,\sigma+2}(z)\hat{W}_{\alpha,\beta}^{\gamma,\sigma}(z) - \frac{\gamma}{\sigma}.(\hat{W}_{\alpha,\beta+\alpha}^{\gamma+1,\sigma+1}(z))^2 \geqslant 0, \ 0 < z < 1, \tag{30}$$

$$\check{W}_{\alpha,\beta}^{\gamma,\sigma}(z) \geqslant \frac{e^{-\frac{\gamma\Gamma(\beta)}{\sigma\Gamma(\beta+\alpha)}z}}{\Gamma(\beta)}, \ z \in (0,1). \tag{31}$$

b. *The function* $\sigma \mapsto W_{\alpha,\beta}^{\gamma,\sigma}(z)$ *is log–convex on* $(0,\infty)$. *Moreover, the following Turán type inequality*

$$W_{\alpha,\beta}^{\gamma,\sigma}(z)W_{\alpha,\beta}^{\gamma,\sigma+2}(z) - \left(W_{\alpha,\beta}^{\gamma,\sigma+1}(z)\right)^2 \geqslant 0, \tag{32}$$

holds true.

Proof. **a.** From Theorem 1 and integral representation of the generalized Wright function $W_{\alpha,\beta}^{\gamma,\sigma}(z)$, we deduce that the function $z \mapsto \check{W}_{\alpha,\beta}^{\gamma,\sigma}(z)$ is completely monotonic on $(0,1)$ and consequently is log–convex. By using the Kimberling's result [3], we obtain the inequality (29). Now, we prove the inequality (30). Since the function $z \mapsto \hat{W}_{\alpha,\beta}^{\gamma,\sigma}(z)$ is log–convex on $(0,1)$ we have $z \mapsto (\hat{W}_{\alpha,\beta}^{\gamma,\sigma}(z))'/\hat{W}_{\alpha,\beta}^{\gamma,\sigma}(z)$ is increasing on $(0,1)$. So, by using the differentiation formula [2, Theorem 19]

$$\frac{d}{dz}W_{\alpha,\beta}^{\gamma,\sigma}(z) = \frac{\gamma}{\sigma}W_{\alpha,\beta+\alpha}^{\gamma+1,\sigma+1}(z), \tag{33}$$

we get

$$\left(\frac{(\hat{W}_{\alpha,\beta}^{\gamma,\sigma}(z))'}{\hat{W}_{\alpha,\beta}^{\gamma,\sigma}(z)}\right)' = \frac{\frac{\gamma(\gamma+1)}{\sigma(\sigma+1)}\hat{W}_{\alpha,\beta+2\alpha}^{\gamma+2,\sigma+2}(z)\hat{W}_{\alpha,\beta}^{\gamma,\sigma}(z) - \frac{\gamma^2}{\sigma^2}.(\hat{W}_{\alpha,\beta+\alpha}^{\gamma+1,\sigma+1}(z))^2}{(\hat{W}_{\alpha,\beta}^{\gamma,\sigma}(z))^2} \geqslant 0, \tag{34}$$

which can be derived easily the inequality (30). Now, we prove the inequality (31). Let $F_1(z) = \log\left[\Gamma(\beta)\hat{W}_{\alpha,\beta}^{\gamma,\sigma}(z)\right]$ and $G_1(z) = z$. Again by using the monotone form of l'Hospital's rule, we deduce that the function $F_1(z)/G_1(z) = (F_1(z) - F_1(0))/(G_1(z) - G_1(0))$ is increasing on $(0,1)$, and consequently

$$\lim_{z \longrightarrow 0} \frac{F_1(z)}{G_1(z)} = -\frac{\gamma\Gamma(\beta)}{\sigma\Gamma(\beta+\alpha)},$$

which completes the proof of inequality (31).

b. For convenience, let us write $A_n(\sigma) = \frac{(\gamma)_n}{(\sigma)_n n!\Gamma(\alpha n+\beta)}$. Since the function ψ' is completely monotonic on $(0,\infty)$ we get

$$\partial^2[\log A_n(\sigma)]/\partial\sigma^2 = \psi'(\sigma) - \psi'(\sigma+n) \geqslant 0,$$

for all $n \geqslant 0$. So, using the fact that sums of log–convex functions are log–convex too, we deduce that the function $\sigma \mapsto W_{\alpha,\beta}^{\gamma,\sigma}(z)$ is log–convex on $(0,\infty)$, for $z > 0$. Now, focus the Turán type inequality (32). Since $\sigma \mapsto W_{\alpha,\beta}^{\gamma,\sigma}(z)$ is log–convex on $(0,\infty)$ for $z > 0$, it follows that for $\sigma_1, \sigma_2 > 0$, $t \in [0,1]$, we have

$$W_{\alpha,\beta}^{\gamma, t\sigma_1+(1-t)\sigma_2}(z) \leqslant \left[W_{\alpha,\beta}^{\gamma,\sigma_1}(z)\right]^t \left[W_{\alpha,\beta}^{\gamma,\sigma}(z)\right]^{1-t}.$$

Choosing $\sigma_1 = \sigma$, $\sigma_2 = \sigma+2$ and $t = 1/2$, the above inequality reduces to the Turán type inequality (32). The proof of Theorem 4 is thus completed. □

THEOREM 5. *Let $\beta,\alpha,\sigma > 0$ and $\gamma > 0$. Then, the following Turán type inequality*

$$W_{\alpha,\beta}^{\gamma,\sigma}(z) W_{\alpha,\beta}^{\gamma+2,\sigma}(z) - \frac{\gamma}{\gamma+1}\left(W_{\alpha,\beta}^{\gamma+1,\sigma}(z)\right)^2 \geqslant 0, \tag{35}$$

holds true for all $z > 0$.

Proof. For convenience, let us write $\mathrm{K}(\gamma) = \frac{\Gamma(\gamma)}{\Gamma(\sigma)} W_{\alpha,\beta}^{\gamma,\sigma}(z)$. By applying the Cauchy product series, we find that

$$\mathrm{K}^2(\gamma+1) - \mathrm{K}(\gamma)\mathrm{K}(\gamma+2) = \sum_{k=0}^{\infty}\sum_{j=0}^{k} \delta_{j,k} T_{j,k} z^k, \tag{36}$$

where $T_{j,k} = ((2j-k)-1)\Gamma(\gamma+j)\Gamma(\gamma+(k-j)+1)$ and $\delta_{j,k} = 1/(j!(k-j)!\Gamma(\sigma+j)\Gamma(\sigma+k-j)\Gamma(\alpha j+\beta)\Gamma(\alpha(k-j)+\beta))$. If k is even, then

$$\begin{aligned}
\sum_{j=0}^{k} \delta_{j,k} T_{j,k} &= \sum_{j=0}^{k/2-1} \delta_{j,k} T_{j,k} + \sum_{j=0}^{k/2+1} \delta_{j,k} T_{j,k} + \delta_{\frac{k}{2},k} T_{\frac{k}{2},k} \\
&= \sum_{j=0}^{k/2-1} \delta_{j,k} T_{j,k} + \sum_{j=0}^{k/2-1} \delta_{j,k} T_{k-j,k} + \delta_{\frac{k}{2},k} T_{\frac{k}{2},k} \\
&= \sum_{j=0}^{[(k-1)/2]} \delta_{j,k}\left(T_{j,k} + T_{k-j,k}\right) - \delta_{\frac{k}{2},k}\Gamma(\gamma+k/2)\Gamma(\gamma+k/2+1),
\end{aligned}$$

where, as usual, $[k]$ denotes the greatest integer part of $k \in \mathbb{R}$. Similarly, if k is odd, then

$$\sum_{j=0}^{k} \delta_{j,k} T_{j,k} = \sum_{j=0}^{[(k-1)/2]} \delta_{j,k}\left(T_{j,k}+T_{k-j,k}\right) - \delta_{\frac{k}{2},k}\Gamma(\gamma+k/2)\Gamma(\gamma+k/2+1).$$

Therefore,

$$\begin{aligned}&\mathrm{K}^2(\gamma+1)-\mathrm{K}(\gamma)\mathrm{K}(\gamma+2)\\ =&\sum_{k=0}^{\infty}\sum_{j=0}^{[(k-1)/2]} \delta_{j,k}\left(T_{j,k}+T_{k-j,k}\right) - \delta_{\frac{k}{2},k}\Gamma(\gamma+k/2)\Gamma(\gamma+k/2+1).\end{aligned}$$

Simplifying, we find that

$$T_{j,k}+T_{k-j,k} = (k-2j)((2j-k)-1)\Gamma(\gamma+j)\Gamma(\gamma+(k-j)) \leqslant 0,$$

for $k < k-j$ (i.e $[(k-1)/2] \geqslant j$), which evidently completes the proof of Theorem 5. □

THEOREM 6. *Let $\beta > \alpha > 0$ and $\sigma > \gamma > 0$. Then the following inequalities hold true:*

$$\frac{\Gamma(\gamma)}{\Gamma(\sigma)} e^{\frac{\gamma}{\sigma}|z|} \leqslant {}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\sigma,1)\end{matrix}\middle| z\right] \leqslant \left(\frac{\Gamma(\gamma)}{\Gamma(\sigma)}\right)\cdot\left(1-\frac{\gamma}{\sigma}(1-e^{|z|})\right),\ z\in\mathbb{R} \tag{37}$$

$$W_{\alpha,\beta}^{\gamma,\sigma}(z) \leqslant \left(\frac{1}{\Gamma(\beta)}\right)\cdot\left[1-\frac{\gamma}{\sigma}\left(1-e^{\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}z}\right)\right], z>0, \tag{38}$$

$$\check{W}_{\alpha,\beta}^{\gamma,\sigma}(z) \geqslant \left(\frac{1}{\Gamma(\beta)}\right).e^{\frac{\gamma\Gamma(\beta)}{\sigma\Gamma(\beta+\alpha)}z}, 0<z<1. \tag{39}$$

Proof. In our case, we have $\psi_0 = \frac{\Gamma(\gamma)}{\Gamma(\sigma)}, \psi_1 = \frac{\Gamma(\gamma+1)}{\Gamma(\sigma+1)}$ and $\psi_2 = \frac{\Gamma(\gamma+2)}{\Gamma(\sigma+2)}$. Since $\sigma > \gamma$, we get $\psi_1 > \psi_2$ and $\psi_1^2 < \psi_0\psi_2$, and consequently the conditions (19) hold. Then, by using (20) we deduce that the inequality (37) holds true. Next, we prove the inequality (38). Combining the inequality (11) and the representation integral of the generalized Wright function (28), we get

$$\begin{aligned}W_{\alpha,\beta}^{\gamma,\sigma}(z) &\leqslant \frac{\Gamma(\sigma)}{\Gamma(\beta)\Gamma(\gamma)\Gamma(\sigma-\gamma)}\int_0^1 t^{\gamma-1}(1-t)^{\sigma-\gamma-1}e^{\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}zt}dt\\ &= \frac{\Gamma(\sigma)}{\Gamma(\beta)\Gamma(\gamma)\Gamma(\sigma-\gamma)}\int_0^1 t^{\gamma+n-1}(1-t)^{\sigma-\gamma-1}\left(\sum_{n=0}^{\infty}\frac{\left((\Gamma(\beta)/\Gamma(\beta+\alpha))z\right)^n}{n!}\right)dt\\ &= \frac{\Gamma(\sigma)}{\Gamma(\beta)\Gamma(\gamma)\Gamma(\sigma-\gamma)}\sum_{n=0}^{\infty}\frac{\left((\Gamma(\beta)/\Gamma(\beta+\alpha))z\right)^n}{n!}\int_0^1 t^{\gamma+n-1}(1-t)^{\sigma-\gamma-1}dt\end{aligned}$$

$$= \frac{\Gamma(\sigma)}{\Gamma(\beta)\Gamma(\gamma)\Gamma(\sigma-\gamma)} \sum_{n=0}^{\infty} \frac{B(\gamma+n,\sigma-\gamma)\Big((\Gamma(\beta)/\Gamma(\beta+\alpha))z\Big)^n}{n!}$$
$$= \frac{\Gamma(\sigma)}{\Gamma(\beta)\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+n)\Big((\Gamma(\beta)/\Gamma(\beta+\alpha))z\Big)^n}{\Gamma(\sigma+n)n!}$$
$$= \frac{\Gamma(\sigma)}{\Gamma(\beta)\Gamma(\gamma)}\,{}_1\Psi_1\left[{}^{(\gamma,1)}_{(\sigma,1)}\Big|\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}z\right].$$

Combining this equation with the right hand side of inequalities (37), we obtain (38). It remains to prove (39). The integral representation (28) of the function $W_{\alpha,\beta}^{\gamma,\sigma}(z)$ and inequality (31) yields that

$$\check{W}_{\alpha,\beta}^{\gamma,\sigma}(z)$$
$$\geqslant \frac{\Gamma(\sigma)}{\Gamma(\beta)\Gamma(\gamma)\Gamma(\sigma-\gamma)} \int_0^1 t^{\gamma-1}(1-t)^{\sigma-\gamma-1} e^{-\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}zt}\,dt$$
$$= \frac{\Gamma(\sigma)}{\Gamma(\beta)\Gamma(\gamma)\Gamma(\sigma-\gamma)} \int_0^1 t^{\gamma+n-1}(1-t)^{\sigma-\gamma-1}\left(\sum_{n=0}^{\infty} \frac{\Big(-(\Gamma(\beta)/\Gamma(\beta+\alpha))z\Big)^n}{n!}\right) dt$$
$$= \frac{\Gamma(\sigma)}{\Gamma(\beta)\Gamma(\gamma)\Gamma(\sigma-\gamma)} \sum_{n=0}^{\infty} \frac{\Big(-(\Gamma(\beta)/\Gamma(\beta+\alpha))z\Big)^n}{n!} \int_0^1 t^{\gamma+n-1}(1-t)^{\sigma-\gamma-1} dt$$
$$= \frac{\Gamma(\sigma)}{\Gamma(\beta)\Gamma(\gamma)}\,{}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\sigma,1)\end{matrix}\Bigg| -\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}z\right].$$

From the above inequality and the left hand side of inequalities (37) we deduce (39) for all $0<z<1$ and $\beta>\alpha>0$ and $\sigma>\gamma>0$. The proof of Theorem 6 is complete. □

REMARK 1. We point out that the inequality (39) complements and improve the inequality (31). Since $e^z \geqslant e^{-z}$ for all $z>0$, we deduce that the inequality (39) is better than (31).

THEOREM 7. *The following inequalities hold true:*
a. *For all* $z>0$, $0<\gamma\leqslant 1$ *and* $\sigma-\gamma\geqslant 1$, *we have*

$$W_{\alpha,\beta}^{\gamma,\sigma}(z) \leqslant \frac{\Gamma(\beta-\alpha)W_{\alpha,\beta-\alpha}(z)-1}{\Gamma(\beta-\alpha)z} = W_{\alpha,\beta}^{1,2}(z). \tag{40}$$

b. *For all* $z>0$, $0<\gamma\leqslant 1$ *and* $\sigma-\gamma\geqslant 2$, *we have*

$$W_{\alpha,\beta}^{\gamma,\sigma+1}(z)W_{\alpha,\beta}^{\gamma,\sigma-1}(z) \leqslant \frac{\Gamma(\sigma-\gamma)\Gamma(\sigma+1)\Gamma(\sigma-1)}{\Gamma(\sigma)\Gamma(\gamma)\Gamma(\sigma-\gamma+1)\Gamma(\sigma-\gamma-1)} W_{\alpha,\beta}^{1,2}(z)W_{\alpha,\beta}^{\gamma,\sigma}(z). \tag{41}$$

Proof. **a.** By again using the Chebyshev integral inequality (13), we consider the functions $p,f,g:[0,1]\longrightarrow\mathbb{R}$ defined by

$$p(t)=1,\ f(t)=(B(\sigma-\gamma,\gamma))^{-1}(1-t)^{\sigma-\gamma-1}t^{\gamma-1} \text{ and } g(t)=W_{\alpha,\beta}(zt).$$

Observe that the function $f(t)$ is decreasing and $g(t)$ is increasing on $[0,1]$, if $0<\gamma\leqslant 1$ and $\sigma-\gamma\geqslant 1$. On the other hand, we have

$$\int_0^1 p(t)f(t)dt=1,$$

and

$$\begin{aligned}\int_0^1 p(t)g(t)&=\frac{1}{z}\int_0^1\left(W_{\alpha,\beta-\alpha}(zt)\right)'dt\\&=\frac{1}{z}\left(W_{\alpha,\beta-\alpha}(z)-\frac{1}{\Gamma(\beta-\alpha)}\right)\\&=W^{1,2}_{\alpha,\beta}(z).\end{aligned}$$

So, the integral representation (28) completes the proof of inequality (40).
b. For the proof of inequality (41), we consider the functions $p,f,g:[0,1]\longrightarrow\mathbb{R}$ defined by

$$p(t)=W_{\alpha,\beta}(zt),\ f(t)=(1-t)^{\sigma-\gamma}t^{\gamma-1},\ \text{ and } g(t)=(1-t)^{\sigma-\gamma-2}t^{\gamma-1}.$$

Thus,

$$\begin{aligned}\int_0^1 p(t)f(t)dt&=\frac{\Gamma(\gamma)\Gamma(\sigma-\gamma+1)}{\Gamma(\sigma+1)}W^{\gamma,\sigma+1}_{\alpha,\beta}(z),\\\int_0^1 p(t)g(t)&=\frac{\Gamma(\gamma)\Gamma(\sigma-\gamma-1)}{\Gamma(\sigma-1)}W^{\gamma,\sigma-1}_{\alpha,\beta}(z),\end{aligned}$$

and

$$\int_0^1 p(t)dt=\int_0^1 W_{\alpha,\beta}(zt)dt=W^{1,2}_{\alpha,\beta}(zt)=\frac{1}{z}\left(W_{\alpha,\beta}(z)-\frac{1}{\Gamma(\beta-\alpha)}\right).$$

On the other hand, the functions $f(t)$ and $g(t)$ are decreasing on $[0,1]$ if $0<\gamma\leqslant 1$ and $\sigma-\gamma\geqslant 2$. Therefore, the Chebyshev integral inequality (13) yields that

$$\begin{aligned}&\frac{\Gamma^2(\gamma)\Gamma(\sigma-\gamma+1)\Gamma(\sigma-\gamma-1)}{\Gamma(\sigma+1)\Gamma(\sigma-1)}W^{\gamma,\sigma+1}_{\alpha,\beta}(z)W^{\gamma,\sigma-1}_{\alpha,\beta}(z)=\\&=\left(\int_0^1 p(t)f(t)dt\right).\left(\int_0^1 p(t)g(t)dt\right)\\&\leqslant\left(\int_0^1 W_{\alpha,\beta}(zt)dt\right).\left(\int_0^1(1-t)^{2\sigma-2\gamma-2}t^{2\gamma-2}W_{\alpha,\beta}(zt)\right)dt\\&\leqslant\left(\int_0^1 W_{\alpha,\beta}(zt)dt\right).\left(\int_0^1(1-t)^{\sigma-\gamma-1}t^{\gamma-1}W_{\alpha,\beta}(zt)\right)dt\\&=\frac{\Gamma(\gamma)\Gamma(\sigma-\gamma)}{\Gamma(\sigma)}W^{1,2}_{\alpha,\beta}(z)W^{\gamma,\sigma}_{\alpha,\beta}(z).\end{aligned}$$

The proof of Theorem 7 is complete. □

REMARK 2. We note that the results obtained in section 3 are not a generalizations of the results obtained in section 2. except Theorem 4, assertion **a.** and equations (29), (30) and (31). Indeed, the results in section 2 follow by using the new integral representation (2) and the results of section 3 follow by using the integral representation (28) which is different from the integral representation (2). Then, in the same way we obtain that the function $W_{\alpha,\beta}^{\sigma,\gamma}(z)$ admits this integral representation

$$W_{\alpha,\beta}^{\gamma,\sigma}(z) = c_{\alpha,\beta}\int_0^1 (1-t^{1/\alpha})^{\beta-\alpha-1} W_{\alpha,\alpha}^{\gamma,\sigma}(zt)dt, \tag{42}$$

which is a generalization of (2), and consequently we can obtain the generalization of our results from section 2.

4. Applications: monotonicity patterns and functional inequalities for the four–parametric Mittag–Leffler functions

The Mittag–Leffler functions with $2n$ parameters are defined for $B_j \in \mathbb{R}$ $(B_1^2 + ... + B_n^2 \neq 0)$ and $\beta_j \in \mathbb{C}$ $(j = 1,...,n \in \mathbb{N})$ by the series

$$E_{(B,\beta)_n}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\prod_{j=1}^n \Gamma(\beta_j + kB_j)},\ z \in \mathbb{C}. \tag{43}$$

When $n = 1$, the definition in (43) coincides with the definition of the two–parametric Mittag–Leffler function

$$E_{(B,\beta)_1}(z) = E_{B,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta + kB)},\ z \in \mathbb{C}, \tag{44}$$

and similarly for $n = 2$, where $E_{(B,\beta)_2}(z)$ coincides with the four–parametric Mittag–Leffler function

$$E_{(B,\beta)_2}(z) = E_{B_1,\beta_1;B_2\beta_2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta_1 + kB_1)\Gamma(\beta_2 + kB_2)},\ z \in \mathbb{C}, \tag{45}$$

and is closer by its properties to the Wright function $W_{B,\beta}(z)$ defined by

$$W_{B,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\beta + kB))},\ z \in \mathbb{C}. \tag{46}$$

The generalized $2n-$ parametric Mittag-Leffler function $E_{(\beta,B)_n}(z)$ can be represented in terms of the Fox–Wright hypergeometric function ${}_p\Psi_q(z)$ by

$$E_{(B,\beta)_n}(z) = {}_1\Psi_n\left[{}^{(1,1)}_{(\beta_1,B_1),...,(\beta_q,B_q)} \middle| z \right],\ z \in \mathbb{C}. \tag{47}$$

Letting $\gamma = 1$ in definition (27) of the generalized Wright function, we obtain that

$$W^{1,\sigma}_{\alpha,\beta}(z) = \Gamma(\sigma)E_{\alpha,\beta;1,\sigma}(z), \tag{48}$$

and consequently we obtain the following assertions for the four–parametric Mittag–Leffler function $E_{\alpha,\beta;1,\sigma}(z)$:

THEOREM 8. ***a.*** *The function $z \mapsto E_{\alpha,\beta;1,\sigma}(-z) = \check{E}_{\alpha,\beta;1,\sigma}(z)$ is completely monotonic and log–convex on $(0,1)$ for all $\beta > \alpha > x^\star$ and $\sigma > 1$. Furthermore, the following inequalities hold true:*

$$\check{E}_{\alpha,\beta;1,\sigma}(x+y) \geqslant \left(\frac{\Gamma(\sigma)}{\Gamma(\beta)}\right).\check{E}_{\alpha,\beta;1,\sigma}(x)\check{E}_{\alpha,\beta;1,\sigma}(y),\ 0 < x+y < 1. \tag{49}$$

$$\frac{2}{\sigma+1}\check{E}_{\alpha,\beta+2\alpha;3,\sigma+2}(z)\check{E}_{\alpha,\beta;1,\sigma}(z) - \frac{1}{\sigma}\left(\check{E}_{\alpha,\beta+\alpha;2,\sigma+1}(z)\right)^2 \geqslant 0,\ 0 < z < 1. \tag{50}$$

$$\check{E}_{\alpha,\beta;1,\sigma}(z) \geqslant \frac{e^{\frac{\Gamma(\beta)}{\sigma\Gamma(\beta+\alpha)}z}}{\Gamma(\sigma)}, 0 < z < 1. \tag{51}$$

b. *The function $\sigma \mapsto \Gamma(\sigma)E_{\alpha,\beta;1,\sigma}(z)$ is log-convex on $(0,\infty)$ for all $z,\alpha,\beta > 0$. Moreover, the following Turán type inequality*

$$E_{\alpha,\beta;1,\sigma+2}(z)E_{\alpha,\beta;1,\sigma}(z) - \frac{\sigma}{\sigma+1}\left(E_{\alpha,\beta;1,\sigma+1}(z)\right)^2 \geqslant 0,$$

holds true for all $z,\alpha,\beta > 0$.
c. *Let $\beta > \alpha > 0$ and $\sigma > 1$. Then, the following inequality*

$$E_{\alpha,\beta;1,\sigma}(z) \leqslant \left(\frac{\Gamma(\sigma)}{\Gamma(\beta)}\right).\left[1 - \frac{1}{\sigma}\left(1 - e^{\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}z}\right)\right],$$

holds true for all $z > 0$.
d. *Let $\beta > \alpha > 0$ and $\sigma > 1$. Then*

$$E_{\alpha,\beta;1,\sigma+1}(z)E_{\alpha,\beta;1,\sigma-1}(z) \leqslant \frac{\Gamma(\sigma-1)}{\Gamma(\sigma)\Gamma(\sigma-2)}E_{\alpha,\beta;1,2}(z)E_{\alpha,\beta;1,\sigma}(z),$$

holds for all $z > 0$ and $\sigma \geqslant 3$.

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