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THE REVERSED HARDY–LITTLEWOOD–SOBOLEV TYPE INTEGRAL SYSTEMS WITH WEIGHTS

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Abstract. This paper is concerned with the existence of positive entire solutions of a weighted integral system. Such a system comes from the conformal properties of the reversed Hardy-Littlewood-Sobolev inequality. Several sufficient conditions of the existence/nonexistence are presented.

1. Introduction

Let $1 < r,s < \infty$, $0 < \lambda < n$, $\alpha+\beta \geqslant 0$ and $\alpha+\beta+\lambda \leqslant n$. Write the $L^p(R^n)$ norm of the function f by $\|f\|_p$. The weighted Hardy-Littlewood-Sobolev (WHLS) inequality states that (cf. [20])

$$\left|\int_{R^n}\int_{R^n}\frac{f(x)g(y)}{|x|^\alpha|x-y|^\lambda|y|^\beta}dxdy\right| \leqslant C_{\alpha,\beta,s,\lambda,n}\|f\|_r\|g\|_s, \tag{1}$$

where

$$1-\frac{1}{r}-\frac{\lambda}{n} < \frac{\alpha}{n} < 1-\frac{1}{r} \text{ and } \frac{1}{r}+\frac{1}{s}+\frac{\lambda+\alpha+\beta}{n} = 2. \tag{2}$$

The extremal functions satisfy the following Euler-Lagrange system

$$\begin{cases} u(x) = \dfrac{1}{|x|^\alpha}\displaystyle\int_{R^n}\frac{v(y)^q}{|y|^\beta|x-y|^\lambda}dy, \\ v(x) = \dfrac{1}{|x|^\beta}\displaystyle\int_{R^n}\frac{u(y)^p}{|y|^\alpha|x-y|^\lambda}dy, \end{cases} \tag{3}$$

where $\alpha+\beta+\lambda \leqslant n$, and

$$\begin{cases} u,v \geqslant 0,\ 0 < p,q < \infty,\ 0 < \lambda < n,\ \alpha+\beta \geqslant 0, \\ \frac{\alpha}{n} < \frac{1}{p+1} < \frac{\lambda+\alpha}{n},\ \frac{1}{p+1}+\frac{1}{q+1} = \frac{\lambda+\alpha+\beta}{n}. \end{cases} \tag{4}$$

Jin and Li [7] used an integral form of the method of moving planes (cf. [4]) to prove the radial symmetry of the solutions. This result implies the best constant of the

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WHLS inequality (cf. [2]). It is also the generalization of another one related to the extremal functions of the reversed weighted Hardy-Littlewood-Sobolev inequality(cf. [3]). Their another paper [8] shows the regularity of the solutions to (3) and the optimal integrability intervals in the case of $p>1$ and $q>1$. The optimal integrability (see also [14]) and the radial symmetry are essential to estimate the asymptotic rates of the solutions (cf. [1], [11], [13] and [15]), and to establish the better regularity results (cf. [12], [22]). Afterwards, Onodera ([19]) generalized the results of radial symmetry, integrability and asymptotic rates to the case of $p>0$ and $q>0$.

In 2015, Dou and Zhu [5] proved the following reversed Hardy-Littlewood-Sobolev (RHLS) inequality (see also [18])

$$|\int_{R^n}\int_{R^n}\frac{f(x)g(y)dxdy}{|x-y|^{\lambda}}|\geqslant C\|f\|_{L^r}\|g\|_{L^s},\quad \forall(f,g)\in L^r(R^n)\times L^s(R^n), \tag{5}$$

and the existence of extremal functions, where $n\geqslant 1$, $\lambda<0$, $0<r,s<1$ and $\frac{1}{r}+\frac{1}{s}+\frac{\lambda}{n}=2$. The Euler-Lagrange integral system is

$$\begin{cases} u(x)=\displaystyle\int_{R^n}\frac{|x-y|^{\lambda}dy}{v^q(y)}, & u>0 \ in\ R^n,\\ v(x)=\displaystyle\int_{R^n}\frac{|x-y|^{\lambda}dy}{u^p(y)}, & v>0 \ in\ R^n. \end{cases} \tag{6}$$

When $u\equiv v$ and $p=q$, (6) is reduced to

$$u(x)=\int_{R^n}\frac{|x-y|^{\lambda}dy}{u^p(y)},\quad u>0\quad in\quad R^n. \tag{7}$$

This equation is related to the study of the conformal geometry and the nonlinear elliptic PDEs. Lieb ([17]), Chen, Li, Ou [4] and Li [16] classified the positive solutions and pointed out that u must be of the form

$$u(x)=a(b^2+|x-x_0|^2)^{\lambda/2} \tag{8}$$

with $a,b>0$ and $x_0\in R^n$. Li ([16]) also studied (7) with exponent $p\in(0,\frac{2n+\lambda}{\lambda}]$, and proved that $p=\frac{2n+\lambda}{\lambda}$. A problem posed by Li is whether or not does (7) admit any positive (regular) solutions for all $n\geqslant 1$, $\lambda>0$ and $p>(2n+\lambda)/\lambda$. Xu gave a positive answer and obtained the following results (cf. [21]).

(Ri) Let $\lambda>0$ and $p>0$. Eq. (7) has a positive solution if and only if $2n+\lambda=p\lambda$. Now, u is given by (8).

(Rii) If $-n<\lambda<0$ and $p>0$, then (7) has no positive solution.

In 2015, Lei ([9]) studied the conformal properties of (6). In particular, under the Kelvin transformation, (6) becomes

$$\begin{cases} u(x)=\displaystyle\int_{R^n}\frac{|x-y|^{\lambda}dy}{|x|^{\alpha}v^q(y)|y|^{\beta}}, & u>0 \ in\ R^n,\\ v(x)=\displaystyle\int_{R^n}\frac{|x-y|^{\lambda}dy}{|x|^{\beta}u^p(y)|y|^{\alpha}}, & v>0 \ in\ R^n. \end{cases} \tag{9}$$

Huang ([6]) obtained a critical condition from the rescaling invariant property of (9) as in [9].

A more general integral system with variational coefficients in [9] is

$$\begin{cases} u(x) = c_1(x)\displaystyle\int_{R^n} \frac{|x-y|^{\lambda} dy}{v^q(y)}, & u > 0 \; in \; R^n, \\ v(x) = c_2(x)\displaystyle\int_{R^n} \frac{|x-y|^{\lambda} dy}{u^p(y)}, & v > 0 \; in \; R^n, \end{cases} \tag{10}$$

where $p,q,\lambda > 0$, and $c_1(x), c_2(x)$ are double bounded functions. A function $k(x)$ is called double bounded, if there is $C > 1$ such that $C^{-1} \leqslant k(x) \leqslant C$ for all $x \in R^n$.

In this paper, we are concerned with the nonexistence of the positive entire super-solution of (9) and the existence of the positive entire solutions of the following weighted system

$$\begin{cases} u(x) = c_1(x)\displaystyle\int_{R^n} \frac{|x-y|^{\lambda}}{|x|^{\alpha} v^q(y)|y|^{\beta}} dy, & u > 0 \; in \; R^n, \\ v(x) = c_2(x)\displaystyle\int_{R^n} \frac{|x-y|^{\lambda}}{|x|^{\beta} u^p(y)|y|^{\alpha}} dy, & v > 0 \; in \; R^n. \end{cases} \tag{11}$$

THEOREM 1. *Assume that either*

$$\lambda \leqslant -n,$$

or

$$-n < \lambda < 0, \min\{p,q\} > 0, \alpha\beta = 0,$$

then (9) has no positive super-solution in $L^{\infty}_{loc}(R^n \setminus \{0\})$.

THEOREM 2. *Let* $\lambda > 0$ *and* $\min\{p,q\} > 0$. *If*

$$\min\{n + (p-1)\alpha, n + (q-1)\beta\} > 0,$$

and

$$\min\{(q-1)(\lambda - \beta), (p-1)(\lambda - \alpha)\} > n,$$

then (11) has positive solutions for some double bounded functions $c_1(x)$ *and* $c_2(x)$.

Theorems 2 and 1 are the corresponding results on system (6) to (Ri) and (Rii) on single equation (7) respectively.

2. Proof of theorems

In this section, we prove theorems 1 and 2.

Proof of theorem 1.

(i) Let $\lambda \leqslant -n$. Suppose (u,v) is a pair of super-solution. For $0<|x|<1/2$, there holds
$$|x|^{\alpha}u(x) \geqslant c(\beta)\min_{B_1(0)}(v^{-q})\int_{B_{|x|/2}(x)}|x-y|^{\lambda}dy=\infty.$$
Thus, $u \notin L^{\infty}_{loc}(R^n\setminus\{0\})$.

(ii) Let $-n<\lambda<0, \min\{p,q\}>0$ and $\alpha\beta=0$. Without loss of generality, we assume $\alpha=0$.

By lemma 3.11.3 in [23], we can find $C>0$ such that for any $\delta>0$,
$$\frac{1}{|B_\delta(x_0)|}\int_{B_\delta(x_0)}u(x)dx=\int_{R^n}\left\{\frac{1}{|B_\delta(x_0)|}\int_{B_\delta(x_0)}|x-y|^{\lambda}dx\right\}\frac{1}{v^q(y)|y|^{\beta}}dy$$
$$\leqslant C\int_{R^n}\frac{|x_0-y|^{\lambda}}{v^q(y)|y|^{\beta}}dy=Cu(x_0).$$

Here $x_0\neq 0$.

Now it follows from Hölder inequality that
$$1=\frac{1}{|B_\delta(x_0)|}\int_{B_\delta(x_0)}u^{-\frac{p}{p+1}}(x)u^{\frac{p}{p+1}}(x)dx$$
$$\leqslant\left(\frac{1}{|B_\delta(x_0)|}\int_{B_\delta(x_0)}u^{-p}(x)dx\right)^{\frac{1}{p+1}}\cdot\left(\frac{1}{|B_\delta(x_0)|}\int_{B_\delta(x_0)}u(x)dx\right)^{\frac{p}{p+1}}.$$

Combining these results yields
$$Cu^{-p}(x_0)\leqslant\left[\frac{1}{|B_\delta(x_0)|}\int_{B_\delta(x_0)}u(x)dx\right]^{-p}\leqslant\frac{1}{|B_\delta(x_0)|}\int_{B_\delta(x_0)}u^{-p}(x)dx.$$

Here C is a positive constant independent of δ.

In view of $\lambda<0$, if $|x-x_0|<\delta$, then $\delta^{\lambda}<|x-x_0|^{\lambda}$. Therefore multiplying the result above by $\frac{\delta^{n+\lambda}}{|x_0|^{\beta}}$, we get
$$\frac{C\delta^{n+\lambda}u^{-p}(x_0)}{|x_0|^{\beta}}\leqslant\int_{B_\delta(x_0)}\frac{|x_0-x|^{\lambda}dx}{|x_0|^{\beta}u^p(x)}\leqslant v(x_0).$$

Noticing $n+\lambda>0$, and letting $\delta\to\infty$, we see a contradiction. □

Proof of theorem 2.
The ideas come from [9] and [10].
Set
$$\begin{cases}u(x)=|x|^{\gamma_1}(1+|x|^2)^{\theta_1},\\ v(x)=|x|^{\gamma_2}(1+|x|^2)^{\theta_2}.\end{cases}\tag{12}$$

Here $\gamma_i, \theta_i \ (i=1,2)$ are constants determined later. We will prove that (11) has the radial solution as (12) for some double bounded functions $c_i(x) \ (i=1,2)$.

When $|x| >> 1$, we have

$$\begin{aligned}\int_{R^n} \frac{|x-y|^\lambda}{|x|^\alpha v^q(y)|y|^\beta} dy =& \int_{B_1(0)} \frac{|x-y|^\lambda}{|x|^\alpha |y|^{\beta+q\gamma_2}(1+|y|^2)^{q\theta_2}} dy \\ &+ \int_{B_{2|x|}(0)\setminus B_1(0)} \frac{|x-y|^\lambda}{|x|^\alpha |y|^{\beta+q\gamma_2}(1+|y|^2)^{q\theta_2}} dy \\ &+ \int_{R^n\setminus B_{2|x|}(0)} \frac{|x-y|^\lambda}{|x|^\alpha |y|^{\beta+q\gamma_2}(1+|y|^2)^{q\theta_2}} dy \\ :=& I_1+I_2+I_3.\end{aligned}$$

Then, there exists $C>0$ such that when $|x| >> 1$,

$$\frac{1}{C}|x|^{\lambda-\alpha} \int_0^1 r^{n-\beta-q\gamma_2} \frac{dr}{r} \leqslant I_1 \leqslant C|x|^{\lambda-\alpha} \int_0^1 r^{n-\beta-q\gamma_2} \frac{dr}{r}.$$

In order to ensure $I_1 < \infty$, we need

$$n-\beta-q\gamma_2 > 0, \tag{13}$$

and hence

$$\frac{1}{C}|x|^{\lambda-\alpha} \leqslant I_1 \leqslant C|x|^{\lambda-\alpha}.$$

Next, we claim $\theta_2 > 0$. Otherwise, there exists $C>0$ such that when $|x| >> 1$,

$$I_3 \geqslant C|x|^{-\alpha} \int_{2|x|}^\infty r^{n+\lambda-\beta-q\gamma_2} \frac{dr}{r}.$$

By (13), we can see $I_3 = \infty$. It is impossible. Thus, by $|y|/2 \leqslant |x-y| \leqslant 3|y|/2$ (implied by $|y| \geqslant 2|x|$), there exists $C>0$ such that when $|x| >> 1$,

$$C^{-1}|x|^{-\alpha} \int_{2|x|}^\infty r^{n+\lambda-\beta-q\gamma_2-2q\theta_2} \frac{dr}{r} \leqslant I_3 \leqslant C|x|^{-\alpha} \int_{2|x|}^\infty r^{n+\lambda-\beta-q\gamma_2-2q\theta_2} \frac{dr}{r}.$$

In order to ensure $I_3 < \infty$, we need

$$n+\lambda-\beta-q\gamma_2-2q\theta_2 < 0, \tag{14}$$

and hence

$$0 \leqslant I_3 \leqslant C|x|^{\lambda-\alpha+(n-\beta-q\gamma_2-2q\theta_2)}.$$

From (14) it follows $n-\beta-q\gamma_2-2q\theta_2 < 0$. Therefore, when $|x| >> 1$, we also get

$$0 \leqslant I_2 \leqslant C|x|^{\lambda-\alpha} \int_1^{2|x|} r^{n-\beta-q\gamma_2-2q\theta_2} \frac{dr}{r} \leqslant C|x|^{\lambda-\alpha}.$$

Thus combining the estimates of $I_1,\ I_2,\ I_3,$ we have

$$\frac{1}{C}|x|^{\lambda-\alpha} \leqslant \int_{R^n} \frac{|x-y|^{\lambda}}{|x|^{\alpha} v^q(y)|y|^{\beta}} dy \leqslant C|x|^{\lambda-\alpha}. \tag{15}$$

When $|x| << 1,$ we have

$$\begin{aligned}\int_{R^n} \frac{|x-y|^{\lambda}}{|x|^{\alpha} v^q(y)|y|^{\beta}} dy =& \int_{B_{2|x|}(0)} \frac{|x-y|^{\lambda}}{|x|^{\alpha}|y|^{\beta+q\gamma_2}(1+|y|^2)^{q\theta_2}} dy \\ &+ \int_{B_1(0)\setminus B_{2|x|}(0)} \frac{|x-y|^{\lambda}}{|x|^{\alpha}|y|^{\beta+q\gamma_2}(1+|y|^2)^{q\theta_2}} dy \\ &+ \int_{R^n\setminus B_1(0)} \frac{|x-y|^{\lambda}}{|x|^{\alpha}|y|^{\beta+q\gamma_2}(1+|y|^2)^{q\theta_2}} dy \\ :=& J_1 + J_2 + J_3.\end{aligned}$$

In view of (13), there exists $C > 0$ such that

$$0 \leqslant J_1 \leqslant C|x|^{\lambda-\alpha} \int_0^{2|x|} r^{n-\beta-q\gamma_2} \frac{dr}{r} \leqslant C|x|^{n+\lambda-\alpha-\beta-q\gamma_2}.$$

When $y \in B_1(0) \setminus B_{2|x|}(0),\ |y|/2 \leqslant |x-y| \leqslant 3|y|/2$. Therefore,

$$C^{-1}|x|^{-\alpha} \int_{2|x|}^1 r^{n+\lambda-\beta-q\gamma_2} \frac{dr}{r} \leqslant J_2 \leqslant C|x|^{-\alpha} \int_{2|x|}^1 r^{n+\lambda-\beta-q\gamma_2} \frac{dr}{r}.$$

Noting (13), we have

$$\frac{1}{C}|x|^{-\alpha} \leqslant J_2 \leqslant C|x|^{-\alpha}.$$

By (14), we also get

$$0 \leqslant J_3 \leqslant C|x|^{-\alpha} \int_1^{\infty} r^{n+\lambda-\beta-q\gamma_2-2q\theta_2} \frac{dr}{r} \leqslant C|x|^{-\alpha}.$$

Thus, combining the estimates of $J_1,\ J_2,\ J_3,$ and noting $n+\lambda-\beta-q\gamma_2 > 0$ (implied by (13)), we have

$$\frac{1}{C}|x|^{-\alpha} \leqslant \int_{R^n} \frac{|x-y|^{\lambda}}{|x|^{\alpha} v^q(y)|y|^{\beta}} dy \leqslant C|x|^{-\alpha}. \tag{16}$$

Let $\gamma_1 = -\alpha,\ \gamma_2 = -\beta,\ 2\theta_1 = 2\theta_2 = \lambda,$ then from the conditions of theorem 2, we know that $\gamma_2,\ \theta_2$ satisfies the (13) and (14). By (15) and (16) we obtain that

$$\frac{1}{C}\int_{R^n} \frac{|x-y|^{\lambda}}{|x|^{\alpha} v^q(y)|y|^{\beta}} dy \leqslant u(x) \leqslant C\int_{R^n} \frac{|x-y|^{\lambda}}{|x|^{\alpha} v^q(y)|y|^{\beta}} dy.$$

Take $K_1(x) = u(x)[\int_{R^n} \frac{|x-y|^\lambda}{|x|^\alpha v^q(y)|y|^\beta} dy]^{-1}$. Then $K_1(x)$ is double bounded and

$$u(x) = K_1(x) \int_{R^n} \frac{|x-y|^\lambda}{|x|^\alpha v^q(y)|y|^\beta} dy.$$

Similarly, we can also deduce that

$$v(x) = K_2(x) \int_{R^n} \frac{|x-y|^\lambda}{|x|^\beta u^p(y)|y|^\alpha} dy,$$

where $K_2 = v(x)[\int_{R^n} \frac{|x-y|^\lambda}{|x|^\beta u^p(y)|y|^\alpha} dy]^{-1}$ is double bounded. Therefore we complete the proof. □

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