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ESTIMATIONS OF THE WEIGHTED POWER MEAN BY THE HERON MEAN AND RELATED INEQUALITIES FOR DETERMINANTS AND TRACES

MASATOSHI ITO

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Abstract. For positive real numbers a and b, the weighted power mean $P_{t,q}(a,b)$ and the weighted Heron mean $K_{t,q}(a,b)$ are defined as follows: For $t \in [0,1]$ and $q \in \mathbb{R}$, $P_{t,q}(a,b) = \{(1-t)a^q + tb^q\}^{\frac{1}{q}}$ and $K_{t,q}(a,b) = (1-q)a^{1-t}b^t + q\{(1-t)a + tb\}$. These means generalize the arithmetic and geometric ones.

In this paper, as a generalization of Wu and Debnath's result on non-weighted means (the case $t = \frac{1}{2}$), we get estimations of the weighted power mean by the weighted Heron mean. In other words, we obtain the greatest value $\alpha_1 = \alpha_1(t,r)$ and the least value $\alpha_2 = \alpha_2(t,r)$ such that the double inequality $K_{t,\alpha_1}(a,b) < P_{t,r}(a,b) < K_{t,\alpha_2}(a,b)$ holds for $t \in (0,1)$ and $r \in \mathbb{R}$. We can also obtain the results for bounded linear operators on a Hilbert space. Moreover, our main results lead some determinant and trace inequalities of matrices.

1. Introduction

For two positive real numbers a and b, the arithmetic, geometric and harmonic means are defined by $\frac{a+b}{2}$, $\sqrt{ab}$ and $\frac{2ab}{a+b}$, respectively. It is well known that these means have their weighted version as follows: For $t \in [0,1]$,

$$\begin{aligned} A_t(a,b) &= (1-t)a + tb \quad \text{(arithmetic mean)},\\ G_t(a,b) &= a^{1-t}b^t \quad \text{(geometric mean)},\\ H_t(a,b) &= \{(1-t)a^{-1} + tb^{-1}\}^{-1} \quad \text{(harmonic mean)}. \end{aligned}$$

If the weight t is equal to $\frac{1}{2}$, then the weighted means coincide with the original (non-weighted) ones, and then we abbreviate the weight t as $A(a,b) = A_{\frac{1}{2}}(a,b)$.

These means are used in various branches, and also their generalizations are known as follows: For $t \in [0,1]$ and $q \in \mathbb{R}$,

$$P_{t,q}(a,b) = \begin{cases} \{(1-t)a^q + tb^q\}^{\frac{1}{q}} & \text{if } q \neq 0,\\ a^{1-t}b^t & \text{if } q = 0, \end{cases} \quad \text{(power mean)},$$

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$$K_{t,q}(a,b) = (1-q)a^{1-t}b^t + q\{(1-t)a+tb\} \quad \text{(Heron mean)}.$$

We remark that we use the notations for non-weighted means that $P_q(a,b) = P_{\frac{1}{2},q}(a,b)$ and $K_q(a,b) = K_{\frac{1}{2},q}(a,b)$.

The weighted means have the properties that $A_t(a,b) = A_{1-t}(b,a)$, $G_t(a,b) = G_{1-t}(b,a)$ and so on. In particular, non-weighted means are symmetric, that is, $A(a,b) = A(b,a)$, $G(a,b) = G(b,a)$ and so on. We remark that

$$\begin{aligned} &H_t(a,b) \leqslant G_t(a,b) \leqslant A_t(a,b), \\ &A_t(a,b) = P_{t,1}(a,b) = K_{t,1}(a,b), \\ &G_t(a,b) = P_{t,0}(a,b) = K_{t,0}(a,b), \\ &H_t(a,b) = P_{t,-1}(a,b), \end{aligned}$$

and also $P_{t,q}(a,b)$ and $K_{t,q}(a,b)$ are monotone increasing on $q \in \mathbb{R}$. The inequality $G_t(a,b) \leqslant A_t(a,b)$ is sometimes called Young's inequality.

It is also known that the non-weighted arithmetic, geometric and harmonic means have many kinds of generalizations besides the power mean and the Heron mean. For example, for $q \in \mathbb{R}$,

$$\begin{aligned} J_q(a,b) &= \frac{q}{q+1}\frac{a^{q+1}-b^{q+1}}{a^q-b^q} \ (q \neq 0,-1) \text{ (power difference mean)}, \\ L_q(a,b) &= \frac{a^{q+1}+b^{q+1}}{a^q+b^q} \quad \text{(Lehmer mean)}. \end{aligned}$$

We note that $J_0(a,b)$ and $J_{-1}(a,b)$ can be defined as the limit, and also $A(a,b) = J_1(a,b) = L_0(a,b)$, $G(a,b) = J_{\frac{-1}{2}}(a,b) = L_{\frac{-1}{2}}(a,b)$ and $H(a,b) = J_{-2}(a,b) = L_{-1}(a,b)$ hold.

Many researchers investigate estimations of these means. For example, recently, we have obtained the results on estimations of several means by the Heron mean. The results for the power difference mean are in [13, 4], and the results for the Lehmer mean are in [5]. For the power mean, Janous [6], Wu and Debnath [12] obtained the following Theorem 1.A.

THEOREM 1.A ([6, 12]) *Let $a,b > 0$ with $a \neq b$.*

(i) *If $0 < r < \frac{1}{2}$ or $1 < r$, then*

$$K_{\left(\frac{1}{2}\right)^{\frac{1}{r}-1}}(a,b) < P_r(a,b) < K_r(a,b).$$

(ii) *If $\frac{1}{2} < r < 1$. Then*

$$K_r(a,b) < P_r(a,b) < K_{\left(\frac{1}{2}\right)^{\frac{1}{r}-1}}(a,b).$$

(iii) *If $r < 0$. Then*

$$K_r(a,b) < P_r(a,b) < K_0(a,b) = G(a,b).$$

The given parameters of $K_\alpha(a,b)$ in each case are best possible.

We remark that Janous [6] has shown Theorem 1.A for $0<r<1$ as the results on estimations of the generalized Heronian mean $\frac{a+w\sqrt{ab}+b}{w+2}$ for $w \geqslant 0$, and also Wu and Debnath [12] got Theorem 1.A as the results on upper and lower bounds of $\frac{P_r(a,b)-G(a,b)}{A(a,b)-G(a,b)}$.

On the other hand, Kittaneh and Manasrah researched improved and reversed Young's inequalities in [8, 9]. As a generalization of their results in [8, 9], for $a,b>0$ with $a \neq b$, Alzer, da Fonseca and Kovačec [1] obtained the inequality

$$\left(\frac{\nu}{\mu}\right)^{\lambda} \leqslant \frac{A_\nu(a,b)^\lambda - G_\nu(a,b)^\lambda}{A_\mu(a,b)^\lambda - G_\mu(a,b)^\lambda} \leqslant \left(\frac{1-\nu}{1-\mu}\right)^{\lambda}, \tag{1.1}$$

where $\lambda \geqslant 1$ and $0<\nu \leqslant \mu <1$. Moreover, Khosravi [7] obtained a generalization of (1.1) of the case $\lambda = 1$, that is,

$$\frac{\nu}{\mu} \leqslant \frac{A_\nu(a,b)-P_{\nu,r}(a,b)}{A_\mu(a,b)-P_{\mu,r}(a,b)} \leqslant \frac{1-\nu}{1-\mu}, \tag{1.2}$$

where $0<\nu\leqslant\mu<1$ and $r\in\mathbb{R}$ with $r\neq 1$. By using (1.1) and (1.2), they obtained some matrix (operator) inequalities, determinant inequalities and trace inequalities in [1, 7].

In this paper, as an extension of Theorem 1.A, we obtain estimations of the weighted power mean by the weighed Heron mean. Our main results immediately lead the results for bounded linear operators on a Hilbert space. Moreover, related to the results in [1, 7], we get some determinant and trace inequalities of matrices.

2. Lemmas

We prepare two lemmas in order to prove our main results. In what follows, we define that

$$\beta(t,r) = \frac{tr}{1-t}\left\{\frac{t(1-2r)}{t-r}\right\}^{\frac{1}{r}-2} \quad \text{and} \quad \widehat{\beta}(t,r) = \min\{\beta(t,r),1\} \tag{2.1}$$

for $t\in(0,1)$ and $r\in\mathbb{R}$ with $r\neq 0,\frac{1}{2},t$.

LEMMA 2.1 *Let $t\in(0,1)$ and $r\in\mathbb{R}$ with $r\neq 0,\frac{1}{2}$. Let $\beta(t,r)$ as in (2.1).*

(i) *If $0<r<t<\frac{1}{2}$, then $r<\beta(t,r)$ holds.*

(ii) *If $r<0<t<\frac{1}{2}$, then $\beta(t,r)<r$ holds.*

(iii) *If $\frac{1}{2}<t<r<1$, then $\beta(t,r)<r$ holds.*

(iv) *If $\frac{1}{2}<t<1<r$, then $r<\beta(t,r)$ holds.*

Proof. We give a proof of (i) and (ii). (iii) and (iv) are shown by the same way. Let

$$f(r) = r\log t - r\log(1-t) + (1-2r)\log(1-2r) - (1-2r)\log(t-r) + (1-2r)\log t.$$

for $r < t < \frac{1}{2}$. Then $r < \beta(t,r)$ holds if and only if $f(r) > 0$ holds. By the derivative calculation, we have

$$f'(r) = 2\log(t-r) - 2\log(1-2r) - \frac{2t-1}{t-r} - \log t(1-t) \quad \text{and}$$
$$f''(r) = \frac{(2t-1)^2}{(1-2r)(t-r)^2} > 0 \quad \text{for } r < t < \tfrac{1}{2}.$$

Since $f'(r)$ is increasing for $r < t$ and $\lim_{r\to-\infty} f'(r) = -\log 4t(1-t) > 0$, $f'(r) > 0$ holds for $r < t$, that is, $f(r)$ is increasing for $r < t$. Therefore we get that $f(r) > 0$ holds for $0 < r < t$ and $f(r) < 0$ holds for $r < 0$ since $f(0) = 0$ holds, so that we have the desired results. $\square$

LEMMA 2.2 *Let* $t, r \in (0,1)$ *with* $r \neq \frac{1}{2}$.

(i) *If* $t \leqslant r \leqslant 1-t$, *then* $t^{\frac{1}{r}-1} < r < (1-t)^{\frac{1}{r}-1}$ *holds.*

(ii) *If* $1-t \leqslant r \leqslant t$, *then* $(1-t)^{\frac{1}{r}-1} < r < t^{\frac{1}{r}-1}$ *holds.*

Proof. Firstly, we show that

$$r < (1-r)^{\frac{1}{r}-1} \text{ for } r \in (0,\tfrac{1}{2}) \quad \text{and} \quad r > (1-r)^{\frac{1}{r}-1} \text{ for } r \in (\tfrac{1}{2},1). \tag{2.2}$$

Put $f(r) = (1-r)\log(1-r) - r\log r$. Then $f'(r) = -\log r(1-r) - 2$, so that $f'(r) = 0$ if and only if $r = \frac{1}{2}(1 \pm \sqrt{1-4e^{-2}})$. Since $\lim_{r\to+0} f(r) = f(\frac{1}{2}) = \lim_{r\to1-0} f(r) = 0$, we get that $f(r) > 0$ holds for $r \in (0,\frac{1}{2})$ and $f(r) < 0$ holds for $r \in (\frac{1}{2},1)$, that is, we have (2.2).

Now we can prove (i) by (2.2) as follows: If $r \in (0,\frac{1}{2})$, then $\frac{1}{r} - 1 > 1$ holds and we have

$$t^{\frac{1}{r}-1} < t \leqslant r < (1-r)^{\frac{1}{r}-1} \leqslant (1-t)^{\frac{1}{r}-1}.$$

If $r \in (\frac{1}{2},1)$, then $0 < \frac{1}{r} - 1 < 1$ holds and we have

$$t^{\frac{1}{r}-1} \leqslant (1-r)^{\frac{1}{r}-1} < r \leqslant 1-t < (1-t)^{\frac{1}{r}-1}.$$

Therefore we obtain the desired result. (ii) is easily shown by putting $t_1 = 1-t$ in (i). $\square$

3. Main results

In this section, we obtain estimations of the weighted power mean of two positive real numbers by the weighted Heron mean.

THEOREM 3.1 *Let $t,r \in (0,1)$. Let $\beta(t,r)$ and $\widehat{\beta}(t,r)$ as in (2.1). For all $a,b > 0$ with $a \neq b$, we have the following.*

(i) *If $t \leqslant r \leqslant 1-t$, then*

$$K_{t,t^{\frac{1}{r}-1}}(a,b) < P_{t,r}(a,b) < K_{t,(1-t)^{\frac{1}{r}-1}}(a,b).$$

(ii) *If $1-t \leqslant r \leqslant t$, then*

$$K_{t,(1-t)^{\frac{1}{r}-1}}(a,b) < P_{t,r}(a,b) < K_{t,t^{\frac{1}{r}-1}}(a,b).$$

(iii) *If $r < t \leqslant 1-t$, then*

$$K_{t,t^{\frac{1}{r}-1}}(a,b) < P_{t,r}(a,b) < K_{t,\widehat{\beta}(t,r)}(a,b).$$

(iv) *If $r < 1-t \leqslant t$, then*

$$K_{t,(1-t)^{\frac{1}{r}-1}}(a,b) < P_{t,r}(a,b) < K_{t,\widehat{\beta}(1-t,r)}(a,b).$$

(v) *If $t \leqslant 1-t < r$, then*

$$K_{t,\beta(1-t,r)}(a,b) < P_{t,r}(a,b) < K_{t,(1-t)^{\frac{1}{r}-1}}(a,b).$$

(vi) *If $1-t \leqslant t < r$, then*

$$K_{t,\beta(t,r)}(a,b) < P_{t,r}(a,b) < K_{t,t^{\frac{1}{r}-1}}(a,b).$$

The given parameters of $K_{t,\alpha}(a,b)$ in each case are best possible on α except the parts $\alpha = \beta(\cdot,r)$ and $\alpha = \widehat{\beta}(\cdot,r)$.

THEOREM 3.2 *Let $\beta(t,r)$ as in (2.1). For all $a,b > 0$ with $a \neq b$, we have the following.*

(i) *If $t \in (0,\frac{1}{2}]$ and $r > 1$, then*

$$K_{t,(1-t)^{\frac{1}{r}-1}}(a,b) < P_{t,r}(a,b) < K_{t,\beta(1-t,r)}(a,b).$$

(ii) *If $t \in [\frac{1}{2},1)$ and $r > 1$, then*

$$K_{t,t^{\frac{1}{r}-1}}(a,b) < P_{t,r}(a,b) < K_{t,\beta(t,r)}(a,b).$$

(iii) *If $t \in (0, \frac{1}{2}]$ and $r < 0$, then*

$$K_{t,\beta(t,r)}(a,b) < P_{t,r}(a,b) < K_{t,0}(a,b) = G_t(a,b).$$

(iv) *If $t \in [\frac{1}{2}, 1)$ and $r < 0$, then*

$$K_{t,\beta(1-t,r)}(a,b) < P_{t,r}(a,b) < K_{t,0}(a,b) = G_t(a,b).$$

The given parameters of $K_{t,\alpha}(a,b)$ in each case are best possible on α except the parts $\alpha = \beta(\cdot, r)$.

Theorems 3.1 and 3.2 imply Theorem 1.A by putting $t = \frac{1}{2}$. We remark that the best possibility of the parts $\alpha = \beta(\frac{1}{2}, r) = r$ is also shown by scrutinizing the proofs of Theorems 3.1 and 3.2.

Proof of Theorem 3.1. We have only to consider the case $(a,b) = (1,x)$ with $x \neq 1$ by easy replacement. Let

$$\begin{aligned} f_t(x) &= P_{t,r}(1,x) - K_{t,\alpha}(1,x) \\ &= \{(1-t)+tx^r\}^{\frac{1}{r}} - (1-\alpha)x^t - \alpha\{(1-t)+tx\}. \end{aligned} \tag{3.1}$$

Now we discuss upper and lower bounds of α to hold the inequalities $K_{t,\alpha}(1,x) < P_{t,r}(1,x)$ and $P_{t,r}(1,x) < K_{t,\alpha}(1,x)$, that is, $f_t(x) > 0$ and $f_t(x) < 0$ for all $x > 0$. Let

$$\begin{aligned} g_t(x) =& \{(1-t)+tx^r\}^{\frac{1}{r}-1}x^{r-t} - (1-\alpha) - \alpha x^{1-t}, \\ h_t(x) =& t(1-r)\{(1-t)+tx^r\}^{\frac{1}{r}-2}x^{2r-1} \\ &+ (r-t)\{(1-t)+tx^r\}^{\frac{1}{r}-1}x^{r-1} - \alpha(1-t) \quad \text{and} \\ k_t(x) =& t(r-1+t)x^r - (1-t)(r-t). \end{aligned} \tag{3.2}$$

Then we have

$$\begin{aligned} f_t'(x) &= tx^{t-1}g_t(x), \\ g_t'(x) &= x^{-t}h_t(x) \quad \text{and} \\ h_t'(x) &= (1-t)(1-r)\{(1-t)+tx^r\}^{\frac{1}{r}-3}x^{r-2}k_t(x) \end{aligned} \tag{3.3}$$

since

$$\begin{aligned} f_t'(x) &= tx^{r-1}\{(1-t)+tx^r\}^{\frac{1}{r}-1} - (1-\alpha)tx^{t-1} - \alpha t \\ &= tx^{t-1}\left[\{(1-t)+tx^r\}^{\frac{1}{r}-1}x^{r-t} - (1-\alpha) - \alpha x^{1-t}\right], \end{aligned}$$

$$\begin{aligned} g_t'(x) =& t(1-r)\{(1-t)+tx^r\}^{\frac{1}{r}-2}x^{2r-1-t} \\ &+ (r-t)\{(1-t)+tx^r\}^{\frac{1}{r}-1}x^{r-t-1} - \alpha(1-t)x^{-t} \end{aligned}$$

$$=x^{-t}\left[t(1-r)\{(1-t)+tx^r\}^{\frac{1}{r}-2}x^{2r-1}\right.$$
$$\left.+(r-t)\{(1-t)+tx^r\}^{\frac{1}{r}-1}x^{r-1}-\alpha(1-t)\right]$$

and

$$\begin{aligned}h_t'(x)=&\,t^2(1-r)(1-2r)\{(1-t)+tx^r\}^{\frac{1}{r}-3}x^{3r-2}\\&+t(1-r)(2r-1)\{(1-t)+tx^r\}^{\frac{1}{r}-2}x^{2r-2}\\&+t(r-t)(1-r)\{(1-t)+tx^r\}^{\frac{1}{r}-2}x^{2r-2}\\&+(r-t)(r-1)\{(1-t)+tx^r\}^{\frac{1}{r}-1}x^{r-2}\\=&(1-r)\{(1-t)+tx^r\}^{\frac{1}{r}-3}x^{r-2}\\&\times\left[(1-2r)t^2x^{2r}+(3r-1-t)\{(1-t)+tx^r\}tx^r-(r-t)\{(1-t)+tx^r\}^2\right]\\=&(1-r)\{(1-t)+tx^r\}^{\frac{1}{r}-3}x^{r-2}\\&\times\left[tx^r-\{(1-t)+tx^r\}\right]\left[(1-2r)tx^r+(r-t)\{(1-t)+tx^r\}\right]\\=&(1-t)(1-r)\{(1-t)+tx^r\}^{\frac{1}{r}-3}x^{r-2}\left[t(r-1+t)x^r-(1-t)(r-t)\right].\end{aligned}$$

Proof of (i). We may except the case $r=t=\frac{1}{2}$ since $P_{\frac{1}{2},\frac{1}{2}}(1,x)=K_{\frac{1}{2},\frac{1}{2}}(1,x)$ holds. Firstly, we consider the case $\alpha\leqslant r$.

(i-a) The case $\alpha\leqslant r$ and $0<x<1$. If $t\leqslant r\leqslant 1-t$ holds, then $h_t'(x)<0$ holds for $0<x\leqslant 1$, that is,

$$h_t(x)\text{ is decreasing for }0<x\leqslant 1\tag{3.4}$$

by (3.2) and (3.3). Since $h_t(1)=(r-\alpha)(1-t)\geqslant 0$, (3.4) implies that $g_t'(x)=x^{-t}h_t(x)>0$ holds for $0<x<1$, that is,

$$g_t(x)\text{ is increasing for }0<x\leqslant 1.$$

Since $g_t(1)=0$, $f_t'(x)=tx^{t-1}g_t(x)<0$ holds for $0<x<1$, that is,

$$f_t(x)\text{ is decreasing for }0<x\leqslant 1.$$

Therefore, since $f_t(1)=0$, we have

$$f_t(x)>0,\text{ that is, }K_{t,\alpha}(1,x)<P_{t,r}(1,x)\text{ for }0<x<1.\tag{3.5}$$

(i-b) The case $\alpha\leqslant r$ and $x>1$. Noting that $K_{t,\alpha}(1,x)=xK_{1-t,\alpha}(1,x^{-1})$ and $P_{t,r}(1,x)=xP_{1-t,r}(1,x^{-1})$, we consider $f_{t_1}(y)$ for $y=x^{-1}\in(0,1)$ and $t_1=1-t$.

If $1-t_1\leqslant r\leqslant t_1$ holds, then $h_{t_1}'(y)>0$ holds for $0<y\leqslant 1$, that is,

$$h_{t_1}(y)\text{ is increasing for }0<y\leqslant 1$$

by (3.2) and (3.3). If $\alpha<r$, then there exists a $\delta_1\in(0,1)$ such that $h_{t_1}(\delta_1)=0$ since

$$\begin{aligned}h_{t_1}(y)=&\{(1-t_1)y^{-r}+t_1\}^{\frac{1-r}{r}}\left[t_1(1-r)\{(1-t_1)y^{-r}+t_1\}^{-1}+(r-t_1)\right]-\alpha(1-t_1)\\&\to-\infty\quad(y\to+0)\end{aligned}$$

and $h_{t_1}(1) = (r-\alpha)(1-t_1) > 0$. This ensures that $g'_{t_1}(y) < 0$ for $0 < y < \delta_1$ and $g'_{t_1}(y) > 0$ for $\delta_1 < y < 1$ hold, that is,

$$g_{t_1}(y) \text{ is decreasing for } 0 < y < \delta_1 \text{ and increasing for } \delta_1 < y < 1.$$

Then there exists a $\delta_2 \in (0,\delta_1)$ such that $g_{t_1}(\delta_2) = 0$ since $\lim_{y \to +0} g_{t_1}(y) = \infty$ holds and $g_{t_1}(1) = 0$ assures that $g_{t_1}(\delta_1) < 0$. So $f'_{t_1}(y) > 0$ holds for $0 < y < \delta_2$ and $f'_{t_1}(y) < 0$ holds for $\delta_2 < y < 1$ hold, that is,

$$f_{t_1}(y) \text{ is increasing for } 0 < y < \delta_2 \text{ and decreasing for } \delta_2 < y < 1.$$

If $\alpha \leqslant (1-t_1)^{\frac{1}{r}-1}$, then $f_{t_1}(0) \geqslant 0$, so that $f_{t_1}(y) > 0$ holds for $0 < y < 1$ since $f_{t_1}(1) = 0$.

If $\alpha = r$, then $f_{t_1}(y) < 0$ for $0 < y < 1$ by the similar argument. We remark that $(1-t_1)^{\frac{1}{r}-1} < r = \alpha$ for $1-t_1 \leqslant r \leqslant t_1$ by (ii) in Lemma 2.2.

Therefore we have $K_{t_1,\alpha}(1,y) < P_{t_1,r}(1,y)$ for $0 < y < 1$ if $\alpha \leqslant (1-t_1)^{\frac{1}{r}-1}$, that is,

$$K_{t,\alpha}(1,x) < P_{t,r}(1,x) \text{ for } x > 1 \text{ holds if } \alpha \leqslant t^{\frac{1}{r}-1}. \tag{3.6}$$

Hence, by (3.5) and (3.6), we get $K_{t,\alpha}(1,x) < P_{t,r}(1,x)$ for all $x > 0$ with $x \neq 1$ if $\alpha \leqslant t^{\frac{1}{r}-1}$. This argument also proves the best possibility of α since $K_{t,\alpha}(1,x) < P_{t,r}(1,x)$ or $P_{t,r}(1,x) < K_{t,\alpha}(1,x)$ does not always hold for $x > 0$ with $x \neq 1$ if $t^{\frac{1}{r}-1} < \alpha \leqslant r$.

Next we consider the case $r \leqslant \alpha$. By the similar way to (i-b), we obtain that $f_t(x) < 0$, that is, $P_{t,r}(1,x) < K_{t,\alpha}(1,x)$ holds for all $0 < x < 1$ if $\alpha \geqslant (1-t)^{\frac{1}{r}-1}$. By applying the similar way to (i-a) for $f_{t_1}(y)$ as in (i-b), we obtain that $f_{t_1}(y) < 0$ holds for $0 < y < 1$, that is, $P_{t,r}(1,x) < K_{t,\alpha}(1,x)$ holds for all $x > 1$. Hence we get $P_{t,r}(1,x) < K_{t,\alpha}(1,x)$ holds for all $x > 0$ with $x \neq 1$ if $\alpha \geqslant (1-t)^{\frac{1}{r}-1}$. We also get the best possibility of α, that is, $P_{t,r}(1,x) < K_{t,\alpha}(1,x)$ or $P_{t,r}(1,x) < K_{t,\alpha}(1,x)$ does not always hold for $x > 0$ if $r \leqslant \alpha < (1-t)^{\frac{1}{r}-1}$.

Proof of (iii). Firstly, we consider the case $\alpha < r$. Let $\delta_0 = \left(\frac{(1-t)(t-r)}{t(1-t-r)}\right)^{\frac{1}{r}}$. We remark that $0 < \delta_0 \leqslant 1$ (resp. $\delta_0 \geqslant 1$) holds for $t, r \in (0,1)$ and $r < t \leqslant 1-t$ (resp. $r < 1-t \leqslant t$).

(iii-a) The case $\alpha < r$ and $0 < x < 1$. If $r < t \leqslant 1-t$ holds, then $h'_t(x) > 0$ holds for $0 < x < \delta_0$ and $h'_t(x) < 0$ holds for $\delta_0 < x < 1$, that is,

$$h_t(x) \text{ is increasing for } 0 < x < \delta_0 \text{ and decreasing for } \delta_0 < x < 1$$

by (3.2) and (3.3). Then there exists a $\delta_1 \in (0,\delta_0)$ such that $h_t(\delta_1) = 0$ since $\lim_{x \to +0} h_t(x) = -\infty$ and $h_t(1) = (r-\alpha)(1-t) \geqslant 0$. This ensures that $g'_t(x) < 0$ for $0 < x < \delta_1$ and $g'_t(x) > 0$ for $\delta_1 < x < 1$, that is,

$$g_t(x) \text{ is decreasing for } 0 < x < \delta_1 \text{ and increasing for } \delta_1 < x < 1.$$

Then there exists a $\delta_2 \in (0,\delta_1)$ such that $g_t(\delta_2) = 0$ since $\lim_{x\to+0} g_t(x) = \infty$ and $g_t(1) = 0$. So $f_t'(x) > 0$ holds for $0 < x < \delta_2$ and $f_t'(x) < 0$ holds for $\delta_2 < x < 1$ hold, that is,

$$f_t(x) \text{ is increasing for } 0 < x < \delta_2 \text{ and decreasing for } \delta_2 < x < 1.$$

If $\alpha \leqslant (1-t)^{\frac{1}{r}-1}$, then $f_t(0) > 0$, so that $f_t(x) > 0$ holds for $0 < x < 1$ since $f_t(1) = 0$.

Therefore we have

$$K_{t,\alpha}(1,x) < P_{t,r}(1,x) \text{ for } 0 < x < 1 \text{ if } \alpha \leqslant (1-t)^{\frac{1}{r}-1}.$$

(iii-b) The case $\alpha < r$ and $x > 1$. Similarly to (i-b), we consider $f_{t_1}(y)$ for $y = x^{-1} \in (0,1)$ and $t_1 = 1-t$. Noting that $r < t \leqslant 1-t$ if and only if $r < 1-t_1 \leqslant t_1$, by the similar way to (i-b), we have that $K_{t_1,\alpha}(1,y) < P_{t_1,r}(1,y)$ for $0 < y < 1$ if $\alpha \leqslant (1-t_1)^{\frac{1}{r}-1}$, that is,

$$K_{t,\alpha}(1,x) < P_{t,r}(1,x) \text{ for } x > 1 \text{ if } \alpha \leqslant t^{\frac{1}{r}-1}.$$

Hence, by (iii-a) and (iii-b), we get $K_{t,\alpha}(1,x) \leqslant P_{t,r}(1,x)$ for all $x > 0$ with $x \neq 1$ if $\alpha \leqslant t^{\frac{1}{r}-1}$ since $t^{\frac{1}{r}-1} \leqslant (1-t)^{\frac{1}{r}-1}$ holds. We remark that $t^{\frac{1}{r}-1} < \left(\frac{1}{2}\right)^{\frac{1}{r}-1} < r$ holds for $r,t \in (0,\frac{1}{2})$. This argument also proves the best possibility of α since $K_{t,\alpha}(1,x) < P_{t,r}(1,x)$ or $P_{t,r}(1,x) < K_{t,\alpha}(1,x)$ does not always hold for $x > 0$ with $x \neq 1$ if $t^{\frac{1}{r}-1} < \alpha < r$.

Next, we consider the case $r \leqslant \alpha$. If $\alpha \geqslant 1$, then we obviously get that $P_{t,r}(1,x) < K_{t,\alpha}(1,x)$ holds for all $x > 0$ with $x \neq 1$ since $K_{t,1}(1,x) = A_t(1,x)$. We remark that $r < \beta(t,r)$ holds for $0 < r < t < \frac{1}{2}$ by (i) in Lemma 2.1.

(iii-c) The case $r \leqslant \beta(t,r) \leqslant \alpha$ and $0 < x < 1$. If $r < t \leqslant 1-t$ holds, then $h_t'(x) > 0$ holds for $0 < x < \delta_0$ and $h_t'(x) < 0$ holds for $\delta_0 < x < 1$, that is,

$$h_t(x) \text{ is increasing for } 0 < x < \delta_0 \text{ and decreasing for } \delta_0 < x < 1.$$

by (3.2) and (3.3). Noting that $h(\delta_0) \leqslant 0$ if and only if $\alpha \geqslant \beta(t,r)$, we get that $g_t'(x) \leqslant 0$ for $0 < x < 1$, that is,

$$g_t(x) \text{ is decreasing for } 0 < x < 1.$$

Since $g_t(1) = 0$, $f_t'(x) > 0$ holds for $0 < x < 1$, that is,

$$f_t(x) \text{ is increasing for } 0 < x < 1.$$

Therefore, since $f_t(1) = 0$, we have

$$f_t(x) < 0, \text{ that is, } P_{t,r}(1,x) < K_{t,\alpha}(1,x) \text{ for } 0 < x < 1 \text{ if } \alpha \geqslant \beta(t,r).$$

(iii-d) The case $r \leqslant \alpha$ and $x > 1$. We consider $f_{t_1}(y)$ for $y = x^{-1} \in (0,1)$ and $t_1 = 1-t$. Noting that $r \leqslant t \leqslant 1-t$ if and only if $r \leqslant 1-t_1 \leqslant t_1$, by the similar way to (i-a), we have that $P_{t_1,r}(1,y) < K_{t_1,\alpha}(1,y)$ for $0 < y < 1$, that is,

$$P_{t,r}(1,x) < K_{t,\alpha}(1,x) \text{ for } x > 1.$$

Hence, by (iii-c) and (iii-d), we get $P_{t,r}(1,x) < K_{t,\alpha}(1,x)$ for all $x > 0$ with $x \neq 1$ if $\alpha \geqslant \widehat{\beta}(t,r)$.

Proof of (v). Firstly, we consider the case $\alpha \leqslant r$.

By the similar way to (i-a), we obtain that $f_t(x) > 0$, that is, $K_{t,\alpha}(1,x) < P_{t,r}(1,x)$ for $0 < x < 1$. By applying the similar way to (iii-c) for $f_{t_1}(y)$ as in (i-b), we obtain that $f_{t_1}(y) > 0$ holds for $0 < y < 1$ if $\alpha \leqslant \beta(t_1,r)$, that is, $K_{t,\alpha}(1,x) < P_{t,r}(1,x)$ for $x > 1$ if $\alpha \leqslant \beta(1-t,r)$. Hence, we get $K_{t,\alpha}(1,x) < P_{t,r}(1,x)$ for all $x > 0$ with $x \neq 1$ if $\alpha \leqslant \beta(1-t,r)$.

Next, we consider the case $r < \alpha$.

By the similar way to (i-b), we have that $P_{t,r}(1,x) < K_{t,\alpha}(1,x)$ for $0 < x < 1$ if $\alpha \geqslant (1-t)^{\frac{1}{r}-1}$. By applying the similar way to (iii-a) for $f_{t_1}(y)$ as in (i-b), we have that $P_{t,r}(1,x) < K_{t,\alpha}(1,x)$ for $x > 1$ if $\alpha \geqslant t^{\frac{1}{r}-1}$. Hence, we get $P_{t,r}(1,x) < K_{t,\alpha}(1,x)$ for all $x > 0$ with $x \neq 1$ if $\alpha \geqslant (1-t)^{\frac{1}{r}-1}$ since $t^{\frac{1}{r}-1} \leqslant (1-t)^{\frac{1}{r}-1}$ holds. We also get the best possibility of α, that is, $K_{t,\alpha}(1,x) < P_{t,r}(1,x)$ or $P_{t,r}(1,x) < K_{t,\alpha}(1,x)$ does not always hold for $x > 0$ with $x \neq 1$ if $r < \alpha < (1-t)^{\frac{1}{r}-1}$.

Since $K_{t,\alpha}(a,b) = K_{1-t,\alpha}(b,a)$ and $P_{t,r}(a,b) = P_{1-t,r}(b,a)$ hold for $a,b > 0$, (ii), (iv) and (vi) are immediately obtained by (i), (iii) and (v), respectively. $\square$

Proof of Theorem 3.2. We can prove Theorem 3.2 by the similar way to Theorem 3.1, so we give proofs of (i) and (iii). In this proof, (i-a), (i-b), (iii-a) and (iii-c) mean the numbers in Theorem 3.1. We have only to consider the case $(a,b) = (1,x)$ with $x \neq 1$ by easy replacement.

Proof of (i). The case $\alpha < r$. By the similar way to (i-b), we have that $K_{t,\alpha}(1,x) < P_{t,r}(1,x)$ for $0 < x < 1$ if $\alpha \leqslant (1-t)^{\frac{1}{r}-1}$. By applying the similar way to (iii-a) for $f_{t_1}(y)$ as in (i-b), we have that $K_{t,\alpha}(1,x) < P_{t,r}(1,x)$ for $x > 1$ if $\alpha \leqslant t^{\frac{1}{r}-1}$. Hence, we get $K_{t,\alpha}(1,x) \leqslant P_{t,r}(1,x)$ for all $x > 0$ with $x \neq 1$ if $\alpha \leqslant (1-t)^{\frac{1}{r}-1}$ since $t^{\frac{1}{r}-1} \geqslant (1-t)^{\frac{1}{r}-1}$ holds. We also get the best possibility of α.

The case $r \leqslant \alpha$. By the similar way to (i-a), we have that $P_{t,r}(1,x) < K_{t,\alpha}(1,x)$ for $0 < x < 1$. By applying the similar way to (iii-c) for $f_{t_1}(y)$ as in (i-b), we have that $P_{t,r}(1,x) < K_{t,\alpha}(1,x)$ for $x > 1$ if $\alpha \geqslant \beta(1-t,r)$. Hence, we get $P_{t,r}(1,x) < K_{t,\alpha}(1,x)$ for all $x > 0$ with $x \neq 1$ if $\alpha \geqslant \beta(1-t,r)$.

Proof of (iii). Firstly, we consider the case $\alpha \leqslant r$. By the similar way to (iii-c), we have that $K_{t,\alpha}(1,x) < P_{t,r}(1,x)$ for $0 < x < 1$ if $\alpha \leqslant \beta(t,r)$. By applying the similar way to (i-a) for $f_{t_1}(y)$ as in (i-b), we have that $K_{t,\alpha}(1,x) < P_{t,r}(1,x)$ for $x > 1$. Hence, we get $K_{t,\alpha}(1,x) < P_{t,r}(1,x)$ for all $x > 0$ with $x \neq 1$ if $\alpha \leqslant \beta(t,r)$.

Next, we consider the case $r < \alpha$. Let $\delta_0 = \left(\frac{(1-t)(t-r)}{t(1-t-r)}\right)^{\frac{1}{r}}$. We remark that $0 < \delta_0 \leqslant 1$ holds for $t \in (0,\frac{1}{2}]$ and $r < 0$.

(a) The case $r < \alpha$ and $0 < x < 1$. Since $f_t(x) < 0$ for $0 < x < 1$ if $\alpha = 0$, we have only to consider the case $\alpha < 0$. If $t \in (0,\frac{1}{2}]$ and $r < 0$, then $h_t'(x) < 0$ holds for $0 < x < \delta_0$ and $h_t'(x) > 0$ holds for $\delta_0 < x < 1$, that is,

$$h_t(x) \text{ is decreasing for } 0 < x < \delta_0 \text{ and increasing for } \delta_0 < x < 1$$

by (3.2) and (3.3). There exists a $\delta_1 \in (0,\delta_0)$ such that $h_t(\delta_1) = 0$ since $\lim_{x\to+0} h_t(x) = (1-t)(t^{\frac{1}{r}} - \alpha) > 0$ and $h_t(1) = (r-\alpha)(1-t) \leqslant 0$. This ensures that $g_t'(x) > 0$ for $0 < x < \delta_1$ and $g_t'(x) < 0$ for $\delta_1 < x < 1$ hold, that is,

$$g_t(x) \text{ is increasing for } 0 < x < \delta_1 \text{ and decreasing for } \delta_1 < x < 1.$$

Then there exists a $\delta_2 \in (0,\delta_1)$ such that $g_t(\delta_2) = 0$ since $\lim_{x\to+0} g_t(x) = -(1-\alpha) < 0$ and $g_t(1) = 0$. So $f_t'(x) < 0$ holds for $0 < x < \delta_2$ and $f_t'(x) > 0$ holds for $\delta_2 < x < 1$ hold, that is,

$$f_t(x) \text{ is decreasing for } 0 < x < \delta_2 \text{ and increasing for } \delta_2 < x < 1.$$

Since $\lim_{x\to+0} f_t(x) = -\alpha(1-t) > 0$ and $f_t(1) = 0$, $f_t(x) < 0$ does not always hold for $0 < x < 1$ if $\alpha < 0$.

Therefore we have

$$P_{t,r}(1,x) < K_{t,\alpha}(1,x) \text{ for } 0 < x < 1 \text{ if } \alpha \geqslant 0.$$

(b) The case $r < \alpha$ and $x > 1$. We consider $f_{t_1}(y)$ for $y = x^{-1} \in (0,1)$ and $t_1 = 1-t$. Noting that $t \in (0,\frac{1}{2}]$ if and only if $t_1 \in [\frac{1}{2},1)$, by the similar way to (i-b) and (a), we have that $P_{t_1,r}(1,y) < K_{t_1,\alpha}(1,y)$ for $0 < y < 1$ if $\alpha \geqslant 0$, that is,

$$P_{t,r}(1,x) < K_{t,\alpha}(1,x) \text{ for } x > 1 \text{ if } \alpha \geqslant 0.$$

Hence, by (a) and (b), we get $P_{t,r}(1,x) \leqslant K_{t,\alpha}(1,x)$ for all $x > 0$ with $x \neq 1$ if $\alpha \geqslant 0$. This argument also proves the best possibility of α. $\square$

4. Operator inequalities

In this section, we get operator inequalities by the results in the previous section.

Here, an operator means a bounded linear operator on a Hilbert space $\mathscr{H}$. An operator T is said to be positive (denoted by $T \geqslant 0$) if $(Tx,x) \geqslant 0$ for all $x \in \mathscr{H}$, and also an operator T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible. A real-valued function f defined on $J \subset \mathbb{R}$ is said to be operator monotone if

$$A \leqslant B \text{ implies } f(A) \leqslant f(B)$$

for selfadjoint operators A and B whose spectra $\sigma(A), \sigma(B) \subset J$, where $A \leqslant B$ means $B - A \geqslant 0$.

Kubo and Ando [10] obtained the general theory on operator means. In [10], they obtained that there exists a one-to-one correspondence between an operator mean $\mathfrak{M}$ and an operator monotone function $f \geqslant 0$ on $[0,\infty)$ with $f(1) = 1$ as follows:

$$\mathfrak{M}(A,B) = A^{\frac{1}{2}} f(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}} \tag{4.1}$$

if $A > 0$ and $B \geqslant 0$. We remark that f is called the representing function of $\mathfrak{M}$, and also it is permitted to consider binary operations given by (4.1) even if f is a general real-valued function.

By (4.1), we can introduce the following weighted operator means for two strictly positive operators A and B. For $t \in [0,1]$ and $q \in \mathbb{R}$,

$$\mathfrak{A}_t(A,B) = (1-t)A + tB \quad \text{(arithmetic mean)},$$
$$\mathfrak{G}_t(A,B) = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^t A^{\frac{1}{2}} \quad \text{(geometric mean)},$$
$$\mathfrak{H}_t(A,B) = \{(1-t)A^{-1} + tB^{-1}\}^{-1} \quad \text{(harmonic mean)},$$
$$\mathfrak{P}_{t,q}(A,B) = \begin{cases} A^{\frac{1}{2}}\{(1-t)I + t(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^q\}^{\frac{1}{q}}A^{\frac{1}{2}} & \text{if } q \neq 0, \\ A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^t A^{\frac{1}{2}} & \text{if } q = 0, \end{cases} \quad \text{(power mean)},$$
$$\mathfrak{K}_{t,q}(A,B) = (1-q)A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^t A^{\frac{1}{2}} + q\{(1-t)A + tB\} \quad \text{(Heron mean)},$$

and then $\mathfrak{P}_{t,q}(A,B)$ is an operator mean if $-1 \leqslant q \leqslant 1$, and also $\mathfrak{K}_{t,q}(A,B)$ is an operator mean if $0 \leqslant q \leqslant 1$. We remark that their representing functions are $A_t(1,x)$, $G_t(1,x)$ and so on, and also notations $A \nabla_t B$, $A \sharp_t B$, $A \,!_t\, B$ and $A \sharp_{t,q} B$ are often used instead of $\mathfrak{A}_t(A,B)$, $\mathfrak{G}_t(A,B)$, $\mathfrak{H}_t(A,B)$ and $\mathfrak{P}_{t,q}(A,B)$, respectively. Refer to [11] for more details.

By Theorem 3.1, we have estimations of the weighted operator power mean by the Heron mean. Theorem 3.2 ensures the similar result, but we omit it.

THEOREM 4.1 *Let $t, r \in (0,1)$. Let $\beta(t,r)$ and $\widehat{\beta}(t,r)$ as in (2.1). For all $A, B > 0$, we have the following.*

(i) *If $t \leqslant r \leqslant 1-t$, then*

$$\mathfrak{K}_{t,t^{\frac{1}{r}-1}}(A,B) \leqslant \mathfrak{P}_{t,r}(A,B) \leqslant \mathfrak{K}_{t,(1-t)^{\frac{1}{r}-1}}(A,B).$$

(ii) *If $1-t \leqslant r \leqslant t$, then*

$$\mathfrak{K}_{t,(1-t)^{\frac{1}{r}-1}}(A,B) \leqslant \mathfrak{P}_{t,r}(A,B) \leqslant \mathfrak{K}_{t,t^{\frac{1}{r}-1}}(A,B).$$

(iii) *If $r < t \leqslant 1-t$, then*

$$\mathfrak{K}_{t,t^{\frac{1}{r}-1}}(A,B) \leqslant \mathfrak{P}_{t,r}(A,B) \leqslant \mathfrak{K}_{t,\widehat{\beta}(t,r)}(A,B).$$

(iv) *If $r < 1-t \leqslant t$, then*

$$\mathfrak{K}_{t,(1-t)^{\frac{1}{r}-1}}(A,B) \leqslant \mathfrak{P}_{t,r}(A,B) \leqslant \mathfrak{K}_{t,\widehat{\beta}(1-t,r)}(A,B).$$

(v) *If $t \leqslant 1-t < r$, then*

$$\mathfrak{K}_{t,\beta(1-t,r)}(A,B) \leqslant \mathfrak{P}_{t,r}(A,B) \leqslant \mathfrak{K}_{t,(1-t)^{\frac{1}{r}-1}}(A,B).$$

(vi) *If* $1-t \leqslant t < r$*, then*

$$\mathfrak{K}_{t,\beta(t,r)}(A,B) \leqslant \mathfrak{P}_{t,r}(A,B) \leqslant \mathfrak{K}_{t,t^{\frac{1}{r}-1}}(A,B).$$

The given parameters of $\mathfrak{K}_{t,\alpha}(A,B)$ *in each case are best possible on* α *except the parts* $\alpha = \beta(\cdot,r)$ *and* $\alpha = \widehat{\beta}(\cdot,r)$*.*

Proof. Put $a=1$ and replace b by $A^{\frac{-1}{2}}BA^{\frac{-1}{2}}$. Then we have Theorem 4.1 by applying the standard operational calculus in Theorem 3.1. □

5. Determinant and trace inequalities

In this section, we get some determinant and trace inequalities of matrices. Let $P_n(\mathbb{C})$ be the set of $n \times n$ positive definite matrices on $\mathbb{C}$.

By using (1.2), Khosravi [7] obtained a generalization of the determinant inequality in [1] as follows: Let $A,B \in P_n(\mathbb{C})$. Then

$$\left(\frac{\nu}{\mu}\right)^p \left[\det\{\mathfrak{A}_\mu(A,B) - \mathfrak{P}_{\mu,r}(A,B)\}\right]^{\frac{p}{n}} \leqslant \left[\det \mathfrak{A}_\nu(A,B)\right]^{\frac{p}{n}} - \left[\det \mathfrak{P}_{\nu,r}(A,B)\right]^{\frac{p}{n}} \quad (5.1)$$

holds for $0 < \nu \leqslant \mu < 1$, $-1 \leqslant r < 1$ and $p \geqslant 1$. We get determinant inequalities related to (5.1) by using Theorem 3.1.

THEOREM 5.1 *Let* $A,B \in P_n(\mathbb{C})$, $r \in (0,1)$ *and* $p \geqslant 1$. *Let* $\widehat{\beta}(t,r)$ *as in* (2.1).

(i) *If* $t \in (0,\frac{1}{2}]$ *and* $t \leqslant r$*, then*

$$\left(1-(1-t)^{\frac{1}{r}-1}\right)^p \left[\det\{\mathfrak{A}_t(A,B) - \mathfrak{G}_t(A,B)\}\right]^{\frac{p}{n}} \leqslant \left[\det \mathfrak{A}_t(A,B)\right]^{\frac{p}{n}} - \left[\det \mathfrak{P}_{t,r}(A,B)\right]^{\frac{p}{n}}.$$

(ii) *If* $t \in (0,\frac{1}{2}]$ *and* $r < t$*, then*

$$\left(1-\widehat{\beta}(t,r)\right)^p \left[\det\{\mathfrak{A}_t(A,B) - \mathfrak{G}_t(A,B)\}\right]^{\frac{p}{n}} \leqslant \left[\det \mathfrak{A}_t(A,B)\right]^{\frac{p}{n}} - \left[\det \mathfrak{P}_{t,r}(A,B)\right]^{\frac{p}{n}}.$$

(iii) *If* $t \in (\frac{1}{2},1)$ *and* $1-t \leqslant r$*, then*

$$\left(1-t^{\frac{1}{r}-1}\right)^p \left[\det\{\mathfrak{A}_t(A,B) - \mathfrak{G}_t(A,B)\}\right]^{\frac{p}{n}} \leqslant \left[\det \mathfrak{A}_t(A,B)\right]^{\frac{p}{n}} - \left[\det \mathfrak{P}_{t,r}(A,B)\right]^{\frac{p}{n}}.$$

(iv) *If* $t \in (\frac{1}{2},1)$ *and* $r < 1-t$*, then*

$$\left(1-\widehat{\beta}(1-t,r)\right)^p \left[\det\{\mathfrak{A}_t(A,B) - \mathfrak{G}_t(A,B)\}\right]^{\frac{p}{n}} \leqslant \left[\det \mathfrak{A}_t(A,B)\right]^{\frac{p}{n}} - \left[\det \mathfrak{P}_{t,r}(A,B)\right]^{\frac{p}{n}}.$$

THEOREM 5.2 *Let* $A,B \in P_n(\mathbb{C})$, $r \in (0,1)$ *and* $p \geqslant 1$. *Let* $\beta(t,r)$ *as in* (2.1).

(i) *If* $t \in (0,\frac{1}{2}]$ *and* $1-t \leqslant r$, *then*

$$\beta(1-t,r)^p\left[\det\{\mathfrak{A}_t(A,B)-\mathfrak{G}_t(A,B)\}\right]^{\frac{p}{n}} \leqslant \left[\det\mathfrak{P}_{t,r}(A,B)\right]^{\frac{p}{n}} - \left[\det\mathfrak{G}_t(A,B)\right]^{\frac{p}{n}}.$$

(ii) *If* $t \in (0,\frac{1}{2}]$ *and* $r < 1-t$, *then*

$$t^{\left(\frac{1}{r}-1\right)p}\left[\det\{\mathfrak{A}_t(A,B)-\mathfrak{G}_t(A,B)\}\right]^{\frac{p}{n}} \leqslant \left[\det\mathfrak{P}_{t,r}(A,B)\right]^{\frac{p}{n}} - \left[\det\mathfrak{G}_t(A,B)\right]^{\frac{p}{n}}.$$

(iii) *If* $t \in (\frac{1}{2},1)$ *and* $t \leqslant r$, *then*

$$\beta(t,r)^p\left[\det\{\mathfrak{A}_t(A,B)-\mathfrak{G}_t(A,B)\}\right]^{\frac{p}{n}} \leqslant \left[\det\mathfrak{P}_{t,r}(A,B)\right]^{\frac{p}{n}} - \left[\det\mathfrak{G}_t(A,B)\right]^{\frac{p}{n}}.$$

(iv) *If* $t \in (\frac{1}{2},1)$ *and* $r < t$, *then*

$$(1-t)^{\left(\frac{1}{r}-1\right)p}\left[\det\{\mathfrak{A}_t(A,B)-\mathfrak{G}_t(A,B)\}\right]^{\frac{p}{n}} \leqslant \left[\det\mathfrak{P}_{t,r}(A,B)\right]^{\frac{p}{n}} - \left[\det\mathfrak{G}_t(A,B)\right]^{\frac{p}{n}}.$$

Let $a_i, b_i > 0$ for $i = 1,2,\ldots,n$. Then Minkowski's product inequality

$$\left(\prod_{i=1}^n a_i\right)^{\frac{1}{n}} + \left(\prod_{i=1}^n b_i\right)^{\frac{1}{n}} \leqslant \left(\prod_{i=1}^n (a_i+b_i)\right)^{\frac{1}{n}}$$

holds (see [3]). We have the following lemma by Minkowski's product inequality and the inequality $a^p + b^p \leqslant (a+b)^p$ for $a,b > 0$ and $p \geqslant 1$.

LEMMA 5.A ([7]) *Let* $a_i, b_i > 0$ *for* $i = 1,2,\ldots,n$. *Then*

$$\left(\prod_{i=1}^n a_i\right)^{\frac{p}{n}} + \left(\prod_{i=1}^n b_i\right)^{\frac{p}{n}} \leqslant \left(\prod_{i=1}^n (a_i+b_i)\right)^{\frac{p}{n}}$$

holds for $p \geqslant 1$.

Proof of Theorem 5.1. Proof of (i). Let $X = A^{\frac{-1}{2}}BA^{\frac{-1}{2}}$ and $k = 1-(1-t)^{\frac{1}{r}-1}$. Then

$$k\{(1-t)+t\lambda_i(X)-\lambda_i(X)^t\} + \{(1-t)+t\lambda_i(X)^r\}^{\frac{1}{r}} \leqslant (1-t)+t\lambda_i(X) \tag{5.2}$$

holds for all eigenvalues $\lambda_i(X)$ $(i = 1,\ldots,n)$ by (i) and (v) in Theorem 3.1. Therefore we have

$$\begin{aligned} &[\det\{(1-t)I+tX\}]^{\frac{p}{n}} \\ &= \left[\prod_{i=1}^n \{(1-t)+t\lambda_i(X)\}\right]^{\frac{p}{n}} \end{aligned}$$

$$\geqslant \left[\prod_{i=1}^{n}\left(k\{(1-t)+t\lambda_i(X)-\lambda_i(X)^t\}+\{(1-t)+t\lambda_i(X)^r\}^{\frac{1}{r}}\right)\right]^{\frac{p}{n}} \quad \text{by (5.2)}$$

$$\geqslant k^p\left[\prod_{i=1}^{n}\{(1-t)+t\lambda_i(X)-\lambda_i(X)^t\}\right]^{\frac{p}{n}}+\left[\prod_{i=1}^{n}\{(1-t)+t\lambda_i(X)^r\}^{\frac{1}{r}}\right]^{\frac{p}{n}}$$

$$= k^p\left[\det\{(1-t)I+tX-X^t\}\right]^{\frac{p}{n}}+\left[\det\{(1-t)I+tX^r\}^{\frac{1}{r}}\right]^{\frac{p}{n}},$$

where the second inequality holds by Lemma 5.A. Multiplying $(\det A^{\frac{1}{2}})^{\frac{p}{n}}$ to both sides, we get the desired inequality by multiplicativity of the determinant.

(ii) is obtained by (iii) in Theorem 3.1 similarly. (iii) and (iv) are immediately shown by considering $\mathfrak{P}_{t,r}(A,B)=\mathfrak{P}_{1-t,r}(B,A)$ in (i) and (ii), respectively. $\square$

Proof of Theorem 5.2. Proof of (i). Let $X=A^{\frac{-1}{2}}BA^{\frac{-1}{2}}$. Then

$$\beta(1-t,r)\{(1-t)+t\lambda_i(X)-\lambda_i(X)^t\}+\lambda_i(X)^t \leqslant \{(1-t)+t\lambda_i(X)^r\}^{\frac{1}{r}} \tag{5.3}$$

holds for all eigenvalues $\lambda_i(X)$ $(i=1,\dots,n)$ by (v) in Theorem 3.1. Therefore we have

$$\begin{aligned}
&\left[\det\{(1-t)I+tX^r\}^{\frac{1}{r}}\right]^{\frac{p}{n}}\\
&=\left[\prod_{i=1}^{n}\{(1-t)+t\lambda_i(X)^r\}^{\frac{1}{r}}\right]^{\frac{p}{n}}\\
&\geqslant \left[\prod_{i=1}^{n}\left(\beta(1-t,r)\{(1-t)+t\lambda_i(X)-\lambda_i(X)^t\}+\lambda_i(X)^t\right)\right]^{\frac{p}{n}} \quad \text{by (5.3)}\\
&\geqslant \beta(1-t,r)^p\left[\prod_{i=1}^{n}\{(1-t)+t\lambda_i(X)-\lambda_i(X)^t\}\right]^{\frac{p}{n}}+\left[\prod_{i=1}^{n}\lambda_i(X)^t\right]^{\frac{p}{n}}\\
&=\beta(1-t,r)^p\left[\det\{(1-t)I+tX-X^t\}\right]^{\frac{p}{n}}+\left[\det X^t\right]^{\frac{p}{n}},
\end{aligned}$$

where the second inequality holds by Lemma 5.A. Multiplying $(\det A^{\frac{1}{2}})^{\frac{p}{n}}$ to both sides, we get the desired inequality by multiplicativity of the determinant.

(ii) is obtained by (i) and (iii) in Theorem 3.1 similarly. (iii) and (iv) are immediately shown by considering $\mathfrak{P}_{t,r}(A,B)=\mathfrak{P}_{1-t,r}(B,A)$ in (i) and (ii), respectively. $\square$

On the other hand, by using (1.2) for $\lambda=1$, Alzer, da Fonseca and Kovačec [1] obtained the trace inequality as follows: Let $A,B\in P_n(\mathbb{C})$. Then

$$\frac{\nu}{\mu}\left\{\operatorname{tr}\mathfrak{A}_\mu(A,B)-(\operatorname{tr}A)^{1-\mu}(\operatorname{tr}B)^{\mu}\right\}\leqslant \operatorname{tr}\mathfrak{A}_\nu(A,B)-\operatorname{tr}A^{1-\nu}B^{\nu}. \tag{5.4}$$

holds for $0<\nu\leqslant\mu<1$. We also get trace inequalities related to (5.4) by using Theorem 3.1.

THEOREM 5.3 *Let $A,B \in P_n(\mathbb{C})$, $r \in (0,1)$ and $p \geqslant 1$. Let $\beta(t,r)$ as in (2.1).*

(i) *If $t \in (0,\frac{1}{2}]$ and $1-t \leqslant r$, then*

$$\beta(1-t,r)\left\{\mathrm{tr}\mathfrak{A}_t(A,B)-(\mathrm{tr}A)^{1-t}(\mathrm{tr}B)^t\right\} \leqslant \{\mathrm{tr}\mathfrak{A}_t(A^r,B^r)\}^{\frac{1}{r}}-\mathrm{tr}A^{1-t}B^t.$$

(ii) *If $t \in (0,\frac{1}{2}]$ and $r < 1-t$, then*

$$t^{\frac{1}{r}-1}\left\{\mathrm{tr}\mathfrak{A}_t(A,B)-(\mathrm{tr}A)^{1-t}(\mathrm{tr}B)^t\right\} \leqslant \{\mathrm{tr}\mathfrak{A}_t(A^r,B^r)\}^{\frac{1}{r}}-\mathrm{tr}A^{1-t}B^t.$$

(iii) *If $t \in (\frac{1}{2},1)$ and $t \leqslant r$, then*

$$\beta(t,r)\left\{\mathrm{tr}\mathfrak{A}_t(A,B)-(\mathrm{tr}A)^{1-t}(\mathrm{tr}B)^t\right\} \leqslant \{\mathrm{tr}\mathfrak{A}_t(A^r,B^r)\}^{\frac{1}{r}}-\mathrm{tr}A^{1-t}B^t.$$

(iv) *If $t \in (\frac{1}{2},1)$ and $r < t$, then*

$$(1-t)^{\frac{1}{r}-1}\left\{\mathrm{tr}\mathfrak{A}_t(A,B)-(\mathrm{tr}A)^{1-t}(\mathrm{tr}B)^t\right\} \leqslant \{\mathrm{tr}\mathfrak{A}_t(A^r,B^r)\}^{\frac{1}{r}}-\mathrm{tr}A^{1-t}B^t.$$

We state two lemmas in order to prove Theorem 5.3. Here, let $\sigma_i(X)$ for $i = 1,\dots,n$ be the singular values of an $n \times n$ matrix X.

LEMMA 5.B ([1, 2, 3]) *The following properties hold.*

(i) *The product of two positive definite matrices is a matrix with only positive eigenvalues.*

(ii) *Let $A > 0$ be of order n. Then, the singular values of A are precisely the eigenvalues of A. In particular,*

$$\mathrm{tr}A = \sum_{i=1}^{n}\lambda_i(A) = \sum_{i=1}^{n}\sigma_i(A),$$

where $\lambda_i(A)$ for $i = 1,\dots,n$ are eigenvalues of A.

(iii) *If A is an $n \times n$ matrix with only real eigenvalues, then*

$$\mathrm{tr}A \leqslant \sum_{i=1}^{n}\sigma_i(A).$$

(iv) *If A and B are $n \times n$ matrices, then there holds the weak majorization property*

$$\sum_{i=1}^{k}\sigma_i(AB) \leqslant \sum_{i=1}^{k}\sigma_i(A)\sigma_i(B), \quad \textit{for } k = 1,\dots,n.$$

LEMMA 5.C *Let $a_i, b_i > 0$ for $i = 1,\dots,n$.*

(i) *The inequality* $\sum_{i=1}^{n} a_i^p \leqslant \left(\sum_{i=1}^{n} a_i\right)^p$ *holds for* $p \geqslant 1$.

(ii) *The inequality* $\sum_{i=1}^{n} a_i b_i \leqslant \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}$ *holds for* $p,q > 1$ *such that* $\frac{1}{p}+\frac{1}{q}=1$ *(Hölder's inequality).*

We remark that (i) in Lemma 5.C is easily obtained by using the inequality $a^p + b^p \leqslant (a+b)^p$ for $a,b>0$ and $p \geqslant 1$ repeatedly, or by Jensen's inequality (see [3]).

Proof of Theorem 5.3. Proof of (i). Since the inequality

$$\begin{aligned}&\beta(1-t,r)\left\{(1-t)\sigma_i(A)+t\sigma_i(B)-\sigma_i(A)^{1-t}\sigma_i(B)^t\right\}+\sigma_i(A)^{1-t}\sigma_i(B)^t\\ \leqslant&\left\{(1-t)\sigma_i(A)^r+t\sigma_i(B)^r\right\}^{\frac{1}{r}}\end{aligned} \tag{5.5}$$

holds by (v) in Theorem 3.1, we have

$$\begin{aligned}&\left\{(1-t)\operatorname{tr}A^r+t\operatorname{tr}B^r\right\}^{\frac{1}{r}}\\ =&\left[\sum_{i=1}^{n}\left\{(1-t)\sigma_i(A)^r+t\sigma_i(B)^r\right\}\right]^{\frac{1}{r}}\\ \geqslant&\sum_{i=1}^{n}\left\{(1-t)\sigma_i(A)^r+t\sigma_i(B)^r\right\}^{\frac{1}{r}} \quad \text{by (i) in Lemma 5.C}\\ \geqslant&\sum_{i=1}^{n}\left[\beta(1-t,r)\left\{(1-t)\sigma_i(A)+t\sigma_i(B)-\sigma_i(A)^{1-t}\sigma_i(B)^t\right\}+\sigma_i(A)^{1-t}\sigma_i(B)^t\right]\\ \geqslant&\beta(1-t,r)\left\{(1-t)\sum_{i=1}^{n}\sigma_i(A)+t\sum_{i=1}^{n}\sigma_i(B)-\left(\sum_{i=1}^{n}\sigma_i(A)\right)^{1-t}\left(\sum_{i=1}^{n}\sigma_i(B)\right)^{t}\right\}\\ &+\sum_{i=1}^{n}\sigma_i(A^{1-t}B^t)\\ \geqslant&\beta(1-t,r)\left\{(1-t)\operatorname{tr}A+t\operatorname{tr}B-(\operatorname{tr}A)^{1-t}(\operatorname{tr}B)^t\right\}+\operatorname{tr}A^{1-t}B^t,\end{aligned}$$

where the second inequality holds by (5.5), and also the third and the last inequalities hold by Lemma 5.B and (ii) in Lemma 5.C. Therefore we get the desired inequality.

(ii) is obtained by (i) and (iii) in Theorem 3.1 similarly. (iii) and (iv) are immediately shown by replacing A, B and t by B, A and $1-t$ in (i) and (ii), respectively. □

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Masatoshi Ito
Maebashi Institute of Technology
460-1 Kamisadorimachi, Maebashi Gunma 371-0816, Japan
e-mail: m-ito@maebashi-it.ac.jp