Mathematical
Inequalities
& Applications
Volume 22, Number 3 (2019), 967–982 doi:10.7153/mia-2019-22-65

ESSENTIAL NORM OF THE WEIGHTED DIFFERENTIATION COMPOSITION OPERATOR BETWEEN BLOCH–TYPE SPACES

CUI CHEN* AND YU-XIA LIANG

(*Communicated by S. Stević*)

Abstract. We give some characterizations for the boundedness and compactness of weighted differentiation composition operator $D^m_{\varphi,u}$ between different Bloch spaces, and also estimate the essential norms of the operator, complementing some recent results in the literature.

1. Introduction

Let $H(\mathbb{D})$ denote the class of holomorphic functions and $S(\mathbb{D})$ be the set of analytic self-maps of the unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$.

For $\varphi \in S(\mathbb{D})$, the composition operator associated to φ is defined by $C_\varphi(f) = f\circ\varphi$ for any $f \in H(\mathbb{D})$. Let $D = D^1$ be the differentiation operator, i.e., $Df = f'$. For $m\in\mathbb{N}$, the operator D^m is defined inductively by $D^0 f = f,\ D^m f = f^{(m)},\ f\in H(\mathbb{D})$.

For $m\in\mathbb{N}_0$, $u\in H(\mathbb{D})$ and $\varphi\in S(\mathbb{D})$, the weighted differentiation composition operator, denoted by $D^m_{\varphi,u}$, is defined by

$$(D^m_{\varphi,u}f)(z) = u(z)f^{(m)}(\varphi(z)),\ \ z\in\mathbb{D},$$

which was studied in some recent papers such as [12, 19, 22, 21, 23, 34, 35]. The operator is a natural generalization of products of differentiation and composition operators previously studied, for example, in [6, 8, 7, 9, 24, 20, 16, 25, 18] (see also the references therein). For related product-type operators see also [26] and [27]. If $m = 0$, then $D^m_{\varphi,u}$ becomes the weighted composition operator uC_φ, defined by

$$uC_\varphi(f)(z) = u(z)f(\varphi(z)),\ \ z\in\mathbb{D}$$

for $f\in H(\mathbb{D})$.

For $0<\alpha<\infty$, the Bloch-type space in $\mathbb{D}$ denoted by $\mathscr{B}^\alpha$ consists of all functions $f\in H(\mathbb{D})$ such that

$$\|f\|_\alpha = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)| < \infty.$$

Mathematics subject classification (2010): 47B38, 30H30, 30H05, 47B33.

Keywords and phrases: Weighted differentiation composition operator, essential norm, Bloch-type space, boundedness, compactness.

The work was supported in part by the National Natural Science Foundation of China (Grant Nos. 11701422) and the Research Project of Tianjin Municipal Education Commission (Grant Nos. 2017KJ124).

* Corresponding author.

Paper MIA-22-65

When $\alpha = 1$, it is the classical Bloch space $\mathscr{B}$. Endowed with the norm $\|f\|_{\mathscr{B}^\alpha} = |f(0)| + \|f\|_\alpha$, the Bolch-type space becomes a Banach space. Besides, for $f \in H(\mathbb{D})$, we say that it belongs to the little Bloch-type space $\mathscr{B}_0^\alpha$, if

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha |f'(z)| = 0.$$

It is well known that the little Bloch-type space $\mathscr{B}_0^\alpha$ is the closure of polynomials in $\mathscr{B}^\alpha$. Many papers study various concrete operators on or to Bloch spaces, see, for example, [2, 3, 5, 6, 8, 7, 9, 10, 11, 13, 14, 32, 24, 20, 16, 22, 21, 28, 29, 30, 33, 34, 35, 31].

The essential norm of a continuous linear operator T is the distance form T to the set of compact operators, that is $\|T\|_e = \inf\{\|T-K\| : K \text{ is compact}\}$. Notice that $\|T\|_e = 0$ if and only if the operator T is compact, so the estimate on $\|T\|_e$ will lead to a condition for the operator T to be compact. The compactness of an operator is a topic of nowadays interest which has been studied in many papers. We refer the interested readers to the recent papers [1, 9, 13, 14, 15, 17, 16, 26, 29, 32, 30, 33, 35, 31]. Among other results in the topic, Wulan et al. [29, Theorem 2] obtained the following one about the compactness of composition operator on the classical Bloch space in the unit disk:

THEOREM 1.1. *Let φ be an analytic self-map of $\mathbb{D}$. Then C_φ is compact on the Bloch space $\mathscr{B}$ if and only if*

$$\lim_{n\to\infty}\|C_\varphi(z^n)\|_{\mathscr{B}} = 0.$$

After that, Ruhan Zhao [32, Corollary 4.4] showed that the essential norm of the composition operator $C_\varphi : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is

$$\|C_\varphi\|_e = \limsup_{n\to\infty}\frac{\|C_\varphi(z^n)\|_\beta}{\|z^n\|_\alpha}.$$

So $C_\varphi : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is compact if and only if $\limsup_{n\to\infty}\frac{\|C_\varphi(z^n)\|_\beta}{\|z^n\|_\alpha} = 0$. Subsequently, several authors have extended the result to differentiation composition operators and weighted composition operators (see, for example, [2, 3, 4, 5, 9, 10, 11, 14, 28, 35]).

In this paper, we are interested in a natural question that arises from those results: is there any similar result about the weighted differentiation composition operators between the Bloch-type spaces? The answer is true, and we will obtain an estimate for their essential norms. For our results, we need the following two integral operators defined by

$$I_u f(z) = \int_0^z f'(\zeta)u(\zeta)d\zeta, \quad J_u f(z) = \int_0^z f(\zeta)u'(\zeta)d\zeta,$$

for every $f \in H(\mathbb{D})$, where $u \in H(\mathbb{D})$.

Throughout this paper, we will use the symbol C to denote a finite positive number, which may differ from one occurrence to the other. The notation $A \asymp B$ means that there is a positive constant C such that $B/C \leqslant A \leqslant CB$.

2. The boundedness of $D^m_{\varphi,u}:\mathscr{B}^\alpha\to\mathscr{B}^\beta$

In this section, we give a new characterization for the boundedness of weighted differentiation composition operators from $\mathscr{B}^\alpha$ to $\mathscr{B}^\beta$. First, we introduce a well-known characterization for the Bloch-type spaces $\mathscr{B}^\alpha$ on the unit disk.

LEMMA 2.1. *For $f\in H(\mathbb{D}), m\in\mathbb{N}$, and $\alpha>0$. Then*

$$f\in\mathscr{B}^\alpha \Leftrightarrow \sup_{z\in\mathbb{D}}(1-|z|^2)^{\alpha+m-1}|f^{(m)}(z)|<\infty.$$

Furthermore, for each $f\in\mathscr{B}^\alpha$,

$$\|f\|_\alpha \asymp \sum_{j=1}^{m-1}|f^{(j)}(0)|+\sup_{z\in\mathbb{D}}(1-|z|^2)^{\alpha+m-1}|f^{(m)}(z)|.$$

The following characterization for the boundedness of operator $D^m_{\varphi,u}:\mathscr{B}^\alpha\to\mathscr{B}^\beta$, which is a natural extension of a result in [20], can be found in [34].

THEOREM 2.1. *Let $0<\alpha,\beta<\infty$, $m\geqslant 1$ be an integer, and $u\in H(\mathbb{D}),\varphi\in S(\mathbb{D})$. Then $D^m_{\varphi,u}:\mathscr{B}^\alpha\to\mathscr{B}^\beta$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta|u'(z)|}{(1-|\varphi(z)|^2)^{\alpha+m-1}}<\infty, \tag{2.1}$$

and

$$\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+m}}<\infty. \tag{2.2}$$

In order to obtain a new characterization, we need the following lemma which can be found in [10].

LEMMA 2.2. *Let $\alpha>0, n\geqslant m+1$, where $m\in\mathbb{N}$. Define the function*

$$H_{n,\alpha}(x)=n(n-1)\cdots(n-m)x^{n-(m+1)}(1-x)^{\alpha+m}, 0\leqslant x\leqslant 1.$$

Then $H_{n,\alpha}$ has the following properties:

(i)

$$\max_{0\leqslant x\leqslant 1}H_{n,\alpha}(x)=H_{n,\alpha}(r_n)$$
$$=\begin{cases}(m+1)!\,, & n=m+1;\\ n(n-1)\cdots(n-m)\left(\frac{n-(m+1)}{n+\alpha-1}\right)^{n-(m+1)}\left(\frac{\alpha+m}{n+\alpha-1}\right)^{\alpha+m}, & n>m+1,\end{cases}$$

where

$$r_n=\begin{cases}0, & n=m+1;\\ \frac{n-(m+1)}{n+\alpha-1}, & n>m+1.\end{cases} \tag{2.3}$$

(ii) For $n \geqslant m+1, H_{n,\alpha}$ is increasing on $[0,r_n]$ and decreasing on $[r_n,1]$.
(iii) For $n \geqslant m+1, H_{n,\alpha}$ is decreasing on $[r_n,r_{n+1}]$, and so

$$\begin{aligned}\min_{x\in[r_n,r_{n+1}]} H_{n,\alpha}(x) &= H_{n,\alpha}(r_{n+1})\\ &= n(n-1)\cdots(n-m)\left(\frac{n-m}{n+\alpha}\right)^{n-(m+1)}\left(\frac{\alpha+m}{n+\alpha}\right)^{\alpha+m}.\end{aligned}$$

Consequently,

$$\lim_{n\to\infty} n^{\alpha-1}\min_{x\in[r_n,r_{n+1}]} H_{n,\alpha}(x) = \left(\frac{\alpha+m}{e}\right)^{\alpha+m}.$$

We can now prove the main result in this section.

THEOREM 2.2. *Let $0<\alpha,\beta<\infty$, $m\geqslant 1$ be an integer, and $u\in H(\mathbb{D}),\varphi\in S(\mathbb{D})$. Then $D^m_{\varphi,u}:\mathscr{B}^\alpha\to\mathscr{B}^\beta$ is bounded if and only if*

$$\sup_{n\geqslant 1} n^{\alpha-1}\|I_uC_\varphi D^m(z^n)\|_\beta<\infty, \tag{2.4}$$

and

$$\sup_{n\geqslant 1} n^{\alpha-1}\|J_uC_\varphi D^m(z^n)\|_\beta<\infty. \tag{2.5}$$

Proof. Suppose that (2.4) and (2.5) hold. For every $f\in\mathscr{B}^\alpha$,

$$\begin{aligned}\|D^m_{\varphi,u}f\|_\beta &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\left|u'(z)f^{(m)}(\varphi(z))+u(z)\varphi'(z)f^{(m+1)}(\varphi(z))\right|\\ &\leqslant \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|u'(z)f^{(m)}(\varphi(z))|+\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|u(z)\varphi'(z)f^{(m+1)}(\varphi(z))|\\ &= I_1+I_2.\end{aligned}$$

To deal with I_2, for any integer $n\geqslant m+1$, let

$$D_n=\{z\in\mathbb{D}: r_n\leqslant|\varphi(z)|\leqslant r_{n+1}\},$$

where r_n is given by (2.3). Let k and l be the smallest and largest positive integers such that $D_k\neq\varnothing$ and $D_l\neq\varnothing$ (l could be ∞). Thus $\mathbb{D}=\cup_{n=k}^{l}D_n$. By Lemma 2.2, for every $k\leqslant n\leqslant l$, we have

$$\begin{aligned}&\min_{z\in D_n} n^{\alpha-1}n(n-1)\cdots(n-m)|\varphi(z)|^{n-(m+1)}(1-|\varphi(z)|)^{\alpha+m}\\ &\geqslant n^{\alpha-1}H_{n,\alpha}(r_{n+1}) = n^{\alpha-1}n(n-1)\cdots(n-m)\left(\frac{n-m}{n+\alpha}\right)^{n-(m+1)}\left(\frac{\alpha+m}{n+\alpha}\right)^{\alpha+m}.\end{aligned}$$

Thus by Lemma 2.2(iii), we have

$$\lim_{n\to\infty}\min_{z\in D_n} n^{\alpha-1}n(n-1)\cdots(n-m)|\varphi(z)|^{n-m-1}(1-|\varphi(z)|)^{\alpha+m}\geqslant\left(\frac{\alpha+m}{e}\right)^{\alpha+m}.$$

From this and since $H_{n,\alpha}(r_n) > 0$ for every $n \geqslant m+1$, there exists a constant $\delta > 0$, independent of n, such that

$$\min_{z\in D_n} n^{\alpha-1} n(n-1)\cdots(n-m)|\varphi(z)|^{n-(m+1)}(1-|\varphi(z)|)^{\alpha+m} \geqslant \delta.$$

From the above mentioned, together with Lemma 2.1, it follows that

$$\begin{aligned}
I_2 &= \sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}|u(z)\varphi'(z)f^{(m+1)}(\varphi(z))| \\
&\leqslant C\|f\|_{\alpha}\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^{\beta}|u(z)\varphi'(z)|}{(1-|\varphi(z)|)^{\alpha+m}} \\
&= C\|f\|_{\alpha}\sup_{k\leqslant n\leqslant l}\sup_{z\in D_n}\frac{n^{\alpha-1}(1-|z|^2)^{\beta}|u(z)\varphi'(z)|n(n-1)\cdots(n-m)|\varphi(z)|^{n-(m+1)}}{n^{\alpha-1}n(n-1)\cdots(n-m)|\varphi(z)|^{n-(m+1)}(1-|\varphi(z)|)^{\alpha+m}} \\
&\leqslant \frac{C\|f\|_{\alpha}}{\delta}\sup_{n\geqslant 1} n^{\alpha-1}\|I_uC_{\varphi}D^m(z^n)\|_{\beta}.
\end{aligned}$$

A similar argument (using $m-1$ instead of m in Lemma 2.1) shows that

$$\begin{aligned}
I_1 &= \sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}|u'(z)f^{(m)}(\varphi(z))| \\
&\leqslant C\|f\|_{\alpha}\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^{\beta}|u'(z)|}{(1-|\varphi(z)|)^{\alpha+m-1}} \\
&= C\|f\|_{\alpha}\sup_{z\in\mathbb{D}}\frac{n^{\alpha-1}(1-|z|^2)^{\beta}|u'(z)|n(n-1)\cdots(n-m+1)|\varphi(z)|^{n-m}}{n^{\alpha-1}n(n-1)\cdots(n-m+1)|\varphi(z)|^{n-m}(1-|\varphi(z)|)^{\alpha+m-1}} \\
&\leqslant \frac{C\|f\|_{\alpha}}{\delta}\sup_{n\geqslant 1} n^{\alpha-1}\|J_uC_{\varphi}D^m(z^n)\|_{\beta}.
\end{aligned}$$

From (2.4), (2.5), and the two inequalities above, we conclude that $D^m_{\varphi,u}:\mathscr{B}^{\alpha}\to\mathscr{B}^{\beta}$ is bounded.

To prove the inverse implication, we assume that $D^m_{\varphi,u}:\mathscr{B}^{\alpha}\to\mathscr{B}^{\beta}$ is bounded, then (2.1) and (2.2) hold from Theorem 2.1. On the other hand, since for $n\geqslant m+1$,

$$\begin{aligned}
&n^{\alpha-1}\|I_uC_{\varphi}D^m(z^n)\|_{\beta} \\
&= n^{\alpha-1}\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}n(n-1)\cdots(n-m)|\varphi(z)|^{n-(m+1)}|u(z)\varphi'(z)| \\
&= n^{\alpha-1}\sup_{z\in\mathbb{D}} n(n-1)\cdots(n-m)|\varphi(z)|^{n-(m+1)} \\
&\quad\times(1-|\varphi(z)|)^{\alpha+m}\frac{(1-|z|^2)^{\beta}|u(z)\varphi'(z)|}{(1-|\varphi(z)|)^{\alpha+m}}.
\end{aligned}\tag{2.6}$$

Besides, applying Lemma 2.2(i), we have

$$\sup_{z\in\mathbb{D}} n(n-1)\cdots(n-m)|\varphi(z)|^{n-(m+1)}(1-|\varphi(z)|)^{\alpha+m}\leqslant H_{n,\alpha}(r_n)$$

and it is easy to see that

$$
\begin{aligned}
&\lim_{n\to\infty} n^{\alpha-1} H_{n,\alpha}(r_n)\\
&= \lim_{n\to\infty} n^{\alpha-1} n(n-1)\cdots(n-m)\Big(\frac{n-(m+1)}{n+\alpha-1}\Big)^{n-(m+1)}\Big(\frac{\alpha+m}{n+\alpha-1}\Big)^{\alpha+m}\\
&= \Big(\frac{\alpha+m}{e}\Big)^{\alpha+m}.
\end{aligned}
$$

Thus there is a constant $C>0$, independent of n, such that

$$n^{\alpha-1}\sup_{z\in\mathbb{D}} n(n-1)\cdots(n-m)|\varphi(z)|^{n-(m+1)}(1-|\varphi(z)|)^{\alpha+m}\leqslant C.$$

This together with (2.2) and (2.6) gives

$$\sup_{n\geqslant m+1} n^{\alpha-1}\|I_uC_\varphi D^m(z^n)\|_\beta<\infty,$$

where we have used the fact $1-x^2\asymp 1-x$ for $x\in[0,1]$. This shows that (2.4) is true.

To prove (2.5), let $n\geqslant m+1$, now we have

$$
\begin{aligned}
&n^{\alpha-1}\|J_uC_\varphi D^m(z^n)\|_\beta\\
&= n^{\alpha-1}\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta n(n-1)\cdots(n-m+1)|\varphi(z)|^{n-m}|u'(z)|\\
&= n^{\alpha-1}\sup_{z\in\mathbb{D}} n(n-1)\cdots(n-m+1)|\varphi(z)|^{n-m}(1-|\varphi(z)|)^{\alpha+m-1}\\
&\quad\times\frac{(1-|z|^2)^\beta|u'(z)|}{(1-|\varphi(z)|)^{\alpha+m-1}}.
\end{aligned}
$$

Using (2.1), (2.5) holds in a similar way. The proof of the theorem is complete. □

3. The essential norm of $D^m_{\varphi,u}:\mathscr{B}^\alpha\to\mathscr{B}^\beta$

In this section, our goal is to estimate the essential norm of the operator $D^m_{\varphi,u}$ acting from $\mathscr{B}^\alpha$ to $\mathscr{B}^\beta$, then the estimation will lead to a condition for the operator to be compact directly. The following lemma is the crucial criterion for compactness, which can be proved similarly to Proposition 3.11 of [1].

LEMMA 3.1. *Let $0<\alpha,\beta<\infty$, $m\geqslant 1$ be an integer, and $u\in H(\mathbb{D}),\varphi\in S(\mathbb{D})$. Then $D^m_{\varphi,u}:\mathscr{B}^\alpha\to\mathscr{B}^\beta$ is compact if and only if it is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathscr{B}^\alpha$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, we have $\|D^m_{\varphi,u}f_k\|_{\mathscr{B}^\beta}\to 0$, as $k\to\infty$.*

In order to prove the upper estimate for the essential norm, we need several lemmas. First, we introduce some notations which will used in the following lemmas.

For $r\in[0,1]$, let $K_rf(z)=f(rz)$. It is known that K_r is a compact operator acting on $\mathscr{B}^\alpha$ (or $\mathscr{B}^\alpha_0$) for $\alpha>0$ with $\|K_r\|\leqslant 1$. The following three lemmas corresponds respectively to the three different cases $0<\alpha<1,\alpha=1$ and $\alpha>1$ of Bloch-type spaces. They can be found in earlier papers, and we omit the proofs here.

LEMMA 3.2. *[13, Lemma 1] Let* $0 < \alpha < 1$. *Then there is a sequence* $\{r_k\}, 0 < r_k < 1$, *tending to 1, such that the compact operator*

$$L_n = \frac{1}{n}\sum_{k=1}^{n} K_{r_k}$$

on $\mathscr{B}_0^\alpha$ *satisfies*

(i) For any $t \in [0,1)$, $\lim\limits_{n\to\infty} \sup\limits_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1} \sup\limits_{|z|\leqslant t} |[(I-L_n)f]'(z)| = 0$.

(ii) $\lim\limits_{n\to\infty} \sup\limits_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1} \sup\limits_{z\in\mathbb{D}} |(I-L_n)f(z)| = 0$.

(iii) $\limsup\limits_{n\to\infty} \|I-L_n\| \leqslant 1$.

Furthermore, these statements hold as well for the sequence of biadjoints L_n^{**} *on* $\mathscr{B}^\alpha$.

LEMMA 3.3. *[13, Lemma 2] There is a sequence* $\{r_k\}, 0 < r_k < 1$, *tending to 1, such that the compact operator*

$$L_n = \frac{1}{n}\sum_{k=1}^{n} K_{r_k}$$

on $\mathscr{B}_0$ *satisfies*

(i) For any $t \in [0,1)$, $\lim\limits_{n\to\infty} \sup\limits_{\|f\|_{\mathscr{B}}\leqslant 1} \sup\limits_{|z|\leqslant t} |((I-L_n)f)'(z)| = 0$.

(iia) $\limsup\limits_{n\to\infty} \sup\limits_{\|f\|_{\mathscr{B}}\leqslant 1} \sup\limits_{|z|>s} |(I-L_n)f(z)|\Big(-\log(1-|z|^2)\Big)^{-1} \leqslant 1$ *for* s *sufficiently close to 1 and*

(iib) $\lim\limits_{n\to\infty} \sup\limits_{\|f\|_{\mathscr{B}}\leqslant 1} \sup\limits_{|z|\leqslant s} |(I-L_n)f(z)| = 0$, *for the above* s.

(iii) $\limsup\limits_{n\to\infty} \|I-L_n\| \leqslant 1$.

Furthermore, the same is true for the sequence of biadjoints L_n^{**} *on* $\mathscr{B}$.

LEMMA 3.4. *[32, Lemma 4.3] Let* $\alpha > 1$. *Then there is a sequence* $\{r_k\}$, *with* $0 < r_k < 1$ *tending to* 1, *such that the compact operator*

$$L_n = \frac{1}{n}\sum_{k=1}^{n} K_{r_k}$$

on $\mathscr{B}_0^\alpha$ *satisfies:*

(i) For any $t \in [0,1)$, $\lim\limits_{n\to\infty} \sup\limits_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1} \sup\limits_{|z|\leqslant t} |((I-L_n)f)'(z)| = 0$.

(ii) For any $s \in [0,1)$, $\lim\limits_{n\to\infty} \sup\limits_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1} \sup\limits_{|z|\leqslant s} |(I-L_n)f(z)| = 0$.

(iii) $\limsup\limits_{n\to\infty} \|I-L_n\| \leqslant 1$.

Furthermore, these statements hold as well for the sequence of biadjoints L_n^{**} *on* $\mathscr{B}^\alpha$.

In order to simplify the inequalities, we use the notations

$$A=\left(\frac{e}{\alpha+m-1}\right)^{\alpha+m-1}\limsup_{n\to\infty} n^{\alpha-1}\|J_uC_\varphi D^m(z^n)\|_\beta,$$

$$B=\left(\frac{e}{\alpha+m}\right)^{\alpha+m}\limsup_{n\to\infty} n^{\alpha-1}\|I_uC_\varphi D^m(z^n)\|_\beta.$$

THEOREM 3.1. *Let $0<\alpha,\beta<\infty$, $m\geqslant 1$ be an integer, and $u\in H(\mathbb{D}),\varphi\in S(\mathbb{D})$. Suppose that the operator $D^m_{\varphi,u}$ is bounded from $\mathscr{B}^\alpha$ to $\mathscr{B}^\beta$. Then*

$$\max\left(\frac{(\alpha+1)(\alpha+2)\cdots(\alpha+m-1)}{2^{\alpha+1}(3\alpha+m+3)}A,\frac{(\alpha+1)(\alpha+2)\cdots(\alpha+m)}{2^{\alpha+1}(3\alpha+m+2)}B\right)$$
$$\leqslant \|D^m_{\varphi,u}\|_e\leqslant A+B. \tag{3.1}$$

Proof. Suppose $D^m_{\varphi,u}$ is bounded from $\mathscr{B}^\alpha$ to $\mathscr{B}^\beta$, that is, there is a constant C such that

$$\|D^m_{\varphi,u}f\|_\beta\leqslant C\|f\|_\alpha,\ \text{ for every } f\in\mathscr{B}^\alpha.$$

By choosing $f(z)=\frac{z^m}{m!}$ and $f(z)=\frac{z^{m+1}}{(m+1)!}$, we have

$$M_1:=\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|u'(z)|<\infty, \tag{3.2}$$

and

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|u'(z)\varphi(z)+u(z)\varphi'(z)|<\infty. \tag{3.3}$$

From (3.2) and (3.3) and the boundedness of function φ, we can easily prove that

$$M_2:=\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|u(z)\varphi'(z)|<\infty. \tag{3.4}$$

Now, we first show that (3.1) is true when $\sup_{z\in\mathbb{D}}|\varphi(z)|<1$. In fact, for this case, there is a number $r\in(0,1)$, such that $\sup_{z\in\mathbb{D}}|\varphi(z)|\leqslant r$. By (3.2) and (3.4), it follows that

$$\begin{aligned}&n^{\alpha-1}\|J_uC_\varphi D^m(z^n)\|_\beta\\&=n^{\alpha-1}\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta n(n-1)\cdots(n-m+1)|\varphi(z)|^{n-m}|u'(z)|\\&\leqslant M_1n^{\alpha-1}n(n-1)\cdots(n-m+1)r^{n-m}\end{aligned}$$

and

$$\begin{aligned}
&n^{\alpha-1}\|I_u C_\varphi D^m(z^n)\|_\beta \\
&= n^{\alpha-1}\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta n(n-1)\cdots(n-m)|\varphi(z)|^{n-(m+1)}|u(z)\varphi'(z)| \\
&\leqslant M_2 n^{\alpha-1} n(n-1)\cdots(n-m)r^{n-(m+1)}.
\end{aligned}$$

Hence

$$\lim_{n\to\infty} n^{\alpha-1}\|J_u C_\varphi D^m(z^n)\|_\beta = \lim_{n\to\infty} n^{\alpha-1}\|I_u C_\varphi D^m(z^n)\|_\beta = 0.$$

That is, $A = B = 0$.

On the other hand, let $(f_k)_{k\in\mathbb{N}}$ be any bounded sequence in $\mathscr{B}^\alpha$, and suppose that $(f_k)_{k\in\mathbb{N}}$ converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k\to\infty$. Then by Cauchy's integral formula, we obtain

$$\begin{aligned}
&\|D^m_{\varphi,u} f_k\|_\beta = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta \left|u'(z)f_k^{(m)}(\varphi(z)) + u(z)\varphi'(z)f_k^{(m+1)}(\varphi(z))\right| \\
&\leqslant \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |u'(z)f_k^{(m)}(\varphi(z))| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |u(z)\varphi'(z)f_k^{(m+1)}(\varphi(z))| \\
&\leqslant M_1 \sup_{z\in\mathbb{D}}|f_k^{(m)}(\varphi(z))| + M_2 \sup_{z\in\mathbb{D}}|f_k^{(m+1)}(\varphi(z))| \\
&\to 0 \ \text{as} \ k\to\infty,
\end{aligned}$$

so $D^m_{\varphi,u}$ is compact from $\mathscr{B}^\alpha$ to $\mathscr{B}^\beta$ by Lemma 3.1, that is, $\|D^m_{\varphi,u}\|_e = 0$. Consequently, for the case $\sup_{z\in\mathbb{D}}|\varphi(z)| < 1$, the essential norm formula is true.

This reduces the proof of the theorem to the case $\sup_{z\in\mathbb{D}}|\varphi(z)| = 1$. First, we intend to get the upper estimate. Let $\{L_n\}$ be the sequence of operators given in Lemmas 3.2-3.4. Since each L_n is compact as an operator from $\mathscr{B}^\alpha$ to $\mathscr{B}^\beta$, then $D^m_{\varphi,u}L_n$ is also compact since $D^m_{\varphi,u} : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is bounded. Thus

$$\begin{aligned}
&\|D^m_{\varphi,u}\|_e \leqslant \limsup_{n\to\infty}\|D^m_{\varphi,u} - D^m_{\varphi,u}L_n\| = \limsup_{n\to\infty}\|D^m_{\varphi,u}(I-L_n)\| \\
&= \limsup_{n\to\infty}\sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1}\|D^m_{\varphi,u}(I-L_n)f\|_{\mathscr{B}^\beta} \\
&\leqslant \limsup_{n\to\infty}\sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1}|u(0)|\left|[(I-L_n)f]^{(m)}(\varphi(0))\right| \\
&\quad + \limsup_{n\to\infty}\sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1}\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|u'(z)|\left|[(I-L_n)f]^{(m)}(\varphi(z))\right| \\
&\quad + \limsup_{n\to\infty}\sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1}\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|u(z)\varphi'(z)|\left|[(I-L_n)f]^{(m+1)}(\varphi(z))\right|. \qquad (3.5)
\end{aligned}$$

By Lemma 3.2(ii), Lemma 3.3(iib) and Lemma 3.4(ii) and Cauchy's integral formula, we obtain that

$$\limsup_{n\to\infty}\sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1}|u(0)|\left|[(I-L_n)f]^{(m)}(\varphi(0))\right| = 0. \qquad (3.6)$$

Next we consider the term

$$J := \sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1} \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |u'(z)| \left|[(I-L_n)f]^{(m)}(\varphi(z))\right|.$$

For each integer $n \geqslant m+1$, denote

$$D_n = \{z \in \mathbb{D} : r_n \leqslant |\varphi(z)| \leqslant r_{n+1}\},$$

where r_n is given by (2.3). Let k be the smallest positive integers such that $D_k \neq \varnothing$. Since $\sup_{z\in\mathbb{D}} |\varphi(z)| = 1, D_n$ is not empty for every integer $n \geqslant k$, then $\mathbb{D} = \cup_{n=k}^{\infty} D_n$. We divide J into two parts:

$$\begin{aligned} J =& \sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1} \sup_{k\leqslant i\leqslant N-1} \sup_{z\in D_i} (1-|z|^2)^\beta |u'(z)| \left|[(I-L_n)f]^{(m)}(\varphi(z))\right| \\ &+ \sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1} \sup_{i\geqslant N} \sup_{z\in D_i} (1-|z|^2)^\beta |u'(z)| \left|[(I-L_n)f]^{(m)}(\varphi(z))\right| \\ =& J_1 + J_2, \end{aligned}$$

where N is a positive integer determined as follows. Consider the term

$$\begin{aligned} &(1-|z|^2)^\beta |u'(z)| \left|[(I-L_n)f]^{(m)}(\varphi(z))\right| \\ =& \frac{i^{\alpha-1} i(i-1)\cdots(i-m+1)|\varphi(z)|^{i-m}(1-|z|^2)^\beta |u'(z)|}{i^{\alpha-1} i(i-1)\cdots(i-m+1)|\varphi(z)|^{i-m}(1-|\varphi(z)|)^{\alpha+m-1}} \\ &\cdot (1-|\varphi(z)|)^{\alpha+m-1} \left|[(I-L_n)f]^{(m)}(\varphi(z))\right|. \end{aligned}$$

By Lemma 2.2, for $z \in D_i$,

$$\begin{aligned} &i^{\alpha-1} i(i-1)\cdots(i-m+1)|\varphi(z)|^{i-m}(1-|\varphi(z)|)^{\alpha+m-1} \\ \geqslant& i^{\alpha-1} i(i-1)\cdots(i-m+1)\left(\frac{i-m+1}{i+\alpha}\right)^{i-m}\left(\frac{\alpha+m-1}{i+\alpha}\right)^{\alpha+m-1}. \end{aligned}$$

An easy calculation shows that

$$\begin{aligned} &\lim_{i\to\infty} i^{\alpha-1} i(i-1)\cdots(i-m+1)\left(\frac{i-m+1}{i+\alpha}\right)^{i-m}\left(\frac{\alpha+m-1}{i+\alpha}\right)^{\alpha+m-1} \\ =& \left(\frac{\alpha+m-1}{e}\right)^{\alpha+m-1}. \end{aligned}$$

Hence, for any $\varepsilon > 0$, there exists $N > m+1$ large enough such that for any $i \geqslant N$,

$$\left[i^{\alpha-1} i(i-1)\cdots(i-m+1)|\varphi(z)|^{i-m}(1-|\varphi(z)|)^{\alpha+m-1}\right]^{-1} < \left(\frac{e}{\alpha+m-1}\right)^{\alpha+m-1} + \varepsilon.$$

For such N it follows that

$$\begin{aligned}
J_2 &= \sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1}\sup_{i\geqslant N}\sup_{z\in D_i}(1-|z|^2)^\beta|u'(z)|\left|[(I-L_n)f]^{(m)}(\varphi(z))\right| \\
&\leqslant \left[\left(\frac{e}{\alpha+m-1}\right)^{\alpha+m-1}+\varepsilon\right]\sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1}\|(I-L_n)f\|_{\mathscr{B}^\alpha} \\
&\quad\cdot\sup_{i\geqslant N}\sup_{z\in D_i} i^{\alpha-1}i(i-1)\cdots(i-m+1)|\varphi(z)|^{i-m}(1-|z|^2)^\beta|u'(z)| \\
&\leqslant \left[\left(\frac{e}{\alpha+m-1}\right)^{\alpha+m-1}+\varepsilon\right]\|I-L_n\|\sup_{i\geqslant N} i^{\alpha-1}\|J_uC_\varphi D^m(z^i)\|_\beta.
\end{aligned}$$

Thus using (iii) of Lemmas 3.2-3.4, we obtain that

$$\limsup_{n\to\infty} J_2 \leqslant \left[\left(\frac{e}{\alpha+m-1}\right)^{\alpha+m-1}+\varepsilon\right]\sup_{i\geqslant N} i^{\alpha-1}\|J_uC_\varphi D^m(z^i)\|_\beta. \tag{3.7}$$

For J_1, by (i) of Lemmas 3.2-3.4, along with Cauchy's integral formula, we have

$$\begin{aligned}
\limsup_{n\to\infty} J_1 &= \limsup_{n\to\infty}\sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1}\sup_{k\leqslant i\leqslant N-1}\sup_{z\in D_i}(1-|z|^2)^\beta|u'(z)|\left|[(I-L_n)f]^{(m)}(\varphi(z))\right| \\
&= \limsup_{n\to\infty}\sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1}\sup_{r_k\leqslant|\varphi(z)|\leqslant r_N}(1-|z|^2)^\beta|u'(z)|\left|[(I-L_n)f]^{(m)}(\varphi(z))\right| \\
&\leqslant M_1\limsup_{n\to\infty}\sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1}\sup_{r_k\leqslant|\varphi(z)|\leqslant r_N}\left|[(I-L_n)f]^{(m)}(\varphi(z))\right| \\
&= 0.
\end{aligned} \tag{3.8}$$

From (3.7) and (3.8), we conclude

$$\limsup_{n\to\infty} J \leqslant \left[\left(\frac{e}{\alpha+m-1}\right)^{\alpha+m-1}+\varepsilon\right]\sup_{i\geqslant N} i^{\alpha-1}\|J_uC_\varphi D^m(z^i)\|_\beta. \tag{3.9}$$

By the same argument for J, we can prove that

$$\begin{aligned}
&\limsup_{n\to\infty}\sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1}\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|u(z)\varphi'(z)|\left|[(I-L_n)f]^{(m+1)}(\varphi(z))\right| \\
&\leqslant \left[\left(\frac{e}{\alpha+m}\right)^{\alpha+m}+\varepsilon\right]\sup_{i\geqslant N} i^{\alpha-1}\|I_uC_\varphi D^m(z^i)\|_\beta.
\end{aligned} \tag{3.10}$$

Then by (3.5), (3.6), (3.9) and (3.10), it is clear that

$$\begin{aligned}
\|D^m_{\varphi,u}\|_e \leqslant &\left[\left(\frac{e}{\alpha+m-1}\right)^{\alpha+m-1}+\varepsilon\right]\sup_{i\geqslant N} i^{\alpha-1}\|J_uC_\varphi D^m(z^i)\|_\beta \\
&+\left[\left(\frac{e}{\alpha+m}\right)^{\alpha+m}+\varepsilon\right]\sup_{i\geqslant N} i^{\alpha-1}\|I_uC_\varphi D^m(z^i)\|_\beta.
\end{aligned}$$

Letting $\varepsilon\to 0$, the desired result of the upper estimate follows.

Now, still under the assumption that $\sup_{z\in\mathbb{D}}|\varphi(z)|=1$, we give a proof for the lower estimate. Let $D^m_{\varphi,u}:\mathscr{B}^\alpha\to\mathscr{B}^\beta$ be bounded. Taking any compact operator $K:\mathscr{B}^\alpha\to\mathscr{B}^\beta$, then for any sequence $\{f_k\}$ in $\mathscr{B}^\alpha$ with $\|f_k\|_{\mathscr{B}^\alpha}\leqslant 1$, and $f_k\to 0$ weakly in $\mathscr{B}^\alpha$, we know that $\lim_{k\to\infty}\|Kf_k\|_{\mathscr{B}^\beta}=0$ (see, for example, [15] or [17]). Hence

$$\|D^m_{\varphi,u}-K\|\geqslant\limsup_{k\to\infty}\|(D^m_{\varphi,u}-K)f_k\|_\beta\geqslant\limsup_{k\to\infty}\|D^m_{\varphi,u}f_k\|_\beta.$$

Thus by the arbitrariness of K,

$$\|D^m_{\varphi,u}\|_e\geqslant\limsup_{k\to\infty}\|D^m_{\varphi,u}f_k\|_\beta.$$

Specially, choosing a sequence $(z_k)_{k\in\mathbb{N}}$ in $\mathbb{D}$ such that $|\varphi(z_k)|\to 1$ as $k\to\infty$, consider the function g_k defined by

$$g_k(z)=\frac{(1-|\varphi(z_k)|^2)^2}{(1-\overline{\varphi(z_k)}z)^{\alpha+1}}-\frac{\alpha+m+1}{\alpha}\frac{1-|\varphi(z_k)|^2}{(1-\overline{\varphi(z_k)}z)^{\alpha}}.$$

It is easy to check that $g_k\in\mathscr{B}^\alpha$ with $\|g_k\|_{\mathscr{B}^\alpha}\leqslant|g_k(0)|+2^{\alpha+1}(3\alpha+m+3)$ and $g_k\to 0$ uniformly on compact subsets of $\mathbb{D}$, which can prove in a similar way of [9].

Let $g^*_k(z)=g_k(z)/\|g_k\|_{\mathscr{B}^\alpha}$. Then it is clearly that $\|g^*_k\|_{\mathscr{B}^\alpha}=1$ and $g^*_k\to 0$ uniformly on compact subsets of $\mathbb{D}$. Together with $g_k^{(m+1)}(\varphi(z_k))=0$ and

$$g_k^{(m)}(\varphi(z_k))=-\frac{(\alpha+1)(\alpha+2)\cdots(\alpha+m-1)\overline{\varphi(z_k)}^m}{(1-|\varphi(z_k)|^2)^{\alpha+m-1}},$$

we get

$$\begin{aligned}\|D^m_{\varphi,u}\|_e&\geqslant\limsup_{k\to\infty}\|D^m_{\varphi,u}g^*_k\|_{\mathscr{B}^\beta}=\limsup_{k\to\infty}\frac{\|D^m_{\varphi,u}g_k\|_{\mathscr{B}^\beta}}{\|g_k\|_{\mathscr{B}^\alpha}}\geqslant\limsup_{k\to\infty}\frac{\|D^m_{\varphi,u}g_k\|_\beta}{\|g_k\|_{\mathscr{B}^\alpha}}\\&\geqslant\limsup_{k\to\infty}\frac{(\alpha+1)(\alpha+2)\cdots(\alpha+m-1)|\varphi(z_k)|^m(1-|z_k|^2)^\beta|u'(z_k)|}{\Big[|g_k(0)|+2^{\alpha+1}(3\alpha+m+3)\Big](1-|\varphi(z_k)|^2)^{\alpha+m-1}}\\&=\frac{(\alpha+1)(\alpha+2)\cdots(\alpha+m-1)}{2^{\alpha+1}(3\alpha+m+3)}\lim_{|\varphi(z)|\to1}\frac{(1-|z|^2)^\beta|u'(z)|}{(1-|\varphi(z)|^2)^{\alpha+m-1}},\end{aligned}\tag{3.11}$$

which we have used the fact that $|g_k(0)|\to 0$ as $k\to\infty$. On the other hand, since

$$\begin{aligned}&n^{\alpha-1}\|J_uC_\varphi D^m(z^n)\|_\beta\\&=\sup_{z\in\mathbb{D}}n^{\alpha-1}n(n-1)\cdots(n-m+1)|\varphi(z)|^{n-m}|u'(z)|(1-|z|^2)^\beta\\&=I^1_n+I^2_n,\end{aligned}\tag{3.12}$$

where

$$I^1_n=\sup_{|\varphi(z)|\leqslant s}n^{\alpha-1}n(n-1)\cdots(n-m+1)|\varphi(z)|^{n-m}|u'(z)|(1-|z|^2)^\beta,$$

$$I_n^2 = \sup_{|\varphi(z)|>s} n^{\alpha-1} n(n-1)\cdots(n-m+1)|\varphi(z)|^{n-m}|u'(z)|(1-|z|^2)^\beta,$$

and $s \in (0,1)$. Using Lemma 2.2(i), we have

$$\begin{aligned} I_n^2 &= \sup_{|\varphi(z)|>s} n^{\alpha-1} n(n-1)\cdots(n-m+1)|\varphi(z)|^{n-m} \\ &\quad \times (1-|\varphi(z)|)^{\alpha+m-1} \frac{(1-|z|^2)^\beta |u'(z)|}{(1-|\varphi(z)|)^{\alpha+m-1}} \\ &\leqslant n^{\alpha-1} n(n-1)\cdots(n-m+1)\Big(\frac{n-m}{n+\alpha-1}\Big)^{n-m}\Big(\frac{\alpha+m-1}{n+\alpha-1}\Big)^{\alpha+m-1} \\ &\quad \times \sup_{|\varphi(z)|>s} \frac{(1-|z|^2)^\beta |u'(z)|}{(1-|\varphi(z)|)^{\alpha+m-1}}. \end{aligned}$$

Noting that

$$\begin{aligned} &\lim_{n\to\infty} n^{\alpha-1} n(n-1)\cdots(n-m+1)\Big(\frac{n-m}{n+\alpha-1}\Big)^{n-m}\Big(\frac{\alpha+m-1}{n+\alpha-1}\Big)^{\alpha+m-1} \\ &= \Big(\frac{\alpha+m-1}{e}\Big)^{\alpha+m-1}, \end{aligned}$$

thus for any fixed $s \in (0,1)$, we have

$$\limsup_{n\to\infty} I_n^2 \leqslant \Big(\frac{\alpha+m-1}{e}\Big)^{\alpha+m-1} \sup_{|\varphi(z)|>s} \frac{(1-|z|^2)^\beta |u'(z)|}{(1-|\varphi(z)|)^{\alpha+m-1}}. \tag{3.13}$$

For I_n^1, it is easy to see that

$$\limsup_{n\to\infty} I_n^1 \leqslant M_1 \limsup_{n\to\infty} n^{\alpha-1} n(n-1)\cdots(n-m+1)s^{n-m} = 0. \tag{3.14}$$

From (3.12)-(3.14) we conclude that

$$\limsup_{n\to\infty} n^{\alpha-1}\|J_u C_\varphi D^m(z^n)\|_\beta \leqslant \Big(\frac{\alpha+m-1}{e}\Big)^{\alpha+m-1} \sup_{|\varphi(z)|>s} \frac{(1-|z|^2)^\beta |u'(z)|}{(1-|\varphi(z)|)^{\alpha+m-1}}$$

for any fixed $s \in (0,1)$. Letting $s \to 1$, we conclude

$$A \leqslant \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |u'(z)|}{(1-|\varphi(z)|^2)^{\alpha+m-1}}. \tag{3.15}$$

Thus from (3.11) and (3.15), it follows that

$$\|D_{\varphi,u}^m\|_e \geqslant \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+m-1)}{2^{\alpha+1}(3\alpha+m+3)} A.$$

At last, we proceed to prove the other lower estimate in a similar way. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. Consider the function h_k defined by

$$h_k(z) = \frac{(1-|\varphi(z_k)|^2)^2}{(1-\overline{\varphi(z_k)}z)^{\alpha+1}} - \frac{\alpha+m}{\alpha}\frac{1-|\varphi(z_k)|^2}{(1-\overline{\varphi(z_k)}z)^{\alpha}}.$$

The fact that $h_k \in \mathscr{B}^\alpha$ with $\|h_k\|_{\mathscr{B}^\alpha} \leqslant |h_k(0)| + 2^{\alpha+1}(3\alpha+m+2)$ and $h_k \to 0$ uniformly on compact subsets of $\mathbb{D}$, can be proved analogously.

Let $h_k^*(z) = h_k(z)/\|h_k\|_{\mathscr{B}^\alpha}$. Then it is clearly that $\|h_k^*\|_{\mathscr{B}^\alpha} = 1$ and $h_k^* \to 0$ uniformly on compact subsets of $\mathbb{D}$. Combine with $h_k^{(m)}(\varphi(z_k)) = 0$ and

$$h_k^{(m+1)}(\varphi(z_k)) = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+m)\overline{\varphi(z_k)}^{m+1}}{(1-|\varphi(z_k)|^2)^{\alpha+m}},$$

then

$$\begin{aligned}\|D_{\varphi,u}^m\|_e &\geqslant \limsup_{k\to\infty} \|D_{\varphi,u}^m h_k^*\|_{\mathscr{B}^\beta} = \limsup_{k\to\infty} \frac{\|D_{\varphi,u}^m h_k\|_{\mathscr{B}^\beta}}{\|h_k\|_{\mathscr{B}^\alpha}} \geqslant \limsup_{k\to\infty} \frac{\|D_{\varphi,u}^m h_k\|_\beta}{\|h_k\|_{\mathscr{B}^\alpha}} \\ &\geqslant \limsup_{k\to\infty} \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+m)|\varphi(z_k)|^{m+1}(1-|z_k|^2)^\beta |u(z_k)\varphi'(z_k)|}{\Big[|h_k(0)| + 2^{\alpha+1}(3\alpha+m+2)\Big](1-|\varphi(z_k)|^2)^{\alpha+m}} \\ &= \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+m)}{2^{\alpha+1}(3\alpha+m+2)} \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+m}}.\end{aligned}$$

Similarly as in the proof of (3.15), we obtain

$$B \leqslant \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+m}}.$$

Thus

$$\|D_{\varphi,u}^m\|_e \geqslant \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+m)}{2^{\alpha+1}(3\alpha+m+2)} B.$$

The proof is complete. □

From the above theorem and the well-know relationship between the compactness of an operator and its essential norm, it is easy to obtain the following corollary.

COROLLARY 3.1. *Let $0 < \alpha, \beta < \infty$, $m \geqslant 1$ be an integer, and $u \in H(\mathbb{D}), \varphi \in S(\mathbb{D})$. Suppose that the operator $D_{\varphi,u}^m$ is bounded from $\mathscr{B}^\alpha$ to $\mathscr{B}^\beta$. Then $D_{\varphi,u}^m : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is compact if and only if*

$$\limsup_{n\to\infty} n^{\alpha-1} \|J_u C_\varphi D^m(z^n)\|_\beta = 0$$

and

$$\limsup_{n\to\infty} n^{\alpha-1} \|I_u C_\varphi D^m(z^n)\|_\beta = 0.$$

REFERENCES

[1] C. C. COWEN AND B. D. MACCLUER, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, FL, 1995.

[2] Z. S. FANG AND Z. H. ZHOU, *Essential norms of composition operators between Bloch type spaces in the polydisk*, Arch. Math., **99**, (2012), 547–556.

[3] Z. S. FANG AND Z. H. ZHOU, *New characterizations of the weighted composition operators between Bloch type spaces in the polydisk*, Canad. Math. Bull., **57**, 4 (2014), 794–802.

[4] O. HYVÄRINEN, M. KEMAPPAINEN, M. LINDSTRÖM, A. RAUTIO AND E. SAUKKO, *The essential norms of weighted composition operators on weighted Banach spaces of analytic function*, Integr. Equ. Oper. Theory, **72**, (2012), 151–157.

[5] O. HYVÄRINEN AND M. LINDSTRÖM, *Estimates of essential norms of weighted composition operators between Bloch-type spaces*, J. Math. Anal. Appl., **393**, (2012), 38–44.

[6] S. LI AND S. STEVIĆ, *Composition followed by differentiation between Bloch type spaces*, J. Comput. Anal. Appl., **9**, 2 (2007), 195–205.

[7] S. LI AND S. STEVIĆ, *Composition followed by differentiation from mixed-norm spaces to α-Bloch spaces*, Sb. Math., **199**, 12 (2008), 1847–1857.

[8] S. LI AND S. STEVIĆ, *Composition followed by differentiation between H^∞ and α-Bloch spaces*, Houston J. Math., **35**, 1 (2009), 327–340.

[9] S. LI AND S. STEVIĆ, *Weighted differentiation composition operators from the logarithmic Bloch space to the weighted-type space*, An. Stiint. Univ. ”Ovidius” Constanta Ser. Mat., **24**, 3 (2016), 223–240.

[10] Y. X. LIANG AND Z. H. ZHOU, *Essential norms of the product of differentiation and composition operators between Bloch-type spaces*, Arch. Math., **100**, 4, (2013), 347–360.

[11] Y. X. LIANG AND Z. H. ZHOU, *New estimate of essential norm of composition followed by differentiation between Bloch-type spces*, Banach J. Math. Anal., **8**, (2014), 118–137.

[12] Y. LIU AND Y. YU, *Weighted differentiation composition operators from mixed-norm to Zygmund spaces*, Numer. Func. Anal. Opt., **31**, 8 (2010), 936–954.

[13] B. D. MACCLUER AND R. ZHAO, *Essential norms of weighted composition operators between Bloch-type spaces*, Rocky Mountain J. Math., **33**, (2003),1437–1458.

[14] J. S. MANHAS AND R. ZHAO, *New estimates of essential norms of weighted composition operators between Bloch type spaces*, J. Math. Anal. Appl., **389**, (2012), 32–47.

[15] S. STEVIĆ, *Essential norms of weighted composition operators from the $\alpha-$Bloch space to a weighted-type space on the unit ball*, Abstr. Appl. Anal., **2008**, (2008), Article ID 279691.

[16] S. STEVIĆ, *Norm and essential norm of composition followed by differentiation from α-Bloch spaces to H_μ^∞*, Appl. Math. Comput., **207**, (2009), 225–229.

[17] S. STEVIĆ, *On a new integral-type operator from the Bloch space to Bloch-type spaces on the unit ball*, J. Math. Anal. Appl., **354**, (2009), 426–434.

[18] S. STEVIĆ, *Products of composition and differentiation operators on the weighted Bergman space*, Bull. Belg. Math. Soc. Simon Stevin, **16**, (2009), 623–635.

[19] S. STEVIĆ, *Weighted differentiation composition operators from mixed-norm spaces to weighted-type spaces*, Appl. Math. Comput., **211**, (2009), 222–233.

[20] S. STEVIĆ, *Composition followed by differentiation from H^∞ and the Bloch space to nth weighted-type spaces on the unit disk*, Appl. Math. Comput., **216**, (2010), 3450–3458.

[21] S. STEVIĆ, *Weighted differentiation composition operators from H^∞ and Bloch spaces to nth weighted-type spaces on the unit disk*, Appl. Math. Comput., **216**, 12 (2010), 3634–3641.

[22] S. STEVIĆ, *Weighted differentiation composition operators from the mixed-norm space to the nth weighted-type space on the unit disk*, Abstr. Appl. Anal., **15**, (2010), Article ID 246287.

[23] S. STEVIĆ, *Weighted iterated radial composition operators between some spaces of holomorphic functions on the unit ball*, Abstr. Appl. Anal., **2010**, (2010), Article ID 801264.

[24] S. STEVIĆ, *Characterizations of composition followed by differentiation between Bloch-type spaces*, Appl. Math. Comput., **218**, (2011), 4312–4316.

[25] S. STEVIĆ, *Weighted radial operator from the mixed-norm space to the nth weighted-type space on the unit ball*, Appl. Math. Comput., **218**, (2012) 9241–9247.

[26] S. STEVIĆ, A.K. SHARMA AND A. BHAT, *Essential norm of products of multiplication composition and differentiation operators on weighted Bergman spaces*, Appl. Math. Comput., **218**, (2011), 2386–2397.

[27] S. STEVIĆ, A. K. SHARMA AND A. BHAT, *Products of multiplication composition and differentiation operators on weighted Bergman spaces*, Appl. Math. Comput., **217**, (2011), 8115–8125.

[28] Y. WU AND H. WULAN, *Products of differentiation and composition operators on the Bloch Space*, Collect. Math., **63**, (2012), 93–107.

[29] H. WULAN, D. ZHENG AND K. ZHU, *Compact composition operators on BMOA and the Bloch space*, Proc. Amer. Math. Soc., **137**, (2009), 3861–3868.

[30] Y. YU AND Y. LIU, *The essential norm of a generalized composition operator between Bloch-type spaces and Q_K type spaces*, Complex Anal. Oper. Th., **6**, (2012) 1231–1240.

[31] H. G. ZENG AND Z. H. ZHOU, *Essential norm estimate of a composition operator between Bloch-type spaces in the unit ball*, Rocky Mountain J. Math., **42**, 3 (2012), 1049–1071.

[32] R. ZHAO, *Essential norms of composition operators between Bloch type spaces*, Proc. Amer. Math. Soc., **138**, (2010), 2537–2546.

[33] Z. H. ZHOU, J. H. SHI, *Compactness of composition operators on the Bloch space in classical bounded symmetric domains*, Michigan Math. J., **50**, (2002), 381–405.

[34] X. L. ZHU, *Generalized weighted composition operators on Bloch-type spaces*, J. Inequal. Appl., 2015, **59** (2015), 1–9.

[35] X. L. ZHU, *Essential norm of generalized weighted composition operators on Bloch-type spaces*, Appl. Math. Comput., **274**, 1 (2016), 133–142.

(Received October 21, 2018)

Cui Chen
Department of Mathematics
Tianjin University of Finance and Economics
Tianjin 300222, P.R. China
e-mail: chencui_cc@126.com

Yu-Xia Liang
School of Mathematical Sciences
Tianjin Normal University
Tianjin 300387, P.R. China
e-mail: liangyx1986@126.com