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(A,m)–SYMMETRIC COMMUTING TUPLES OF OPERATORS ON A HILBERT SPACE

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Abstract. Let $\mathbf{T} = (T_1, \cdots, T_d)$ and A be a commuting d-tuple of operators and a positive operator on a complex Hilbert space, respectively. We introduce an (A,m)-symmetric commuting tuple of operators and characterize the joint approximate point spectrum of (A,m)-symmetric commuting tuple $\mathbf{T}$. Next we introduce an (A,m)-expansive symmetric commuting tuple of operators and show basic properties of (A,m)-expansive symmetric commuting tuple.

1. Introduction

Throughout this paper $\mathscr{H}$ stands for a complex separable Hilbert space with inner product $\langle \cdot \mid \cdot \rangle$ and $\mathscr{L}(\mathscr{H})$ is the Banach algebra of all bounded linear operators on $\mathscr{H}$. $\mathscr{L}(\mathscr{H})^+$ is the cone of positive (semi-definite) operators, i.e.,

$$\mathscr{L}(\mathscr{H})^+ = \{A \in \mathscr{L}(\mathscr{H}) : \langle Au \mid u \rangle \geqslant 0, \ \forall\, u \in \mathscr{H}\ \}.$$

For every $T \in \mathscr{L}(\mathscr{H})$, we write $\sigma_p(T)$ and $\sigma_{ap}(T)$ respectively, for the point spectrum and the approximate point spectrum of T.

In 1970, J.W. Helton [11] initiated the study of operators $T \in \mathscr{L}(\mathscr{H})$ which satisfies an identity of the following form

$$\sum_{0 \leqslant k \leqslant m} (-1)^{m-k} \binom{m}{k} T^{*m-k} T^k = 0. \tag{1.1}$$

(See [1, 2, 4, 12, 18]). Let R and S be in $\mathscr{L}(\mathscr{H})$. In [15], the authors studied the operator

$$C(R,S) : \mathscr{L}(\mathscr{H}) \longrightarrow \mathscr{L}(\mathscr{H})$$

defined by $C(R,S)(A) = RA - AS$. Then

$$C(R,S)^k(I) = \sum_{0 \leqslant j \leqslant k} (-1)^{k-j} \binom{k}{j} R^j S^{k-j}. \tag{1.2}$$

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In [15], the authors introduced the class of Hilton operators as follows: an operator $R \in \mathcal{L}(\mathcal{H})$ is said to be in the nth Helton class of S and write $R \in \text{Helton}_n(S)$ if $C(R,S)^n = 0$.

Let $A \in \mathcal{L}(\mathcal{H})$ be a positive operator and let m be a positive integer. An operator $T \in \mathcal{L}(\mathcal{H})$ is said an (A,m)-isometry if

$$\sum_{0 \leqslant k \leqslant m} (-1)^{m-k} \binom{m}{k} T^{*k} A T^k = 0. \tag{1.3}$$

If $m = 1$, it is called A-isometry, that is, T is an A-isometry if $T^*AT = A$. The class of (A,m)-isometries has been introduced by Sid Ahmed and Saddi [16], and studied by other authors. $\big($See [3, 5, 19]$\big)$.

In this paper, A will denote a positive operator.

The motivation for the present paper comes from the intensive study for considerable literature on tuples of commuting operators on infinite dimensional Hilbert space $\mathcal{H}$ (refer to [6, 7, 8, 10, 13, 17]). It is natural to look for the higher-dimensional analogs of (A,m)-symmetric operators.

A commuting d-tuple of operators $\mathbf{T} = (T_1, \cdots, T_d)$ of bounded linear operators on a Hilbert space $\mathcal{H}$ is called an m-isometry (also called spherical m-isometry) if

$$\sum_{0 \leqslant k \leqslant m} (-1)^{m-k} \binom{m}{k} Q_{\mathbf{T}}^k(I) = 0, \tag{1.4}$$

where

$$Q_{\mathbf{T}}(X) = \sum_{1 \leqslant j \leqslant d} T_j^* X T_j \ \big(X \in \mathcal{L}(\mathcal{H})\big) \text{ and } Q_{\mathbf{T}}^k(I) = \sum_{|\alpha| = k} \frac{k!}{\alpha!} \mathbf{T}^{*\alpha} \mathbf{T}^{\alpha}.$$

Note that $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}^d, |\alpha| = \alpha_1 + \cdots + \alpha_d$, $\mathbf{T}^{\alpha} = T_1^{\alpha_1} . \cdots . T_d^{\alpha_d}$ and $\mathbf{T}^* = (T_1^*, \cdots, T_d^*)$. (See [7, 8, 10, 13]).

Recently, the authors [6] have introduced m-symmetric commuting tuple of operators as follows: a tuple of operators $\mathbf{T} = (T_1, \cdots, T_d) \in \mathcal{L}(\mathcal{H})^d := \mathcal{L}(\mathcal{H}) \times \cdots \times \mathcal{L}(\mathcal{H})$ is said to be an m-symmetric commuting tuple of operators if $\mathbf{T}$ satisfies

$$\sum_{0 \leqslant k \leqslant m} (-1)^{m-k} \binom{m}{k} \big(T_1^* + \cdots + T_d^*\big)^{m-k} \big(T_1 + \cdots + T_d\big)^k = 0.$$

In this paper we are interested to the classes of tuple of commuting operators $\mathbf{T} = (T_1, \cdots, T_d) \in \mathcal{L}(\mathcal{H})^d$, which satisfy the following equation

$$\Delta_m^A(\mathbf{T}) := \sum_{0 \leqslant k \leqslant m} (-1)^{m-k} \binom{m}{k} \big(T_1^* + \cdots + T_d^*\big)^{m-k} A \big(T_1 + \cdots + T_d\big)^k = 0$$

$\big($resp. $\Delta_m^A(\mathbf{T}) \leqslant 0\big)$. Such operators are called (A,m)-symmetric commuting tuple (resp. (A,m)-expansive symmetric commuting tuple). We give some basic properties concerning these classes of operators.

The outline of the paper is as follows. In Section 2, we investigate various structural properties of the class of (A,m)-symmetric single operators. In particular, we prove that the class of (A,m)-symmetric single operator is translation invariant and further if T is (A,m)-symmetric operator then e^{itT} is (A,m)-isometric operator for $t \in \mathbb{R}$. In Section 3 and Section 4, we introduce the class of (A,m)-symmetric commuting tuple of operators. Some of their algebraic and spectral properties are studied. The main result in Section 4 is Theorem 4.1 which describes the structure of the joint approximate spectrum of some (A,m)-symmetric commuting tuple. In Section 5, we introduce the class of (A,m)-expansive symmetric commuting tuple of operators. We establish some general facts about this class of tuple of operators.

2. (A,m)-symmetric operators

DEFINITION 2.1. For $T \in \mathscr{L}(\mathscr{H})$, T is said to be an (A,m)-symmetric operator if

$$\Delta_m^A(T) := \sum_{0 \leqslant k \leqslant m} (-1)^{m-k} \binom{m}{k} T^{*m-k} A T^k = 0.$$

REMARK 2.1. If $A = I$ (the identity operator), every (I,m)-symmetric operator is called m-symmetric operator ([18]).

THEOREM 2.1. *For an operator $T \in \mathscr{L}(\mathscr{H})$, if $A \geqslant 0$ is invertible and T is (A,m)-symmetric, then $\sigma(T) \subset \mathbb{R}$.*

Proof. Let $\lambda \in \sigma_{ap}(T)$ and $\{x_n\}$ be a sequence of unit vectors such that $(T - \lambda)x_n \to 0$ $(n \to \infty)$. Then

$$0 = \lim_{n\to\infty} \langle \Big(\sum_{0 \leqslant k \leqslant m} (-1)^{m-k} \binom{m}{k} T^{*m-k} A T^k \Big) x_n \,|\, x_n \rangle = (\lambda - \overline{\lambda})^m \lim_{n\to\infty} \langle A x_n \,|\, x_n \rangle.$$

If $\lim_{n\to\infty} \langle A x_n \,|\, x_n \rangle = 0$, then $0 \in \overline{W(A)}$, where $W(A)$ denotes the numerical range of A. Since A is positive, $\overline{W(A)} = \mathrm{co}\,\sigma(A)$, where $\mathrm{co}\,\sigma(A)$ denotes the convex hull of $\sigma(A)$. Hence, $0 \in \sigma(A)$. Since A is invertible, it's a contradiction. Therefore, $\lambda = \overline{\lambda}$ and λ is a real number. Since the boundary of $\sigma(T)$ is included in $\mathbb{R}$, $\sigma(T) \subset \mathbb{R}$. $\square$

We prepare a symbol. We define a polynomial $\{(x-y)^m\}_a$ by

$$\{(x-y)^m\}_a = \Big\{ \sum_{0 \leqslant k \leqslant m} (-1)^{m-k} \binom{m}{k} y^{m-k} x^k \Big\}_a := \sum_{0 \leqslant k \leqslant m} (-1)^{m-k} \binom{m}{k} y^{m-k} a x^k.$$

For $T \in \mathscr{L}(\mathscr{H})$ and $A \geqslant 0$, we define

$$\Big(\Big\{ \sum_{0 \leqslant k \leqslant m} (-1)^{m-k} \binom{m}{k} y^{m-k} x^k \Big\}_a \Big)(T,\, A) := \sum_{0 \leqslant k \leqslant m} (-1)^{m-k} \binom{m}{k} y^{m-k} a x^k \Big|_{y=T^*, x=T, a=A}$$

$$= \sum_{0 \leqslant k \leqslant m} (-1)^{m-k} \binom{m}{k} T^{*m-k} A T^k.$$

Then we have

$$\Delta^A_{m+1}(T) = \Big(\{(x-y)^{m+1}\}_a\Big)(T,\,A) = \Big(\big(\{(x-y)^m\}_a\big)x - y\big(\{(x-y)^m\}_a\big)\Big)(T,\,A)$$
$$= \Delta^A_m(T)T - T^*\,\Delta^A_m(T).$$

Hence we have an equation

$$\Delta^A_{m+1}(T) = \Delta^A_m(T)\,T - T^*\,\Delta^A_m(T). \tag{2.1}$$

Therefore, if T is (A,m)-symmetric, then T is (A,n)-symmetric for every n $(n \geqslant m)$.

The following theorem shows that the class of (A,m)-symmetric operators is translation invariant.

THEOREM 2.2. *If $T \in \mathscr{L}(\mathscr{H})$ is (A,m)-symmetric and $A \geqslant 0$, then, for a real number t, $T-t$ is (A,m)-symmetric.*

Proof. By the previous symbol since $\Big(\{(x-y)^m\}_a\Big)(T,\,A) = \Delta^m_A(T) = 0$, the proof follows from

$$\Big(\{\big((x-t)-(y-t)\big)^m\}_a\Big)(T,\,A) = \Big(\{(x-y)^m\}_a\Big)(T,\,A). \quad \square$$

For $T \in \mathscr{L}(\mathscr{H})$ and $t \in \mathbb{R}$,

$$e^{itT} = I + itT + \frac{(it)^2}{2!}T^2 + \frac{(it)^3}{3!}T^3 + \cdots$$

and

$$(e^{itT})^* = I - itT^* + \frac{(-it)^2}{2!}T^{*2} + \frac{(-it)^3}{3!}T^{*3} + \cdots.$$

Hence

$$(e^{itT})^*\,A\,e^{itT} = A - it(T^*A - AT) + \frac{(-it)^2}{2!}\Big(T^{*2}A - 2T^*AT + AT^2\Big) + \cdots$$
$$= A - it\Delta^A_1(T) + \frac{(-it)^2}{2!}\Delta^A_2(T) + \frac{(-it)^3}{3!}\Delta^A_3(T) + \cdots.$$

Therefore, if T is (A,m)-symmetric, then

$$(e^{itT})^*\,A\,e^{itT} = A - it\Delta^A_1(T) + \frac{(-it)^2}{2!}\Delta^A_2(T) + \cdots + \frac{(-it)^{m-1}}{(m-1)!}\Delta^A_{m-1}(T).$$

Hence, in this case,

$$\begin{aligned}&(e^{itT})^{*k}A\,(e^{itT})^k=(e^{iktT})^*A\,e^{iktT}\\&=A-ikt\Delta_1^A(T)+\frac{(-it)^2}{2!}k^2\Delta_2^A(T)+\cdots+\frac{(-it)^{m-1}}{(m-1)!}k^{m-1}\Delta_{m-1}^A(T).\end{aligned}$$

THEOREM 2.3. *If $T\in\mathscr{L}(\mathscr{H})$ is (A,m)-symmetric and $A\geqslant 0$, then*

$$(-i)^{m-1}\Delta_{m-1}^A(T)\geqslant 0.$$

Proof. Let t be a real number. Then it holds

$$\begin{aligned}e^{-itT^*}A\,e^{itT}=&A+(-it)\Delta_1^A(T)+\frac{(-it)^2}{2!}\Delta_2^A(T)+\cdots\\&+\frac{(-it)^{m-1}}{(m-1)!}\Delta_{m-1}^A(T)+\frac{(-it)^m}{m!}\Delta_m^A(T)+\cdots.\end{aligned}$$

Since T is (A,m)-symmetric, by equation (2.1) it holds $\Delta_n^A(T)=0$ for every $n\geqslant m$. Hence we have

$$e^{-itT^*}A\,e^{itT}=A+(-it)\Delta_1^A(T)+\frac{(-it)^2}{2!}\Delta_2^A(T)+\cdots+\frac{(-it)^{m-1}}{(m-1)!}\Delta_{m-1}^A(T).$$

Therefore it holds

$$\begin{aligned}&\frac{(-i)^{m-1}}{(m-1)!}\Delta_{m-1}^A(T)=\frac{1}{t^{m-1}}e^{-itT^*}A\,e^{itT}\\&-\frac{1}{t^{m-1}}\left(A+(-it)\Delta_1^A(T)+\frac{(-it)^2}{2!}\Delta_2^A(T)+\cdots+\frac{(-it)^{m-2}}{(m-2)!}\Delta_{m-2}^A(T)\right).\end{aligned}$$

Since, for $t>0$, $\frac{1}{t^{m-1}}e^{-itT^*}A\,e^{itT}\geqslant 0$ and

$$\frac{1}{t^{m-1}}\left(A+(-it)\Delta_1^A(T)+\frac{(-it)^2}{2!}\Delta_2^A(T)+\cdots+\frac{(-it)^{m-2}}{(m-2)!}\Delta_{m-2}^A(T)\right)\longrightarrow 0\ \ (t\to\infty),$$

we have $(-i)^{m-1}\Delta_{m-1}^A(T)\geqslant 0$. $\square$

For $T\in\mathscr{L}(\mathscr{H})$ and $A\geqslant 0$, we define $B_m^A(T)$ by

$$B_m^A(T):=\sum_{0\leqslant k\leqslant m}(-1)^k\binom{m}{k}T^{*m-k}A\,T^{m-k}.$$

Recall that T is said to be (A,m)-isometric if $B_m^A(T)=0$ $\big([16]\big)$.

THEOREM 2.4. ([16]) *The following properties hold.*

(i) If $T \in \mathscr{L}(\mathscr{H})$ is (A,m)-isometric and $A \geqslant 0$, then $B_{m-1}^{A}(T) \geqslant 0$.

(ii) If $T \in \mathscr{L}(\mathscr{H})$ is (A,m)-isometric and T is invertible, then T^{-1} is (A,m)-isometric.

THEOREM 2.5. *If $T \in \mathscr{L}(\mathscr{H})$ is (A,m)-isometric, m is even and T is invertible, then T is $(A,m-1)$-isometric.*

Proof. By Theorem 2.4, it holds $B_{m-1}^{A}(T) \geqslant 0$ and $B_{m-1}^{A}(T^{-1}) \geqslant 0$. Hence it holds

$$0 \leqslant T^{*m-1} B_{m-1}^{A}(T^{-1}) T^{m-1} = (-1)^{m-1} B_{m-1}^{A}(T).$$

Since $m-1$ is an odd number, it holds $B_{m-1}^{A}(T) \leqslant 0$. Therefore we have $B_{m-1}^{A}(T) = 0$. $\square$

For the next result, we need the following lemma.

LEMMA 2.1. *([9]) Let $m \in \mathbb{N}$. For every $j = 0,1,\ldots,m-1$, it holds*

$$\sum_{0 \leqslant k \leqslant m} (-1)^k \binom{m}{k} k^j = 0.$$

THEOREM 2.6. *If $T \in \mathscr{L}(\mathscr{H})$ is (A,m)-symmetric and $A \geqslant 0$, then e^{itT} is (A,m)-isometric for every $t \in \mathbb{R}$.*

Proof. By the previous lemma, we have

$$\begin{aligned}
\Delta_m^A(e^{itT}) &= \sum_{0 \leqslant k \leqslant m} (-1)^k \binom{m}{k} (e^{itT})^{*k} A (e^{itT})^k = \sum_{0 \leqslant k \leqslant m} (-1)^k \binom{m}{k} (e^{-iktT^*}) A (e^{iktT}) \\
&= \left(\sum_{0 \leqslant k \leqslant m} (-1)^k \binom{m}{k} \right) A + (-it) \left(\sum_{0 \leqslant k \leqslant m} (-1)^k \binom{m}{k} k \right) \Delta_1^A(T) + \cdots \\
&\quad + (-it)^{m-1} \left(\sum_{0 \leqslant k \leqslant m} (-1)^k \binom{m}{k} k^{m-1} \right) \Delta_{m-1}^A(T) \\
&= (1-1)^m A + (-it) \cdot 0 \cdot \Delta_1^A(T) + \cdots + (-it)^{m-1} \cdot 0 \cdot \Delta_{m-1}^A(T) = 0.
\end{aligned}$$

So e^{itT} is (A,m)-isometric for every $t \in \mathbb{R}$. It completes the proof. $\square$

REMARK 2.2. Let $T \in \mathscr{L}(\mathscr{H})$. If e^{itT} is (A,m)-symmetric and m is even, then e^{itT} is invertible (A,m)-isometric and m is even. Hence e^{itT} is $(A,m-1)$-isometric. Therefore, e^{itT} is $(A,m-1)$-symmetric by Theorem 2.5.

3. (A,m)-symmetric commuting tuple of operators

In this section, we give a basic result about (A,m)-symmetric tuple of commuting operators.

Let $\mathbf{T} = (T_1,\cdots,T_d) \in \mathscr{L}(\mathscr{H})^d$ be a commuting tuple of operators.

Set

$$\Delta_m^A(\mathbf{T}) := \sum_{0\leqslant k\leqslant m} (-1)^{m-k}\binom{m}{k}\big(T_1^*+\cdots+T_d^*\big)^{m-k}A\big(T_1+\cdots+T_d\big)^k. \tag{3.1}$$

DEFINITION 3.1. Let $\mathbf{T} = (T_1,\cdots,T_d) \in \mathscr{L}(\mathscr{H})^d$ be a commuting tuple of bounded linear operators. $\mathbf{T}$ is said to be an (A,m)-symmetric tuple if

$$\sum_{0\leqslant k\leqslant m} (-1)^{m-k}\binom{m}{k}\big(T_1^*+\cdots+T_d^*\big)^{m-k}A\big(T_1+\cdots+T_d\big)^k = 0$$

or equivalently $\Delta_m^A(\mathbf{T}) = 0$.

REMARK 3.1. When $A = I$, Definition 3.1 coincides with [6, Definition 4.2].

REMARK 3.2. The following are trivial examples of (A,m)-symmetric commuting tuple of operators.

(i) If $A := I$, then $\mathbf{T}$ is an m-symmetric commuting tuple if and only if $\mathbf{T}$ is an (A,m)-symmetric commuting tuple.

(ii) If $A := 0$, any commuting tuple of operators is an (A,m)-symmetric commuting tuple.

EXAMPLE 1. Let $S \in \mathscr{L}(\mathscr{H})$ be an (A,m)-symmetric operator and let $\mathbf{T} = (S,\cdots,S) \in \mathscr{L}(\mathscr{H})^d$. Then $\mathbf{T}$ is an (A,m)-symmetric tuple.

In fact, a simple computation shows that

$$\begin{aligned}&\sum_{0\leqslant k\leqslant m} (-1)^{m-k}\binom{m}{k}\big(T_1^*+\cdots+T_d^*\big)^{m-k}A\big(T_1+\cdots+T_d\big)^k\\ &= d^m \sum_{0\leqslant k\leqslant m} (-1)^{m-k}\binom{m}{k} S^{*m-k}AS^k = 0.\end{aligned}$$

REMARK 3.3. Let $\mathbf{T} = (T_1,\cdots,T_d) \in \mathscr{L}(\mathscr{H})^d$. Then

(i) $\mathbf{T}$ is an $(A,1)$-symmetric tuple if

$$A\big(T_1+\cdots+T_d\big) - \big(T_1^*+\cdots+T_d^*\big)A = 0. \tag{3.2}$$

(ii) $\mathbf{T}$ is an $(A,2)$-symmetric tuple if

$$\big(T_1^*+\cdots+T_d^*\big)^2A-2\big(T_1^*+\cdots+T_d^*\big)A\big(T_1+\cdots+T_d\big)+A\big(T_1+\cdots+T_d\big)^2=0. \tag{3.3}$$

In the following proposition, we give a recursive formula for $\Delta_m^A(\mathbf{T})$.

PROPOSITION 3.1. *Let $\mathbf{T}=(T_1,\cdots,T_d)\in\mathscr{L}(\mathscr{H})^d$ be a commuting tuple of operators. Then the following identity holds*

$$\Delta_{m+1}^A(\mathbf{T})=\Delta_m^A(\mathbf{T})\big(T_1+\cdots+T_d\big)-\big(T_1^*+\cdots+T_d^*\big)\Delta_m^A(\mathbf{T}). \tag{3.4}$$

In particular, if $\mathbf{T}$ is an (A,m)-symmetric commuting tuple of operators, then $\mathbf{T}$ is an (A,n)-symmetric commuting tuple of operators of all $n\geqslant m$.

Proof. By applying (3.1) we observe that

$$\begin{aligned}
&\Delta_{m+1}^A(\mathbf{T})\\
&=\sum_{0\leqslant k\leqslant m+1}(-1)^{m+1-k}\binom{m+1}{k}\big(T_1^*+\cdots+T_d^*\big)^{m+1-k}A\big(T_1+\cdots+T_d\big)^k\\
&=(-1)^{m+1}\big(T_1^*+\cdots+T_d^*\big)^{m+1}A+A\big(T_1+\cdots+T_d\big)^{m+1}\\
&\quad+\sum_{1\leqslant k\leqslant m}(-1)^{m+1-k}\left(\binom{m}{k}+\binom{m}{k-1}\right)\big(T_1^*+\cdots+T_d^*\big)^{m+1-k}A\big(T_1+\cdots+T_d\big)^k\\
&=(-1)^{m+1}\big(T_1^*+\cdots+T_d^*\big)^{m+1}A+A\big(T_1+\cdots+T_d\big)^{m+1}\\
&\quad+\sum_{1\leqslant k\leqslant m}(-1)^{m+1-k}\binom{m}{k}\big(T_1^*+\cdots+T_d^*\big)^{m+1-k}A\big(T_1+\cdots+T_d\big)^k\\
&\quad+\sum_{1\leqslant k\leqslant m}(-1)^{m+1-k}\binom{m}{k-1}\big(T_1^*+\cdots+T_d^*\big)^{m+1-k}A\big(T_1+\cdots+T_d\big)^k\\
&=\Delta_m^A(\mathbf{T})\big(T_1+\cdots+T_d\big)-\big(T_1^*+\cdots+T_d^*\big)\Delta_m^A(\mathbf{T}),
\end{aligned}$$

which completes the derivation of (3.4). Applying the preceding result to an (A,m)-symmetric commuting tuple of operators $\mathbf{T}$, we immediately obtain that $\mathbf{T}$ is an $(A,m+1)$-symmetric commuting tuple of operators. $\square$

REMARK 3.4. If $\mathbf{T}=(T_1,\cdots,T_d)\in\mathscr{L}(\mathscr{H})^d$ is an (A,m)-symmetric tuple of operators, then for every integer $k\geqslant 0$, $\Delta_{m-1}^A(\mathbf{T})\big(T_1+\cdots+T_d\big)^k=\big(T_1^*+\cdots+T_d^*\big)^k\Delta_{m-1}^A(\mathbf{T})$.

PROPOSITION 3.2. *Let $\mathbf{T}=(T_1,\cdots,T_d)\in\mathscr{L}(\mathscr{H})^d$ be an (A,m)-symmetric commuting tuple of operators. Then the operator $\mathbf{S}:=(X^{-1}T_1X,\cdots,X^{-1}T_dX)$ is an (X^*AX,m)-symmetric commuting tuple of operators, where $X\in\mathscr{L}(\mathscr{H})$ is an invertible operator.*

Proof. It is easy to see from (3.1) that

$$\begin{aligned}
&\Delta_m^{X^*AX}(\mathbf{S})\\
&=\sum_{0\leqslant k\leqslant m}(-1)^{m-k}\binom{m}{k}\Big(S_1+\cdots+S_d\Big)^{*m-k}X^*AX\Big(S_1+\cdots+S_d\Big)^k\\
&=\sum_{0\leqslant k\leqslant m}(-1)^{m-k}\binom{m}{k}\Big(\big(X^{-1}T_1X+\cdots+X^{-1}T_dX\big)^{m-k}\Big)^*X^*AX\\
&\quad\Big(X^{-1}T_1X+\cdots+X^{-1}T_dX\Big)^k\\
&=\sum_{0\leqslant k\leqslant m}(-1)^{m-k}\binom{m}{k}\Big(X^{-1}\big(T_1+\cdots+T_d\big)^{m-k}X\Big)^*X^*AXX^{-1}\Big(T_1+\cdots+T_d\Big)^kX\\
&=X^*\Big(\sum_{0\leqslant k\leqslant m}(-1)^{m-k}\binom{m}{k}\big(T_1^*+\cdots+T_d^*\big)^{m-k}A\big(T_1+\cdots+T_d\big)^k\Big)X\\
&=X^*\Delta_m^A(\mathbf{T})X=0.
\end{aligned}$$

This completes the proof of the proposition. $\square$

Recall that the symbol $\{(x-y)^m\}_a$, i.e.,

$$\{(x-y)^m\}_a=\Big\{\sum_{k=0}^m(-1)^{m-k}\binom{m}{k}y^{m-k}x^k\Big\}_a:=\sum_{k=0}^m(-1)^{m-k}\binom{m}{k}y^{m-k}ax^k.$$

Let $\mathbf{T}=(T_1,\cdots,T_d)\in\mathscr{L}(\mathscr{H})^d$ and $A\geqslant 0$. We define $\Big(\{(x-y)^m\}_a\Big)(\mathbf{T},\,A)$ by

$$\begin{aligned}
\Big(\{(x-y)^m\}_a\Big)(\mathbf{T},\,A)&:=\sum_{k=0}^m(-1)^{m-k}\binom{m}{k}y^{m-k}ax^k\Big|_{y=T_1^*+\cdots+T_d^*,\,x=T_1+\cdots+T_d,\,a=A}\\
&=\sum_{k=0}^m(-1)^{m-k}\binom{m}{k}(T_1^*+\cdots+T_d^*)^{m-k}A\,(T_1+\cdots+T_d)^k.
\end{aligned}$$

Then we have $\Big(\Big\{(x-y)^m\Big\}_a\Big)(\mathbf{T},\,A)=\Delta_m^A(\mathbf{T})$. Since, for some constants ξ_k $(k=0,\ldots,m(n-1))$, it holds

$$\big(x^n-y^n\big)^m=\Big((x-y)\big(\sum_{j=0}^{n-1}y^{n-1-j}x^j\big)\Big)^m=\sum_{k=0}^{m(n-1)}\xi_k\,y^{m(n-1)-k}(x-y)^m x^k,$$

we have

$$\left\{\left(x^n-y^n\right)^m\right\}_a=\sum_{k=0}^{m(n-1)}\xi_k\,y^{m(n-1)-k}\left(\{(x-y)^m\}_a\right)x^k$$

and

$$\left(\left\{(x^n-y^n)^m\right\}_a\right)(\mathbf{T},\,A)=\sum_{k=0}^{m(n-1)}\xi_k(T_1^*+\cdots+T_d^*)^{m(n-1)-k}\,\Delta_m^A(\mathbf{T})\,(T_1+\cdots+T_d)^k. \tag{3.5}$$

By (3.5), we have

$$\Delta_m^A((T_1+\cdots+T_d)^n)=\sum_{k=0}^{m(n-1)}\xi_k(T_1^*+\cdots+T_d^*)^{m(n-1)-k}\,\Delta_m^A(\mathbf{T})\,(T_1+\cdots+T_d)^k, \tag{3.6}$$

where ξ_k $(k=0,...,m(n-1))$ are constants. Hence, by (3.6) we have the following proposition.

PROPOSITION 3.3. *If $\mathbf{T}=(T_1,\cdots,T_d)\in\mathcal{L}(\mathcal{H})^d$ be an (A,m)-symmetric commuting tuple of operators, then the operator $(T_1+\cdots+T_d)^n$ is (A,m)-symmetric for any $n\in\mathbb{N}$.*

PROPOSITION 3.4. *Let $\mathbf{T}=(T_1,\cdots,T_d)\in\mathcal{L}(\mathcal{H})^d$ and $\mathbf{S}=(S_1,\cdots,S_d)\in\mathcal{L}(\mathcal{H})^d$ be commuting d-tuples of operators such that $T_jS_k=S_kT_j$ for all $j,k=1,\cdots,d$. If $\mathbf{T}$ is an (A,m)-symmetric commuting tuple and $\mathbf{S}$ is an (A,n)-symmetric commuting tuple, then e^{itR} is an $(A,m+n-1)$-isometric operator, where $R=\sum_{1\leqslant j\leqslant d}(T_j+S_j)$.*

Proof. Since $\mathbf{T}$ is an (A,m)-symmetric commuting tuple and $\mathbf{S}$ is an (A,n)-symmetric commuting tuple , it follows that $\Delta_m^A(\mathbf{T})=0$ and $\Delta_n^A(\mathbf{S})=0$. From which we deduce that $(T_1+\cdots+T_d)$ is an (A,m)-symmetric single operator and $(S_1+\cdots+S_d)$ is an (A,n)-symmetric single operator. By Theorem 2.6 we have for all $t\in\mathbb{R}$, $e^{it\left(T_1+\cdots+T_d\right)}$ is an (A,m)-isometric operator and $e^{it\left(S_1+\cdots+S_d\right)}$ is an (A,n)-isometric operator.

Applying [3, Theorem 3], we obtain that $e^{it\left(T_1+S_1+\cdots+T_d+S_d\right)}$ is an $(A,m+n-1)$-isometric operator. $\square$

THEOREM 3.1. *Let $\left(\mathbf{T}_n=(T_{1n},\cdots,T_{dn})\right)_n$ be a sequence of an (A,m)-symmetric tuple of operators with $A\geqslant 0$ such that $T_{jn}\to T_j$ for each $j=1,\cdots,d$ as $n\to\infty$ in the strong topology of $\mathcal{L}(\mathcal{H})$. Then $\mathbf{T}:=(T_1,\cdots,T_d)$ is an (A,m)-symmetric commuting tuple.*

Proof. Assume that $\left(\mathbf{T}_n=(T_{1n},\cdots,T_{dn})\right)_n$ is a sequence of an (A,m)-symmetric commuting tuple of operators such that $\|T_{jn}-T_j\|\to 0$ as $n\to\infty$ for each $j=1,\cdots,d$.

Set $S_n=T_{1n}+\cdots+T_{dn}$ and $S=T_1+\cdots+T_d$.

It is obvious that $S_n \to S$ and adjoint operation is continuous, $S_n^* \to S^*$ in $\mathscr{L}(\mathscr{H})$. Also, as multiplication is jointly continuous, $S_n^k \to S^k$ and $S_n^{*k} \to S^{*k}$ in $\mathscr{L}(\mathscr{H})$.

Since $\big(\mathbf{T}_n = (T_{1n}, \cdots, T_{dn})\big)_n$ is an (A,m)-symmetric tuple of operators, we get

$$\|\Delta_m^A(\mathbf{T})\| = \|\sum_{0\leqslant j\leqslant m} (-1)^{m-j}\binom{m}{j}\big(T_1^* + \cdots + T_d^*\big)^{m-j} A\big(T_1 + \cdots + T_d\big)^j\|$$

$$= \|\Delta_m^A(\mathbf{T}) - \Delta_m^A(\mathbf{T_n})\|$$

$$= \|\sum_{0\leqslant j\leqslant m} (-1)^{m-j}\binom{m}{j} S_n^{*m-j}AS_n^j - \sum_{0\leqslant j\leqslant m} (-1)^{m-j}\binom{m}{j} S^{*m-j}AS^j\|$$

$$\leqslant \|\sum_{0\leqslant j\leqslant m} (-1)^{m-j}\binom{m}{j} S_n^{*m-j}AS_n^j - \sum_{0\leqslant j\leqslant m} (-1)^{m-j}\binom{m}{j} S_n^{*m-j}AS^j\|$$

$$+\|\sum_{0\leqslant j\leqslant m} (-1)^{m-j}\binom{m}{j} S_n^{*m-j}AS^j - \sum_{0\leqslant j\leqslant m} (-1)^{m-j}\binom{m}{j} S^{*n-j}AS^j\|$$

$$\leqslant \|\sum_{0\leqslant j\leqslant m} (-1)^{m-j}\binom{m}{j} S_n^{*m-j}A\Big(S_n^j - S^j\Big)\|$$

$$+\|\sum_{0\leqslant j\leqslant m} (-1)^{m-j}\binom{m}{j}\Big(S_n^{*m-j} - S^{*m-j}\Big)AS^j\|.$$

Hence we conclude that $\Delta_m^A(\mathbf{T}) = 0$ by taking $n \to \infty$. □

4. Spectral properties of (A,m)-symmetric commuting tuple of operators

For a commuting tuple of operators $\mathbf{T} = (T_1, \cdots, T_d) \in \mathscr{L}(\mathscr{H})^d$, $\sigma_{ja}(\mathbf{T})$ and $\sigma_{jp}(\mathbf{T})$ denote the joint approximate point spectrum and the joint point spectrum of $\mathbf{T}$, that is,

$$\mu = (\mu_1, ..., \mu_d) \in \sigma_{ja}(\mathbf{T}) \iff \exists\,\{x_n\} : \text{unit vctors}; (T_j - \mu_j)x_n \to 0\ (n \to \infty),$$

and

$$\mu = (\mu_1, ..., \mu_d) \in \sigma_{jp}(\mathbf{T}) \iff \exists\, x \neq 0;\ (T_j - \mu_j)x = 0,$$

for all j $(j = 1, \cdots, d)$, respectively.

THEOREM 4.1. *Let $\mathbf{T} = (T_1, \cdots, T_d) \in \mathscr{L}(\mathscr{H})^d$ be an (A,m)-symmetric commuting tuple of operators. If $0 \notin \sigma_{ap}(A)$, then the following statements hold:*

(i) $\sigma_{ja}(\mathbf{T}) \subset \Big\{(\mu_1, \cdots, \mu_d) \in \mathbb{C}^d :\ \mathrm{Im}\Big(\sum_{1\leqslant \mathrm{j}\leqslant \mathrm{d}} \mu_{\mathrm{j}}\Big) = 0.$

(ii) $\sigma_{jp}(\mathbf{T}) \subset \Big\{(\mu_1, \cdots, \mu_d) \in \mathbb{C}^d :\ \mathrm{Im}\Big(\sum_{1\leqslant \mathrm{j}\leqslant \mathrm{d}} \mu_{\mathrm{j}}\Big) = 0\Big\}.$

(iii) Eigenvectors u and v of $\mathbf{T}$ corresponding to two joint eigenvalues $\lambda=(\lambda_1,\cdots,\lambda_d)$ and $\mu=(\mu_1,\cdots,\mu_d)$ respectively such that $\sum_{1\leqslant j\leqslant d}\big(\lambda_j-\mu_j\big)\neq 0$ satisfies

$$\langle Au \mid v\rangle = 0.$$

(iv) If $\lambda=(\lambda_1,\cdots,\lambda_d)$ and $\mu=(\mu_1,\cdots,\mu_d)$ are two joint eigenvalues of $\mathbf{T}$ such that $\sum_{1\leqslant j\leqslant d}\big(\lambda_j-\mu_j\big)\neq 0$. If $\{u_n\}_n$, $\{v_n\}_n$ are two sequences of unit vectors such that

$$(T_j-\lambda_j)u_n \longrightarrow 0 \ \textit{and}\ (T_j-\mu_j)v_n \longrightarrow 0 \ (\textit{as}\ n\longrightarrow +\infty) \quad (j=1,\cdots,d),$$

then we have

$$\langle Au_n \,|\, v_n\rangle \longrightarrow 0 \ (\textit{as}\ n\longrightarrow +\infty).$$

Proof. (ii) and (iii) follow from (i) and (iv), respectively. So we show (i) and (iv).

(i) Let $\mu=(\mu_1,\cdots,\mu_d)\in\sigma_{ja}(\mathbf{T})$. Then there exists a sequence $\{z_n\}_n$ of unit vectors in $\mathscr{H}$ such that

$$(T_j-\mu_j)z_n \longrightarrow 0 \quad (\text{as}\ \ n\longrightarrow\infty).$$

Since $\mathbf{T}$ is an (A,m)-symmetric tuple, it follows that

$$\begin{aligned} 0 &= \langle\Big(\sum_{0\leqslant k\leqslant m}(-1)^{m-k}\binom{m}{k}\big(T_1^*+\cdots+T_d^*\big)^{m-k}A\big(T_1+\cdots+T_d\big)^k\Big)z_n \mid z_n\rangle \\ &= \sum_{0\leqslant k\leqslant m}(-1)^{m-k}\binom{m}{k}\Big(\langle A\big(T_1+\cdots+T_d\big)^k z_n \mid \big(T_1+\cdots+T_d\big)^{m-k}z_n\rangle\Big). \end{aligned}$$

By taking $n\longrightarrow\infty$ we get

$$\sum_{0\leqslant k\leqslant m}(-1)^{m-k}\binom{m}{k}\big(\mu_1+\cdots+\mu_d\big)^k\big(\overline{\mu_1}+\cdots+\overline{\mu_d}\big)^{m-k}\langle Az_n \mid z_n\rangle \longrightarrow 0,$$

or equivalently

$$\Big(\big(\mu_1+\cdots+\mu_d\big)-\big(\overline{\mu_1+\cdots+\mu_d}\big)\Big)^m\langle Az_n \mid z_n\rangle \longrightarrow 0.$$

Since $0\notin\sigma_{ap}(A)$, it follows that $\big(\mu_1+\cdots+\mu_d\big)-\big(\overline{\mu_1}+\cdots+\overline{\mu_d}\big)=0$ and so that

$$\operatorname{Im}\Big(\sum_{1\leqslant \mathrm{j}\leqslant \mathrm{d}}\mu_{\mathrm{j}}\Big)=0.$$

(iv) Let $\lambda = (\lambda_1, \cdots, \lambda_d)$ and $\mu = (\mu_1, \cdots, \mu_d)$ be in $\sigma_{ja}(T)$ and satisfy

$$\sum_{1 \leqslant j \leqslant d} (\lambda_j - \mu_j) \neq 0.$$

Let $\{u_n\}_n, \{v_n\}_n \subset \mathscr{H}$ be sequences of unit vectors such that

$$(T_k - \lambda_k)u_n \longrightarrow 0 \text{ and } (T_k - \mu_k)v_n \longrightarrow 0 \quad (n \to \infty).$$

By repeating the process as in the statement (iii) it holds that

$$\begin{aligned} 0 &= \lim_{n \longrightarrow \infty} \langle \sum_{0 \leqslant k \leqslant m} (-1)^{m-k} \binom{m}{k} \langle A\big(T_1 + \cdots + T_d\big)^k u_n \mid \big(T_1 + \cdots + T_d\big)^{m-k} v_n \rangle \\ &= \Big(\sum_{1 \leqslant j \leqslant d} (\lambda_j - \mu_j) \Big)^m \lim_{n \longrightarrow \infty} \langle Au_n \mid v_n \rangle, \end{aligned}$$

which allows to conclude. □

PROPOSITION 4.1. *Let $\mathbf{T} = (T_1, \cdots, T_d) \in \mathscr{L}(\mathscr{H})^d$ be an (A,m)-symmetric commuting tuple of operators. If $0 \notin \sigma_{ap}(A)$, then the following statements hold.*

(i) If $\mu = (\mu_1, \cdots, \mu_d) \in \sigma_{ja}(\mathbf{T})$, then $\mu_1 + \cdots + \mu_d \in \sigma_{ap}\big(T_1^* + \cdots + T_d^*\big)$.

(ii) If $\mu = (\mu_1, \cdots, \mu_d) \in \sigma_{jp}(\mathbf{T})$, then $\mu_1 + \cdots + \mu_d \in \sigma_p\big(T_1^* + \cdots + T_d^*\big)$.

Proof.

(i) Let $\mu = (\mu_1, \cdots, \mu_d) \in \sigma_{ja}(\mathbf{T})$. Then there exists a sequence $\{z_n\}_n$ of unit vectors in $\mathscr{H}$ such that

$$(T_j - \mu_j)z_n \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty).$$

Since $\mathbf{T}$ is an (A,m)-symmetric tuple it follows that

$$0 = \sum_{0 \leqslant k \leqslant m} (-1)^{m-k} \binom{m}{k} \big(T_1^* + \cdots + T_d^*\big)^{m-k} A \big(T_1 + \cdots + T_d\big)^k z_n.$$

Hence we obtain

$$0 = \lim_{n \longrightarrow \infty} \Big((\mu_1 + \cdots + \mu_d) - \big(T_1^* + \cdots + T_d^*\big) \Big)^m Az_n.$$

As $0 \notin \sigma_{ap}(A)$ we obtain also that

$$0 = \lim_{n \longrightarrow \infty} \Big((\mu_1 + \cdots + \mu_d) - \big(T_1^* + \cdots + T_d^*\big) \Big)^m \frac{Az_n}{\|Az_n\|}.$$

This shows that $\Big((\mu_1 + \cdots + \mu_d) - \big(T_1^* + \cdots + T_d^*\big) \Big)$ is not bounded below. Consequently, $(\mu_1 + \cdots + \mu_d) \in \sigma_{ap}\big(T_1^* + \cdots + T_d^*\big)$. This proves the statement in (i).

(ii) The remaining statement in (ii) also holds by a similar method. $\square$

THEOREM 4.2. *Let* $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{L}(\mathcal{H})^d$ *be* (A,m)*-symmetric commuting tuple and let* $A \geqslant 0$ *be invertible. If* $(z_1, \ldots, z_d) \in \sigma_T(\mathbf{T})$*, then* $z_1 + \cdots + z_d \in \mathbb{R}$*, where* $\sigma_T(\mathbf{T})$ *is the Taylor spectrum of* $\mathbf{T}$.

Proof. Let $S = T_1 + \cdots + T_d$. Then by the Definition 3.1, S is (A,m)-symmetric. Hence by Theorem 2.1 we have $\sigma(S) \subset \mathbb{R}$. Let $f(x_1, \ldots, x_d) = x_1 + \cdots + x_d$. Then by the spectral mapping theorem of the Taylor spectrum, we have $f(\sigma_{\mathbf{T}}(\mathbf{T})) = \sigma_{\mathbf{T}}(f(\mathbf{T})) = \sigma(S) \subset \mathbb{R}$. It completes the proof. $\square$

5. (A,m)-expansive symmetric tuple

According to the paper of Jung, Kim, Ko and Lee [14], we introduce (A,m)-expansive symmetric tuples.

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be (A,m)-expansive if $B_m^A(T) \leqslant 0$ for some positive integer m. We refer the interested reader to [14] for more details.

In the following definition, we introduce de notion of (A,m)-expansive symmetric for tuple of commuting operators.

DEFINITION 5.1. Let $\mathbf{T} = (T_1, \cdots, T_d) \in \mathcal{L}(\mathcal{H})^d$ be a commuting tuple of bounded linear operators and $A \geqslant 0$. $\mathbf{T}$ is said to be an (A,m)-expansive symmetric tuple if

$$\Delta_m^A(\mathbf{T}) = \sum_{0 \leqslant k \leqslant m} (-1)^{m-k} \binom{m}{k} \big(T_1^* + \cdots + T_d^*\big)^{m-k} A \big(T_1 + \cdots + T_d\big)^k \leqslant 0.$$

By equation (3.4), it does not hold that if $\mathbf{T}$ is (A,m)-expansive symmetric tuple, then T is $(A,m+1)$-expansive symmetric tuple. We have the following proposition.

PROPOSITION 5.1. *Let* $\mathbf{T} = (T_1, \cdots, T_d) \in \mathcal{L}(\mathcal{H})^d$ *be an* (A,m)*-expansive symmetric commuting tuple of operators and let* $X \in \mathcal{L}(\mathcal{H})$ *be an invertible operator. Then the operator* $\mathbf{S} := (X^{-1}T_1X, \cdots, X^{-1}T_dX)$ *is an* (X^*AX, m)*-expansive symmetric commuting tuple of operators.*

Proof. By the proof of Proposition 3.2, since it holds

$$\Delta_m^{X^*AX}(\mathbf{S}) = X^* \Delta_m^A(\mathbf{T}) X$$

and $\Delta_m^A(\mathbf{T}) \leqslant 0$, we have $\Delta_m^{X^*AX}(\mathbf{S}) \leqslant 0$ and it completes the proof. $\square$

THEOREM 5.1. *Let $\mathbf{T}=(T_1,\cdots,T_d)\in\mathscr{L}(\mathscr{H})^d$ be an (A,m)-expansive symmetric tuple and $0\notin\sigma_{ap}(A)$. The following statements hold:*

(i) If $m\in\{2k+1,\ k=0,1,\cdots\}$, then

$$\sigma_{ja}(\mathbf{T})\subset\Big\{(\mu_1,\cdots,\mu_d)\in\mathbb{C}^d:\ \mathrm{Im}\Big(\sum_{1\leqslant j\leqslant d}\mu_j\Big)=0\Big\}$$
$$=\{(\mu_1,\cdots,\mu_d)\in\mathbb{C}^d:\ \sum_{1\leqslant j\leqslant d}\mu_j\in\mathbb{R}\}.$$

(ii) If $m\in\{4k,\ k=1,2,\cdots\}$, then

$$\sigma_{ja}(\mathbf{T})\subset\Big\{(\mu_1,\cdots,\mu_d)\in\mathbb{C}^d:\ \mathrm{Im}\Big(\sum_{1\leqslant j\leqslant d}\mu_j\Big)=0\Big\}$$
$$=\{(\mu_1,\cdots,\mu_d)\in\mathbb{C}^d:\ \sum_{1\leqslant j\leqslant d}\mu_j\in\mathbb{R}\}.$$

(iii) If $m\in\{4k+2,\ k=0,1,\cdots\}$, then

$$\sigma_{ja}(\mathbf{T})\subset\{(\mu_1,\cdots,\mu_d)\in\mathbb{C}^d:\ \sum_{1\leqslant j\leqslant d}\mu_j\in\mathbb{C}\}.$$

Proof. Let $\mu=(\mu_1,\cdots,\mu_d)\in\sigma_{ja}(\mathbf{T})$. Then there exists a sequence $\{z_n\}_n$ of unit vectors in $\mathscr{H}$ such that

$$(T_j-\mu_j)z_n\longrightarrow 0\quad(\text{as } n\longrightarrow\infty).$$

Since $\mathbf{T}$ is an (A,m)-expansive symmetric tuple, it follows that

$$0\geqslant\langle\Big(\sum_{0\leqslant k\leqslant m}(-1)^{m-k}\binom{m}{k}\big(T_1^*+\cdots+T_d^*\big)^{m-k}A\big(T_1+\cdots+T_d\big)^k\Big)z_n\mid z_n\rangle$$
$$=\sum_{0\leqslant k\leqslant m}(-1)^{m-k}\binom{m}{k}\Big(\langle A\big(T_1+\cdots+T_d\big)^k z_n\mid\big(T_1+\cdots+T_d\big)^{m-k}z_n\rangle\Big).$$

Therefore we get

$$\lim_{n\to\infty}\sum_{0\leqslant k\leqslant m}(-1)^{m-k}\binom{m}{k}\Big(\mu_1+\cdots+\mu_d\Big)^k\Big(\overline{\mu_1}+\cdots+\overline{\mu_d}\Big)^{m-k}\langle Az_n\mid z_n\rangle\ \leqslant 0,$$

or equivalently

$$\lim_{n\to\infty}\Big(\big(\mu_1+\cdots+\mu_d\big)-\big(\overline{\mu_1}+\cdots+\overline{\mu_d}\big)\Big)^m\langle Az_n\mid z_n\rangle$$
$$=(2i)^m\Big(\mathrm{Im}\Big(\sum_{1\leqslant j\leqslant d}\mu_j\Big)\Big)^m\lim_{n\to\infty}\langle Az_n\mid z_n\rangle\ \leqslant\ 0.$$

Since $A\geqslant 0$, $0\notin\sigma_{ap}(A)$ we have:

(i) If m is odd, it follows that

$$(2)^m i\mathrm{Im}\Big(\sum_{1\leqslant j\leqslant d}\mu_j\Big)^m \lim_{n\to\infty}\langle Az_n \mid z_n\rangle \leqslant 0.$$

Hence we get

$$\mathrm{Im}\Big(\sum_{1\leqslant j\leqslant d}\mu_j\Big) = 0.$$

(ii) Since $m=4k$, by the similar calculation we have

$$(2)^m\Big(\mathrm{Im}\Big(\sum_{1\leqslant j\leqslant d}\mu_j\Big)\Big)^m \lim_{n\to\infty}\langle Az_n \mid z_n\rangle \leqslant 0.$$

Hence we obtain

$$\sigma_{ja}(\mathbf{T})\subset\Big\{(\mu_1,\cdots,\mu_d)\in\mathbb{C}^d:\ \mathrm{Im}\Big(\sum_{1\leqslant j\leqslant d}\mu_j\Big)=0\Big\}.$$

(iii) If $m=4k+2$,we can obtain

$$-2^m\Big(\mathrm{Im}\Big(\sum_{1\leqslant j\leqslant d}\mu_j\Big)\Big)^m \lim_{n\to\infty}\langle Az_n \mid z_n\rangle \leqslant 0.$$

This completes the proof. □

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