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NORMS OF SUMMABILITY AND HAUSDORFF MEAN MATRICES ON DIFFERENCE SEQUENCE SPACES

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Dedicated to Prof. Maryam Mirzakhani who in spite of a short lifetime, left a long-standing impact on mathematics

(*Communicated by M. Praljak*)

Abstract. In this paper, we compute the norms of summability and Hausdorff mean matrices on difference sequence space bv_p. Moreover, as an application, we derive the main result of [5].

1. Introduction

The idea of difference sequence spaces was introduced by Kizmaz [4]. The backward difference matrix $\Delta = (\delta_{j,k})$ and its inverse $\Delta^{-1} = (\delta^{-1}_{j,k})$ are

$$\delta_{j,k} = \begin{cases} 1 & k = j, \\ -1 & k = j-1, \\ 0 & otherwise, \end{cases} \qquad and \qquad \delta^{-1}_{j,k} = \begin{cases} 1 & 0 \leqslant k \leqslant j, \\ 0 & otherwise. \end{cases}$$

The difference sequence space bv_p associated with matrix Δ is

$$bv_p = \left\{ x = (x_n) \ : \ \sum_{n=1}^{\infty} |x_n - x_{n-1}|^p < \infty \right\}, \qquad (1 \leqslant p < \infty),$$

with norm

$$\|x\|_{bv_p} = \left(\sum_{n=1}^{\infty} |x_n - x_{n-1}|^p \right)^{\frac{1}{p}}.$$

Recall the Hausdorff matrix $H^{\mu} = (h_{j,k})$, that has entries of the form:

$$h_{j,k} = \begin{cases} \binom{j}{k} \int_0^1 \theta^k (1-\theta)^{j-k} d\mu(\theta) & 0 \leqslant k \leqslant j, \\ 0 & k > j, \end{cases}$$

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where μ is a probability measure on $[0,1]$. The Hausdorff matrix contains some famous classes of matrices. These classes are as follows:

(i) Choice $d\mu(\theta) = \alpha(1-\theta)^{\alpha-1}d\theta$ gives the Cesàro matrix of order α;
(ii) Choice $d\mu(\theta) = \alpha\theta^{\alpha-1}d\theta$ gives the Gamma matrix of order α;
(iii) Choice $d\mu(\theta) = \frac{|\log\theta|^{\alpha-1}}{\Gamma(\alpha)}d\theta$ gives the Hölder matrix of order α;
(iv) Choice $d\mu(\theta) = point\ evaluation\ at\ \theta = \alpha$ gives the Euler matrix of order α.

By letting $d\mu(\theta) = \alpha(1-\theta)^{\alpha-1}d\theta$, $d\mu(\theta) = \alpha\theta^{\alpha-1}d\theta$ and $d\mu(\theta) =$ point evaluation at $\theta = \alpha$ in the definition of the Hausdorff matrix, the Cesàro matrix of order α, $C^\alpha = (c^\alpha_{j,k})$, the Gamma matrix of order α, $\Gamma^\alpha = (\gamma^\alpha_{j,k})$ and the Euler matrix of order α, $(0 < \alpha < 1)$, $E^\alpha = (e^\alpha_{j,k})$ are

$$c^\alpha_{j,k} = \begin{cases} \frac{\binom{\alpha+j-k-1}{j-k}}{\binom{\alpha+j}{j}} & 0 \leqslant k \leqslant j, \\ 0 & otherwise, \end{cases}$$

$$\gamma^\alpha_{j,k} = \begin{cases} \frac{\binom{\alpha+k-1}{k}}{\binom{\alpha+j}{j}} & 0 \leqslant k \leqslant j, \\ 0 & otherwise, \end{cases}$$

and

$$e^\alpha_{j,k} = \begin{cases} \binom{j}{k}\alpha^k(1-\alpha)^{j-k} & 0 \leqslant k \leqslant j, \\ 0 & otherwise, \end{cases}$$

respectively.

Let T be an operator. Throughout this paper, $\|T\|_p$ denotes the norm of T as an operator from l_p to itself, and $\|T\|_{bv_p}$ the norm as an operator from bv_p to itself. In this study, we investigate $\|T\|_{bv_p}$ for summability and Hausdorff mean operators.

The problem of finding the norm of matrix operators on the sequence space l_p have been studied extensively by many mathematicians and abundant literature exists on the topic. Although topological properties and inclusion relations of bv_p have largely been explored [1], computing the norm of matrix operators on this space has not been investigated to date, except Lashkaripour and Fathi [5]. They only obtained the norm of weighted mean matrix in special case. More recently, the authors investigated this problem for the sequence space $l_p(w,\Delta)$ and $l_p(\Delta^n)$, [2, 6].

2. Norms of summability and Hausdorff mean matrices on bv_p

In this section, we investigate the norm of well-known operators, Cesàro, Gamma and Euler, from bv_p into bv_p. In so doing, the following lemma and the Schur's theorem are needed.

LEMMA 2.1. *Let T be a matrix and $U = \Delta T\Delta^{-1}$. If U is a bounded operator on l_p, then T is a bounded operator on bv_p and*

$$\|T\|_{bv_p} = \|U\|_p.$$

Proof. The map $x \to \Delta x$ is an isomorphism between bv_p and l_p spaces. Now since

$$\|T\|_{bv_p} = \sup_{x\in bv_p} \frac{\|Tx\|_{bv_p}}{\|x\|_{bv_p}} = \sup_{x\in bv_p} \frac{\|\Delta Tx\|_p}{\|\Delta x\|_p} = \sup_{x\in bv_p} \frac{\|\Delta T\Delta^{-1}\Delta x\|_p}{\|\Delta x\|_p} = \sup_{y\in l_p} \frac{\|Uy\|_p}{\|y\|_p} = \|U\|_p,$$

hence we have the desired result. $\square$

The following theorem is known as Schur's theorem.

THEOREM 2.2. [3, theorem 275] *Let $p \geqslant 1$ and $T = (t_{m,k})$ be a matrix operator with $t_{m,k} \geqslant 0$ for all m,k. Suppose that K, R are two strictly positive numbers such that*

$$\sum_{m=0}^{\infty} t_{m,k} \leqslant K \quad for\ all\ k, \qquad \sum_{k=0}^{\infty} t_{m,k} \leqslant R \quad for\ all\ m,$$

(bounds for column and row sums respectively). Then

$$\|T\|_p \leqslant R^{(p-1)/p}K^{1/p}.$$

We say that $T = (t_{n,k})$ is a lower triangular, if $t_{n,k} = 0$ for $k > n$. A non-negative lower triangular matrix is called a summability matrix if $\sum_{k=0}^{n} t_{n,k} = 1$ for all n.

THEOREM 2.3. *Suppose that $T = (t_{n,k})$ is a summability matrix with decreasing $r_{n,k}(T) = \sum_{j=0}^{k} t_{n,j}$ respect to n for each k. Let $R_n = \sum_{k=0}^{n}(k+1)t_{n,k}$. If $R_n - R_{n-1} \leqslant M$ for all n, then T is a bounded operator on bv_p and*

$$\|T\|_{bv_p} \leqslant M^{1-\frac{1}{p}}.$$

In particular, for $M = 1$, we have $\|T\|_{bv_p} = 1$.

Proof. By applying lemma 2.1, we have $\|T\|_{bv_p} = \|U\|_p$, where $U = \Delta T\Delta^{-1}$. If $S = T\Delta^{-1}$, by assuming $S = (s_{i,j})$ and $U = (u_{i,j})$, we have $s_{i,j} = \sum_{k=j}^{i} t_{i,k}$. Since $\sum_{k=0}^{i} t_{i,k} = 1$, we have $s_{i,j} = 1 - r_{i,j-1}$ and $u_{i,j} = (\Delta S)_{i,j} = s_{i,j} - s_{i-1,j}$, which is non-negative by the hypothesis. Thus

$$\sum_{i=j}^{k} u_{i,j} = \sum_{i=j}^{k}(s_{i,j} - s_{i-1,j}) = s_{k,j} = \sum_{i=j}^{k} t_{k,i} = t_{k,j} + \cdots + t_{k,k} \leqslant 1 \quad (k = 0,1,\cdots),$$

hence $\sum_{i=0}^{\infty} u_{i,j} \leqslant 1$. Also

$$\sum_{j=0}^{n} s_{n,j} = \sum_{j=0}^{n}\sum_{k=j}^{n} t_{n,k} = \sum_{k=0}^{n}(k+1)t_{n,k} = R_n,$$

and

$$\sum_{j=0}^{\infty} u_{n,j} = \sum_{j=0}^{n} u_{n,j} = \sum_{j=0}^{n}(s_{n,j} - s_{n-1,j}) = R_n - R_{n-1}.$$

Since $R_n - R_{n-1} \leqslant M$, Schur's theorem implies that $\|T\|_{bv_p} \leqslant M^{1-\frac{1}{p}}$. For the case $M = 1$, letting $x = (1,1,\cdots)$ we have $Tx = x$, and therefore $\|T\|_{bv_p} = 1$. $\square$

THEOREM 2.4. *The Hausdorff operator H^μ is a bounded operator on bv_p and*

$$\|H^\mu\|_{bv_p} = 1.$$

Proof. Consider the Euler matrix E^θ. Let $r_{n,k}(\theta) = r_{n,k}(E^\theta) = \sum_{j=0}^{k} e_{n,j}(\theta)$. Using Pascal's identity, one finds easily that $e_{n+1,k}(\theta) = (1-\theta)e_{n,k}(\theta) + \theta e_{n,k-1}(\theta)$, and hence $r_{n+1,k}(\theta) = r_{n,k}(\theta) - \theta e_{n,k}(\theta)$, which shows that $r_{n,k}(\theta)$ is decreasing. Now, consider a general Hausdorff matrix H^μ , with $h_{n,k} = \int_0^1 e_{n,k}(\theta)d\mu(\theta)$. Then

$$r_{n,k}(H^\mu) = \sum_{j=0}^{k} h_{n,j} = \int_0^1 r_{n,k}(\theta)d\mu(\theta).$$

Since $r_{n,k}(\theta)$ decreases with n, so does $r_{n,k}(H^\mu)$. Also, according to theorem 2.3

$$\begin{aligned} R_n = \sum_{k=0}^{n}(k+1)h_{n,k} = & \sum_{k=0}^{n}(k+1)\int_0^1 \binom{n}{k}(1-\theta)^{n-k}\theta^k d\mu(\theta) \\ = & \int_0^1 \sum_{k=0}^{n} k\binom{n}{k}(1-\theta)^{n-k}\theta^k d\mu(\theta) \\ & + \int_0^1 \sum_{k=0}^{n} \binom{n}{k}(1-\theta)^{n-k}\theta^k d\mu(\theta) \\ = & \; n\int_0^1 \theta \sum_{k=1}^{n}\binom{n-1}{k-1}(1-\theta)^{n-k}\theta^{k-1}d\mu(\theta) + 1 \\ = & \; n\int_0^1 \theta d\mu(\theta) + 1, \end{aligned}$$

hence

$$R_n - R_{n-1} = \int_0^1 \theta d\mu(\theta) \leqslant \int_0^1 d\mu(\theta) = 1. \quad \square$$

COROLLARY 2.5. *The Cesàro, Gamma, Hölder and Euler matrices of order α are bounded operators on bv_p and*

$$\|C^\alpha\|_{bv_p} = \|\Gamma^\alpha\|_{bv_p} = \|H^\alpha\|_{bv_p} = \|E^\alpha\|_{bv_p} = 1.$$

COROLLARY 2.6. *Let H^μ and H^ν be two Hausdorff operators. Then we have*

$$\|H^\mu H^\nu\|_{bv_p} = \|H^\mu\|_{bv_p}\|H^\nu\|_{bv_p}.$$

Proof. According to theorem 2.4

$$\|H^\mu H^\nu\|_{bv_p} \leqslant \|H^\mu\|_{bv_p}\|H^\nu\|_{bv_p} = 1.$$

Now, since the product of two summability matrices is a summability matrix, hence we have the desired result. $\square$

In the last part of this study, we derive the main result of [5] as an application of theorem 2.3. In so doing, we need the definition of weighted mean matrices.

Suppose that $a=(a_j)_{j=0}^{\infty}$ is a non-negative sequence with $a_0>0$ and $A_j=a_0+a_1+\cdots+a_j$. The weighted mean matrix $M_a=(a_{j,k})$ is a lower triangular matrix which is defined as

$$a_{j,k}=\begin{cases}\frac{a_k}{A_j} & 0\leqslant k\leqslant j,\\ 0 & otherwise.\end{cases}$$

THEOREM 2.7. *([5, theorem 2.1]) Suppose that $a_n\leqslant ma_r$ for all $r\leqslant n$. Then $\|M_a\|_{bv_p}\leqslant m^{1-1/p}$. In particular, the norms equals 1 when $p=1$ (for any m) and when (a_n) is decreasing (for any p).*

Proof. Here $t_{j,k}=\frac{a_k}{A_j}$, which decreases with j, and $R_n=S_n/A_n$, where $S_n=\sum_{k=1}^{n}(k+1)a_k$. Writing $S_n=S_{n-1}+(n+1)a_n$,

$$S_nA_{n-1}-S_{n-1}A_n=[S_{n-1}+(n+1)a_n]A_{n-1}-S_{n-1}(A_{n-1}+a_n)\leqslant(n+1)a_nA_{n-1},$$

hence $R_n-R_{n-1}\leqslant(n+1)a_n/A_n$. Under the hypothesis of [5], this is not greater than m which completes the proof. □

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