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EXTREMAL FUNCTIONS FOR THE MODIFIED TRUDINGER–MOSER INEQUALITIES IN TWO DIMENSIONS

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Abstract. Let $\Omega\subset\mathbb{R}^2$ be a smooth bounded domain, $W_0^{1,2}(\Omega)$ be the standard Sobolev space. Assuming certain conditions on a function $g:\mathbb{R}\to\mathbb{R}$, we prove that the supremum

$$\sup_{u\in W_0^{1,2}(\Omega),\,\|\nabla u\|_2\leqslant 1}\int_\Omega (1+g(u))e^{4\pi u^2}dx,$$

can be attained by some function $u_0\in W_0^{1,2}(\Omega)$ with $\|\nabla u_0\|_2=1$. The proof is based on the usual blow-up analysis. Also we consider the same problem for the supremum

$$\sup_{u\in W_0^{1,2}(\Omega),\,\|\nabla u\|_2\leqslant 1}\int_\Omega h(1+g(u))e^{4\pi u^2}dx,$$

where h is continuous in $\overline{\Omega}$, $h\geqslant 0$ and $h\not\equiv 0$.

1. Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^2$, and $W_0^{1,2}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ under the norm $\|u\|_{W_0^{1,2}(\Omega)}=(\int_\Omega|\nabla u|^2dx)^{1/2}$. For $1\leqslant p<2$, the standard Sobolev embedding theorem states that $W_0^{1,p}\hookrightarrow L^q(\Omega)$ for all $1<q\leqslant 2p/(2-p)$; while if $p>2$, there holds $W_0^{1,p}(\Omega)\hookrightarrow C^0(\overline{\Omega})$. As a borderline of the Sobolev embeddings, the classical Trudinger-Moser inequality [31, 21, 20, 19, 24] says

$$\sup_{u\in W_0^{1,2}(\Omega),\|\nabla u\|_2\leqslant 1}\int_\Omega e^{\alpha u^2 dx}<+\infty,\quad \forall\alpha\leqslant 4\pi. \tag{1}$$

Moreover, these integrals are still finite for any $\alpha>4\pi$, but the supremum is infinity. Here in the sequel, for any real number $q\geqslant 1$, $\|\cdot\|_q$ denotes the $L^q(\Omega)$-norm with respect to the Lebesgue measure.

A function u_0 is called an extremal function for the Trudinger-Moser inequality (1) if u_0 belongs to $W_0^{1,2}(\Omega)$, $\|\nabla u_0\|_2\leqslant 1$ and

$$\int_\Omega e^{\alpha u_0^2}dx=\sup_{u\in W_0^{1,2}(\Omega),\|\nabla u\|_2\leqslant 1}\int_\Omega e^{\alpha u^2}dx.$$

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Existence of extremal functions for (1) was solved by Carleson-Chang [3] and Lions [15] when Ω is a unit ball, by Flucher [7] when Ω is a smooth bounded domain, and by Lin [14] when Ω is an arbitrary dimensional domain.

Recently, by a method of energy estimates developed by Malchiodi-Martinazzi [17], Mancini-Martinazzi [18] reproved Carleson-Chang's result. Namely the supremum

$$\sup_{u\in W_0^{1,2}(\mathbb{B}),\|\nabla u\|_2\leqslant 1}\int_{\mathbb{B}} e^{\alpha u^2}dx,$$

can be attained, where $\mathbb{B}$ is a unit disc in $\mathbb{R}^2$. Then the result was extended by Yang [28]. For applications of the energy estimate, they proved that the supremum

$$\sup_{u\in W_0^{1,2}(\mathbb{B}),\|\nabla u\|_2\leqslant 1}\int_{\mathbb{B}}(1+g(u))e^{4\pi u^2}dx,$$

can also be attained for certain smooth function $g:\mathbb{R}\to\mathbb{R}$. In our paper, we extend the previous results and study the existence of the extremal functions for such inequalities in (2). Clearly, for any bounded function g, there holds

$$\sup_{u\in W_0^{1,2}(\Omega),\|\nabla u\|_2\leqslant 1}\int_{\Omega}(1+g(u))e^{4\pi u^2}dx<\infty. \tag{2}$$

In this paper, unlike the previous energy estimate procedure in [17, 18, 28], we mainly employ method of blow-up analysis in [6, 16, 13, 11] to prove the supremum in (2) can be achieved. In order to prove the critical Trudinger-Moser inequality, we firstly discuss the existence of extremal functions for subcritical one, which is based on a direct method in the calculus of variations. We derive a different Euler-Lagrange equation on which the analysis is performed. However, the essential problem is the presence of the function g. To meet the necessary of our proof, we assume g satisfies certain conditions. And then we deduce a dedicate upper bound through blow-up analysis. For works in this direction, we refer the reader to Y. Yang [26, 27], Y. Li [13], Lu and Yang [16], Carleson-Chang [3], Struwe [22], Ding, Jost, Li and Wang [6].

Motivated by Mancini-Martinazzi [18] (see pages 3 and 4), we assume the function g in (2) satisfies:

$$g\in C^1(\mathbb{R}),\ \inf_{\mathbb{R}} g>-1,\ g(-t)=g(t),\ \lim_{|t|\to\infty}t^2g(t)=0,\ g'(t)>0\,(\forall t>0). \tag{3}$$

In the proof, we derive

$$-\Delta u_\varepsilon=\frac{1}{\lambda_\varepsilon}(1+g(u)+\frac{g'(u)}{2(4\pi-\varepsilon)u})u_\varepsilon e^{(4\pi-\varepsilon)u_\varepsilon^2}=\frac{1}{\lambda_\varepsilon}\left(1+\omega(u_\varepsilon)\right)u_\varepsilon e^{(4\pi-\varepsilon)u_\varepsilon^2},$$

for some $\lambda_\varepsilon\in\mathbb{R}$, where we set

$$\omega(t):=g(t)+\frac{g'(t)}{2(4\pi-\varepsilon)t},\quad t\in\mathbb{R}\setminus\{0\}. \tag{4}$$

We further assume

$$\inf_{(0,+\infty)} \omega(t) > -1, \quad \sup_{(0,+\infty)} \omega(t) < +\infty, \textit{ and } \lim_{t\to\infty} t^2\omega(t) = 0. \tag{5}$$

Comparing the conditions on function g in Mancini-Martinazzi [18], one can see some differences. In this note, we assume $g'(t) > 0\,(\forall t > 0)$, which will be used in the lemma 1. Moreover, the assumptions in (3) and (5) implies that $\lim_{|t|\to\infty} g(t) = 0$ in [18]. The main conclusions can be stated as the following two theorems respectively.

Our first result is the existence of extremal functions for the modified Trudinger-Moser inequality (2), namely

THEOREM 1. *Let Ω be a smooth bounded domain in $\mathbb{R}^2$ and $W_0^{1,2}(\Omega)$ be the usual Sobolev space. Suppose $g \in C^1(\mathbb{R})$ satisfies the hypotheses in (3) and (5). Then the supremum in (2) can be attained.*

For simplicity, we introduce the notations:

$$\Lambda_{4\pi-\varepsilon} := \sup_{u\in W_0^{1,2}(\Omega), \|\nabla u\|_2 \leqslant 1} \int_\Omega (1+g(u))e^{(4\pi-\varepsilon)u^2} dx,$$

and

$$\Lambda_{4\pi} := \sup_{u\in W_0^{1,2}(\Omega), \|\nabla u\|_2 \leqslant 1} \int_\Omega (1+g(u))e^{4\pi u^2} dx.$$

By means of blow-up analysis, we divide the argument into three steps:

Step 1. For any $\varepsilon > 0$, the supremum $\Lambda_{4\pi-\varepsilon}$ in subcritical circumstance can be attained by some nonnegative function u_ε. The Euler-Lagrange equation of u_ε is an elliptic one.

Step 2. Denote $c_\varepsilon = u_\varepsilon(x_\varepsilon) = \max_\Omega u_\varepsilon$. If c_ε is a bounded sequence, then applying elliptic estimates to the equation of u_ε, we conclude that u_ε converges to a desired extremal function. While if $c_\varepsilon \to \infty$, by a delicate analysis on u_ε, we derive

$$\Lambda_{4\pi} = \lim_{\varepsilon\to 0} \int_\Omega (1+g(u_\varepsilon))e^{(4\pi-\varepsilon)u_\varepsilon^2} dx \leqslant |\,\Omega\,|\,(1+g(0)) + \pi e^{1+4\pi A_{x_0}},$$

where

$$A_{x_0} = \lim_{x\to x_0} (G + \frac{1}{2\pi}\log|x-x_0|),$$

G is a Green function satisfying $-\Delta G = \delta_{x_0}$ in $\mathbb{R}^2$ and δ_{x_0} is a Dirac measure centered at x_0.

Step 3. Construct a sequence of test functions $\phi_\varepsilon \in W_0^{1,2}(\Omega)$ satisfying $\|\nabla\phi_\varepsilon\|_2 = 1$ and if ε is sufficiently small, there holds

$$\int_\Omega (1+g(\phi_\varepsilon))e^{4\pi\phi_\varepsilon^2} dx > |\Omega|(1+g(0)) + \pi e^{1+4\pi A_{x_0}}.$$

Comparing *Steps* 2 and 3, we are led to the conclusion that c_ε must be bounded and thus the existence of the extremal function follows from elliptic estimates. It should be remarked that in Step 2, we shall use an estimate of Carleson-Chang [3]. This completes the proof of theorem 1.

Motivated by works of Yang-Zhu [29] and Hou [9], we have the following:

THEOREM 2. *Let Ω be a smooth bounded domain in $\mathbb{R}^2$, $g \in C^1(\mathbb{R})$ satisfies (3) and (5), $h \in C^0(\overline{\Omega})$ satisfies $h \geqslant 0$ and $h \not\equiv 0$ in Ω. Then the supremum*

$$\sup_{u \in W_0^{1,2}(\Omega), \|\nabla u\|_2 \leqslant 1} \int_\Omega h(1+g(u))e^{4\pi u^2}dx,$$

can be attained by some $u_0 \in W_0^{1,2}(\Omega) \cap C^0(\overline{\Omega})$ satisfying $\|\nabla u_0\|_2 = 1$.

The proof of theorem 2 is different from that of theorem 1 in that we must exclude the following possibility: x_0 is the blow-up point and $h(x_0) = 0$. Hence we use the different scaling when define the maximizing sequences of functions.

Similar problems may also be raised concerning the singular Trudinger-Moser inequalities [2, 5, 10, 23, 30]: Let h and g be given as in theorem 2, and $0 \leqslant \gamma < 1$. One may ask whether the supremum

$$\sup_{u \in W_0^{1,2}(\Omega), \|\nabla u\|_2 \leqslant 1} \int_\Omega h(1+g(u)) \frac{e^{4\pi(1-\gamma)u^2}}{|x|^{2\gamma}}dx,$$

can be attained or not.

The remaining part of the paper is arranged as follows: In section 2, we complete the proof of the theorem 1, mainly by adopting the blow-up analysis; In section 3, we prove theorem 2. Throughout this paper, we do not distinguish sequence and subsequence, the reader can recognize it easily from the context.

2. Proof of theorem 1

We divide the proof into several subsections.

2.1. Maximizers for subcritical functionals

We first show that maximizers for the subcritical functions exist. Namely, for any $0 < \varepsilon < 4\pi$, there exists some $u_\varepsilon \in W_0^{1,2}(\Omega)$ such that

$$\int_\Omega (1+g(u_\varepsilon))e^{(4\pi-\varepsilon)u_\varepsilon^2}dx = \sup_{u \in W_0^{1,2}(\Omega), \|\nabla u\|_2 \leqslant 1} \int_\Omega (1+g(u))e^{(4\pi-\varepsilon)u^2}dx. \tag{6}$$

Take a function sequence $u_j \in W_0^{1,2}(\Omega)$, satisfying $\|\nabla u_j\|_2 \leqslant 1$ and

$$\lim_{j\to\infty} \int_\Omega (1+g(u_j))e^{(4\pi-\varepsilon)u_j^2}dx = \Lambda_{4\pi-\varepsilon}. \tag{7}$$

Since

$$\int_\Omega |\nabla u_\varepsilon|^2 dx = \lim_{j\to\infty}\int_\Omega \nabla u_\varepsilon \nabla u_j dx \leqslant \lim_{j\to\infty}\sup\left(\int_\Omega |\nabla u_\varepsilon|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla u_j|^2 dx\right)^{\frac{1}{2}},$$

it yields $\|\nabla u_\varepsilon\|_2 \leqslant 1,$ which implies that u_ε is bounded in $W_0^{1,2}(\Omega)$. Thus there exists some $u_\varepsilon \in W_0^{1,2}(\Omega)$ such that up to a subsequence, assuming

$$\begin{aligned} u_j &\rightharpoonup u_\varepsilon \quad \text{weakly in} \quad W_0^{1,2}(\Omega),\\ u_j &\to u_\varepsilon \quad \text{strongly in } L^p(\Omega),\ \forall p \geqslant 1,\\ u_j &\to u_\varepsilon \quad a.e \text{ in} \quad \Omega. \end{aligned}$$

For any $1 < p \leqslant \frac{1}{s}$, $\delta > 0$, $s > 1$ and $s' = s/(s-1)$, we have by the Hölder's inequality,

$$\begin{aligned} \int_\Omega (1+g(u_j))^p e^{(4\pi-\varepsilon)pu_j^2} dx &\leqslant \int_\Omega (1+g(u_j))^p e^{(4\pi-\varepsilon)p(1+\delta)(u_j-u_\varepsilon)^2+(4\pi-\varepsilon)p(1+\frac{1}{4\delta})u_\varepsilon^2} dx\\ &\leqslant \left(\int_\Omega (1+g(u_j))^p e^{(4\pi-\varepsilon)p(1+\delta)s(u_j-u_\varepsilon)^2} dx\right)^{\frac{1}{s}}\\ &\quad\times\left(\int_\Omega (1+g(u_j))^p e^{(4\pi-\varepsilon)p(1+\frac{1}{4\delta})s'u_\varepsilon^2} dx\right)^{\frac{1}{s'}}. \end{aligned} \tag{8}$$

Choose $p,\ 1+\delta$ and s sufficiently close to 1, we have

$$p(1+\delta)s < 1. \tag{9}$$

In view of $\|\nabla u_\varepsilon\|_2 \leqslant 1$, it follows that

$$\int_\Omega |\nabla(u_\varepsilon - u_j)|^2 dx = \int_\Omega |\nabla u_\varepsilon|^2 dx - \int_\Omega |\nabla u_j|^2 + o_j(1) \leqslant 1 - \int_\Omega |\nabla u_\varepsilon|^2 + o_j(1). \tag{10}$$

Inserting (9) and (10) into (8), we obtain $(1+g(u_j))e^{(4\pi-\varepsilon)u_j^2}$ is bounded in $L^p(\Omega)$ for some $p > 1$ by (1) and (3). Since

$$\begin{aligned} &|(1+g(u_j))e^{(4\pi-\varepsilon)u_j^2} - (1+g(u_\varepsilon))e^{(4\pi-\varepsilon)u_\varepsilon^2}|\\ \leqslant& C(e^{4\pi(1-\gamma-\varepsilon)u_j^2} + e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2})|u_j^2 - u_\varepsilon^2| + \max\{g'(u_j), g'(u_\varepsilon)\}|u_j - u_\varepsilon| e^{4\pi(1-\gamma-\varepsilon)u_j^2}, \end{aligned}$$

and $u_j \to u_\varepsilon$ strongly in $L^p(\Omega)$ for all $p > 1$ as $j \to \infty$, there have

$$\lim_{j\to\infty}\int_\Omega (1+g(u_j))e^{(4\pi-\varepsilon)u_j^2} dx = \int_\Omega (1+g(u_\varepsilon))e^{(4\pi-\varepsilon)u_\varepsilon^2} dx. \tag{11}$$

Combing (7) and (11), we conclude that u_ε attains the supremum $\Lambda_{4\pi-\varepsilon}$ so that (6) holds. Suppose $\|\nabla u_\varepsilon\|_2 < 1$. Denote $\widetilde{u}_\varepsilon = \frac{u_\varepsilon}{\|\nabla u_\varepsilon\|_2}$. Obviously, there have $\|\nabla \widetilde{u}_\varepsilon\|_2 = 1$. Since $0 \leqslant u_\varepsilon < \widetilde{u}_\varepsilon$ and $u_\varepsilon \not\equiv 0$, it follows from (3) that

$$\Lambda_{4\pi-\varepsilon} = \int_\Omega (1+g(u_\varepsilon))e^{(4\pi-\varepsilon)u_\varepsilon^2}dx < \int_\Omega (1+g(\widetilde{u}_\varepsilon))e^{(4\pi-\varepsilon)\widetilde{u}_\varepsilon^2}dx \leqslant \Lambda_{4\pi-\varepsilon},$$

which leads to a contradiction. Consequently, $\|\nabla u_\varepsilon\|_2 = 1$ holds. Furthermore, one can also check that $|u_\varepsilon|$ attains the supremum $\Lambda_{4\pi-\varepsilon}$. Thus, u_ε can be chosen so that $u_\varepsilon \geqslant 0$. A careful calculation shows that u_ε satisfies the following Euler-Lagrange equation:

$$\begin{cases} -\Delta u_\varepsilon = \frac{1}{\lambda_\varepsilon}(1+\omega(u_\varepsilon))u_\varepsilon e^{(4\pi-\varepsilon)u_\varepsilon^2} & \text{in } \Omega \subset \mathbb{R}^2, \\ u_\varepsilon \geqslant 0,\ \|\nabla u_\varepsilon\|_2 = 1 & \text{in } \Omega \subset \mathbb{R}^2, \\ \lambda_\varepsilon = \int_\Omega (1+\omega(u_\varepsilon))u_\varepsilon^2 e^{(4\pi-\varepsilon)u_\varepsilon^2}dx, \end{cases} \tag{12}$$

where ω is defined as in (4).

2.2. Elementary properties of u_ε

Note that $\Lambda_{4\pi}$ is finite. To prove theorem 1, it suffices to find some $u_0 \in W_0^{1,2}(\Omega)$ satisfying $\|\nabla u_0\|_2 = 1$ and

$$\int_\Omega (1+g(u_0))e^{4\pi u_0^2}dx = \sup_{u\in W_0^{1,2}(\Omega), \|\nabla u\|_2 \leqslant 1} \int_\Omega (1+g(u))e^{4\pi u^2}dx. \tag{13}$$

We first show $\Lambda_{4\pi-\varepsilon} \to \Lambda_{4\pi}$ as $\varepsilon \to 0$. In fact, for any $u \in W_0^{1,2}(\Omega)$ with $\|\nabla u\|_2 \leqslant 1$, there holds

$$\int_\Omega (1+g(u))e^{4\pi u^2}dx \leqslant \lim_{\varepsilon\to 0}\int_\Omega (1+g(u_\varepsilon))e^{(4\pi-\varepsilon)u_\varepsilon^2}dx \leqslant \lim_{\varepsilon\to 0}\Lambda_{4\pi-\varepsilon}.$$

This leads to

$$\Lambda_{4\pi} = \sup_{u\in W_0^{1,2}(\Omega), \|\nabla u\|_2 \leqslant 1} \int_\Omega (1+g(u))e^{4\pi u^2}dx \leqslant \lim_{\varepsilon\to 0}\Lambda_{4\pi-\varepsilon}. \tag{14}$$

On the other hand, it is evident that

$$\Lambda_{4\pi-\varepsilon} = \int_\Omega (1+g(u_\varepsilon))e^{(4\pi-\varepsilon)u_\varepsilon^2}dx \leqslant \Lambda_{4\pi}. \tag{15}$$

Together with (14) and (15), we get the desired result. Since u_ε is bounded in $W_0^{1,2}(\Omega)$, we can assume without loss of generality,

$$\begin{aligned} &u_\varepsilon \rightharpoonup u_0 \quad \text{weakly in} \quad W_0^{1,2}(\Omega), \\ &u_\varepsilon \to u_0 \quad \text{strongly in} \quad L^p(\Omega),\ \forall p \geqslant 1, \\ &u_\varepsilon \to u_0 \quad a.e \text{ in} \quad \Omega. \end{aligned} \tag{16}$$

Denote $c_\varepsilon = \max_\Omega u_\varepsilon$. Assuming c_ε is bounded, then for any $u \in W_0^{1,2}(\Omega)$ with $u \geqslant 0$, $\|\nabla u\|_2 = 1$, we have by the Lebesgue dominated convergence theorem

$$\int_\Omega (1+g(u))e^{4\pi u^2}dx \leqslant \lim_{\varepsilon\to 0}\int_\Omega (1+g(u_\varepsilon))e^{(4\pi-\varepsilon)u_\varepsilon^2}dx = \int_\Omega (1+g(u_0))e^{4\pi u_0^2}dx.$$

By the arbitrariness of $u \in W_0^{1,2}(\Omega)$, it follows that

$$\sup_{u\in W_0^{1,2}(\Omega),\|\nabla u\|_2\leqslant 1}\int_\Omega (1+g(u))e^{4\pi u^2}dx \leqslant \int_\Omega (1+g(u_0))e^{4\pi u_0^2}dx. \tag{17}$$

Thus, (17) makes it clear that u_0 is the desired extremal function, or equivalently (13) holds. Define $c_\varepsilon = u_\varepsilon(x_\varepsilon) = \max_\Omega u_\varepsilon \to \infty$ and we distinguish two cases to proceed.

Case 1. If $u_0 \not\equiv 0$, the supremum in (2) can be attained by u_0 without difficulty. And the proof will just be divided into several simple steps.

Step 1. According to the classical Trudinger-Moser inequality (1), we know $e^{(4\pi-\varepsilon)u_\varepsilon^2}$ is bounded in $L^p(\Omega)$ $(p>1)$.

Step 2. By mean value theorem and Hölder inequality (1), it can be easily verified that

$$e^{(4\pi-\varepsilon)u_\varepsilon^2} \to e^{4\pi u_0^2} \quad \text{in } L^1(\Omega) \text{ as } \quad \varepsilon\to 0.$$

Step 3. On the basis of the above steps, there have

$$\begin{aligned}&\int_\Omega |(1+g(u_\varepsilon))e^{(4\pi-\varepsilon)u_\varepsilon^2} - (1+g(u_0))e^{4\pi u_0^2}|dx\\ \leqslant &\int_\Omega |g(u_\varepsilon)-g(u_0)|e^{(4\pi-\varepsilon)u_\varepsilon^2}dx + |g(u_0)+1|\int_\Omega |e^{(4\pi-\varepsilon)u_\varepsilon^2} - e^{4\pi u_0^2}|dx = o_\varepsilon(1).\end{aligned}$$

Hence

$$\int_\Omega (1+g(u_\varepsilon))e^{(4\pi-\varepsilon)u_\varepsilon^2} \to \int_\Omega (1+g(u_0))e^{4\pi u_0^2} \text{ in } \Omega,$$

as $\varepsilon \to 0$. This together with (13), we proof the existence of the extremal function.

Case 2. If $u_0 \equiv 0$, in view of the equation (12), the following discussion is crucial to our analysis:

LEMMA 1. *Let λ_ε be as in (12). Then there holds* $\liminf_{\varepsilon\to 0}\lambda_\varepsilon > 0$.

Proof. By an inequality $e^t \leqslant 1+te^t$ for $t \geqslant 0$ and $g'(t) > 0$ in (3), we get

$$\begin{aligned}\lambda_\varepsilon &\geqslant \frac{1}{4\pi-\varepsilon}\int_\Omega (1+\omega(u_\varepsilon))(e^{(4\pi-\varepsilon)u_\varepsilon^2}-1)dx\\ &\geqslant \frac{1}{4\pi-\varepsilon}\left(\Lambda_{4\pi-\varepsilon} + \int_\Omega \frac{g'(u_\varepsilon)}{2u_\varepsilon(4\pi-\varepsilon)}(e^{(4\pi-\varepsilon)u_\varepsilon^2}-1)dx - \int_\Omega (1+g(u_\varepsilon))dx\right)\\ &\geqslant \frac{1}{4\pi-\varepsilon}\left(\Lambda_{4\pi-\varepsilon} - \int_\Omega (1+g(u_\varepsilon))dx\right).\end{aligned}$$

Passing to the limit $\varepsilon \to 0$, since $u_\varepsilon \not\equiv 0$, together with (14) and (15) leads to

$$\liminf_{\varepsilon\to 0} \lambda_\varepsilon \geqslant \frac{1}{4\pi}\left(\Lambda_{4\pi} - (1+g(0))|\Omega|\right) > 0.$$

Hence, it gives

$$\frac{1}{\lambda_\varepsilon} \leqslant C. \tag{18}$$

Namely, $\frac{1}{\lambda_\varepsilon}$ is uniformly bounded. This ends the proof of the lemma. □

2.3. Blow-up analysis

In the following discussion, we will use blow-up analysis to understand the asymptotic behavior of the maximizers u_ε, when u_ε is not uniformly bounded in Ω. We first claim that x_0 can not lie on the boundary $\partial\Omega$.

Using the equation (12), we have

$$-\Delta u_\varepsilon = \frac{1}{\lambda_\varepsilon}(1+\omega(u_\varepsilon))u_\varepsilon e^{(4\pi-\varepsilon)u_\varepsilon^2}, \quad u_\varepsilon > 0 \quad \text{in} \quad \Omega, \quad u_\varepsilon = 0 \quad \text{on} \quad \partial\Omega,$$

where $\lambda_\varepsilon, 1+\omega(u_\varepsilon)$ are both positive constants depending on ε as defined in (3) and (5). Thus, u_ε satisfies

$$-\Delta u_\varepsilon = f_\varepsilon(u),$$

where

$$f_\varepsilon(u) = \frac{1}{\lambda_\varepsilon}(1+\omega(u_\varepsilon))u_\varepsilon e^{(4\pi-\varepsilon)u_\varepsilon^2} > 0 \quad \text{in} \quad \Omega.$$

Then the proof $x_0 \notin \partial\Omega$ follows from Lu-Yang [14] (page 970) and Gidas-Ni-Nireberg [8] (page 223). Thus, we exclude the boundary to blow-up.

From now on, we assume $x_0 \in \Omega$.

2.3.1. Energy concentration phenomenon

Followed by [27], we study the concentration phenomenon, which is crucial in our blow-up analysis:

LEMMA 2. *For the sequence $\{u_\varepsilon\}$, we have that $u_\varepsilon \rightharpoonup 0$ weakly in $W_0^{1,2}(\Omega)$ and $u_\varepsilon \to 0$ strongly in $L^q(\Omega)$ for any $q > 1$. Moreover, $|\nabla u_\varepsilon|^2 dx \rightharpoonup \delta_{x_0}$ in a sense of measure, where δ_{x_0} is the usual Dirac measure centered at the point x_0.*

Proof. Since $\|\nabla u_\varepsilon\|_2 = 1$ and $u_\varepsilon \in W_0^{1,2}(\Omega)$, there have the same assumptions as in (16). Assume $u_0 \not\equiv 0$. Since $(1+g(u))e^{(4\pi-\varepsilon)u^2}$ is bounded in $L^p(\Omega)$ for some $p > 1$ provided that ε is sufficiently small, together with (18) and (16) implies that Δu_ε is bounded in $L^q(\Omega)$ for some $q > 1$. Applying elliptic estimate to (12), one gets u_ε is uniformly bounded in Ω, which contradicts to $c_\varepsilon \to \infty$. Therefore $u_0 \equiv 0$.

We next prove $|\nabla u_\varepsilon|^2 dx \rightharpoonup \delta_{x_0}$. If the statements were false, suppose $|\nabla u_\varepsilon|^2 dx \rightharpoonup \eta$ in a sense of measure. In view of $\eta \neq \delta_{x_0}$, there exists $r_0 > 0$ such that

$$\lim_{\varepsilon \to 0} \int_{B_{r_0}(x_0)} |\nabla u_\varepsilon|^2 dx \leqslant \frac{\eta + 1}{2} < 1.$$

Choose a cut-off function $\phi \in C_0^1(\Omega)$, which is supported in $B_{r_0}(x_0) \in \Omega$.

$$\begin{cases} \phi(x) = 1 & \text{in} \quad B_{r_0/2}(x_0), \\ \phi(x) = 0 & \text{on} \quad \partial B_{r_0}(x_0). \end{cases}$$

For some $\eta > 0$, we can find proper $r_0 > 0$ and sufficient small ε such that

$$\int_{B_{r_0}(x_0)} |\nabla(\phi u_\varepsilon)|^2 dx \leqslant 1 - \eta.$$

By the classical Trudinger-Moser inequality (1), $e^{(4\pi-\varepsilon)(\phi u_\varepsilon)^2}$ is bounded in $L^r(\Omega)$ for some $r > 1$. Then the elliptic estimate on the Euler-Lagrange equation (12) indicates that u_ε is uniformly bounded in $B_{r_0/2}(x_0)$, contradicting to $c_\varepsilon \to \infty$ again. Consequently, $|\nabla u_\varepsilon|^2 dx \rightharpoonup \delta_{x_0}$. $\square$

2.3.2. Asymptotic behavior of u_ε near the concentration point x_0

Let

$$r_\varepsilon = \sqrt{\lambda_\varepsilon} c_\varepsilon^{-1} e^{-(2\pi - \varepsilon/2)c_\varepsilon^2}. \tag{19}$$

For any $0 < \delta < 4\pi$, together with (1) and (5), we have by using the Hölder inequality,

$$\lambda_\varepsilon = \int_\Omega (1 + \omega(u_\varepsilon)) u_\varepsilon^2 e^{(4\pi-\varepsilon)u_\varepsilon^2} dx \leqslant C e^{\delta c_\varepsilon^2} \int_\Omega u_\varepsilon^2 e^{(4\pi-\varepsilon-\delta)u_\varepsilon^2} dx \leqslant C e^{\delta c_\varepsilon^2},$$

for some constant C depending only on δ. This leads to

$$r_\varepsilon^2 \leqslant C c_\varepsilon^{-2} e^{-(4\pi-\varepsilon-\delta)c_\varepsilon^2} \to 0, \quad \text{for} \quad \forall 0 < \delta < 4\pi, \tag{20}$$

as $\varepsilon \to 0$. Define on $\Omega_\varepsilon = \{x \in \mathbb{R}^2 : x_\varepsilon + r_\varepsilon x \in \Omega\}$,

$$\alpha_\varepsilon(x) = c_\varepsilon^{-1} u_\varepsilon(x_\varepsilon + r_\varepsilon x), \tag{21}$$

$$\beta_\varepsilon(x) = c_\varepsilon(u_\varepsilon(x_\varepsilon + r_\varepsilon x) - c_\varepsilon). \tag{22}$$

A straightforward calculation gives

$$-\Delta \alpha_\varepsilon(x) = c_\varepsilon^{-2} \alpha_\varepsilon(x)(1 + \omega(u_\varepsilon)) e^{(4\pi-\varepsilon)({u_\varepsilon}^2(x_\varepsilon + r_\varepsilon) - c_\varepsilon^2)} \quad \text{in} \quad \Omega_\varepsilon, \tag{23}$$

$$-\Delta \beta_\varepsilon(x) = \alpha_\varepsilon(x)(1 + \omega(u_\varepsilon)) e^{(4\pi-\varepsilon)(1+\alpha_\varepsilon(x))\beta_\varepsilon(x)} \quad \text{in} \quad \Omega_\varepsilon. \tag{24}$$

We now investigate the convergence behavior of $\alpha_\varepsilon(x)$ and $\beta_\varepsilon(x)$. In view of (20), $r_\varepsilon \to 0$ and thus $\Omega_\varepsilon \to \mathbb{R}^2$ as $\varepsilon \to 0$. Since $0 < |\alpha_\varepsilon(x)| \leqslant 1$ and $\alpha_\varepsilon(x)$ is L^p bounded in $B_R\,(\forall R > 0)$, one has $\alpha_\varepsilon(x) \to \alpha$ in $C_{loc}^1(\mathbb{R}^2)$ by applying elliptic estimates to (23), where α is a bounded harmonic function on $\mathbb{R}^2$. Thus it gives $\alpha_\varepsilon \to \alpha$ in $C^1_{loc}(\mathbb{R}^2)$ uniformly as $\varepsilon \to 0$. If we set $-\Delta\alpha_\varepsilon(x) = g_\varepsilon(x)$, then for any $\alpha \in C^1_0(\mathbb{R}^2)$, there holds

$$\int_{B_R(0)} \nabla\alpha\nabla\alpha_\varepsilon dx = \int_{B_R(0)} g_\varepsilon \alpha dx.$$

Letting $\varepsilon \to 0$, by the arbitrary of α, we obtain

$$\Delta\alpha = 0.$$

Since $|\alpha| \leqslant 1$, the Liouville theorem implies that

$$\alpha = \alpha(0) = \lim_{\varepsilon\to 0} \frac{u_\varepsilon(x_\varepsilon + r_\varepsilon \cdot 0)}{c_\varepsilon} = 1.$$

Therefore $\alpha_\varepsilon \to 1$ in $C^1_{loc}(\mathbb{R}^2)$. Similarly, we have $\beta_\varepsilon \to \beta$ in $C^1_{loc}(\mathbb{R}^2)$ by elliptic estimates, where β satisfies

$$\begin{cases} \beta(0) = 0 = \sup_{\mathbb{R}^2} \beta, \\ \Delta\beta = -e^{8\pi\beta} \quad \text{in} \quad \Omega. \end{cases} \tag{25}$$

For $\forall R > 1$, we know when $\varepsilon \to 0$,

$$\alpha_\varepsilon(x) = \frac{u_\varepsilon(x_\varepsilon + r_\varepsilon x)}{c_\varepsilon} \to 1 \text{ in } C^1_{loc}(\mathbb{R}^2),$$

which together with (5) and Fatou's lemma lead to

$$\begin{aligned}\int_{\mathbb{B}_R(0)} e^{8\pi\beta} dx &\leqslant \limsup_{\varepsilon\to 0} \int_{\mathbb{B}_R(0)} e^{(4\pi-\varepsilon)(1+\alpha_\varepsilon(x))\beta_\varepsilon(x)} dx \\ &\leqslant \limsup_{\varepsilon\to 0} \lambda_\varepsilon^{-1} \int_{\mathbb{B}_{Rr_\varepsilon}(x_\varepsilon)} (1+\omega(u_\varepsilon))u_\varepsilon^2 e^{(4\pi-\varepsilon)u_\varepsilon^2(y)} dy \leqslant 1.\end{aligned}$$

Let $R \to \infty$, and this leads to

$$\int_{\mathbb{R}^2} e^{8\pi\beta} dx \leqslant 1.$$

More precisely, the uniqueness theorem obtained in [4] implies that

$$\beta(x) = -\frac{1}{4\pi}\log(1+\pi|x|^2),$$

and

$$\int_{\mathbb{R}^2} e^{8\pi\beta} dx = 1. \tag{26}$$

Until this, we have gave the convergence behavior of u_ε near the point x_0. To reveal the convergence behavior of u_ε away from x_0, we define $u_{\varepsilon,\sigma} = \min\{\sigma c_\varepsilon, u_\varepsilon\}$ as in [1, 13]. Then we have the following:

2.3.3. Convergence away from the concentration point

LEMMA 3. *For any* $0<\sigma<1$, *there holds*

$$\lim_{\varepsilon\to 0}\int_\Omega |\nabla u_{\varepsilon,\sigma}|^2 dx=\sigma.$$

Proof. In view of the equation (12), we deduce the following through using a suitable change of variable $y=x_\varepsilon+r_\varepsilon x$,

$$\begin{aligned}\int_\Omega |\nabla u_{\varepsilon,\sigma}|^2 dx &= -\int_\Omega u_{\varepsilon,\sigma}\Delta u_\varepsilon dx=\int_\Omega \frac{u_{\varepsilon,\sigma}}{\lambda_\varepsilon}(1+\omega(u_\varepsilon))u_\varepsilon e^{(4\pi-\varepsilon)u_\varepsilon^2}dx\\ &\geqslant \int_{B_{Rr_\varepsilon}(x_\varepsilon)}\frac{\sigma c_\varepsilon}{\lambda_\varepsilon}u_\varepsilon(1+\omega(u_\varepsilon))e^{(4\pi-\varepsilon)u_\varepsilon^2}dx\\ &=\int_{B_R(0)}\sigma\alpha_\varepsilon(y)(1+\omega(u_\varepsilon(y)))e^{(4\pi-\varepsilon)(1+\alpha_\varepsilon)\beta_\varepsilon}dy.\end{aligned}$$

Obviously, there have

$$\liminf_{\varepsilon\to 0}\int_\Omega |\nabla u_{\varepsilon,\sigma}|^2 dx\geqslant \sigma\int_{B_R(0)} e^{8\pi\beta}dx,\ \forall R>0.$$

Passing to the limit $R\to+\infty$ in the above inequality, we have from (26)

$$\liminf_{\varepsilon\to 0}\int_\Omega |\nabla u_{\varepsilon,\sigma}|^2 dx\geqslant \sigma. \tag{27}$$

Note that

$$|\nabla(u_\varepsilon-\sigma c_\varepsilon)^+|^2=\nabla(u_\varepsilon-\sigma c_\varepsilon)^+\cdot\nabla u_\varepsilon \quad \text{on} \quad \Omega,$$

and

$$(u_\varepsilon-\sigma c_\varepsilon)^+=(1+o_\varepsilon(1))(1-\sigma)c_\varepsilon \quad \text{in} \quad B_{Rr_\varepsilon}(x_0).$$

Similarly as above, we calculate

$$\begin{aligned}\int_\Omega |\nabla(u_\varepsilon-\sigma c_\varepsilon)^+|^2 dx &= \int_\Omega (u_\varepsilon-\sigma c_\varepsilon)^+\frac{u_\varepsilon}{\lambda_\varepsilon}(1+\omega_\varepsilon(u_\varepsilon))e^{(4\pi-\varepsilon)u_\varepsilon^2}dx\\ &\geqslant \int_{B_{Rr_\varepsilon}(x_\varepsilon)}(u_\varepsilon-\sigma c_\varepsilon)^+\frac{u_\varepsilon}{\lambda_\varepsilon}(1+\omega_\varepsilon(u_\varepsilon))e^{(4\pi-\varepsilon)u_\varepsilon^2}dx\\ &=\int_{B_R(x_0)}(1+o_\varepsilon(1))(1-\sigma)\alpha_\varepsilon(1+\omega_\varepsilon(u_\varepsilon))e^{(4\pi-\varepsilon)u_\varepsilon^2}dx.\end{aligned}$$

Let $\varepsilon\to 0$ and then $R\to\infty$, we have by (26)

$$\liminf_{\varepsilon\to 0}\int_\Omega |\nabla(u_\varepsilon-\sigma c_\varepsilon)^+|^2 dx\geqslant 1-\sigma. \tag{28}$$

Since $|\nabla u_\varepsilon|^2=|\nabla u_{\varepsilon,\sigma}|^2+|\nabla(u_\varepsilon-\sigma c_\varepsilon)^+|^2$ almost everywhere, there have

$$\int_\Omega |\nabla(u_\varepsilon-\sigma c_\varepsilon)^+|^2 dx+\int_\Omega|\nabla u_{\varepsilon,\sigma}|^2dx=\int_\Omega|\nabla u_\varepsilon|^2dx=1+o_\varepsilon(1). \tag{29}$$

Combining (27), (28) and (29), we end the proof of the lemma. □

Though the following estimate is not used in this step, it is a byproduct of lemma 3 and will be employed in the next section.

LEMMA 4. *There holds*

$$\lim_{\varepsilon\to 0}\int_\Omega (1+g(u_\varepsilon))e^{(4\pi-\varepsilon)u_\varepsilon^2}dx = \mid\Omega\mid(1+g(0))+\lim_{\varepsilon\to 0}\frac{\lambda_\varepsilon}{c_\varepsilon^2}. \tag{30}$$

Proof. Note that $0\leqslant u_{\varepsilon,\sigma}\leqslant u_\varepsilon$. As $\varepsilon\to 0$, we have $u_{\varepsilon,\sigma}$ converges to 0 in $C^1_{loc}(\overline{\Omega}\setminus\{x_0\})$. According to the Hölder inequality and Lagrange theorem, it follows

$$\begin{aligned}
&\int_{\{x\in\Omega|u_\varepsilon\leqslant\sigma c_\varepsilon\}}(1+g(u_\varepsilon))e^{(4\pi-\varepsilon)u_\varepsilon^2}dx-|\Omega|(1+g(0))\\
\leqslant&\int_\Omega(1+g(u_{\varepsilon,\sigma}))e^{(4\pi-\varepsilon)u_{\varepsilon,\sigma}^2}dx-|\Omega|(1+g(0))\\
\leqslant&\int_\Omega|g(u_{\varepsilon,\sigma})-g(0)|e^{(4\pi-\varepsilon)u_{\varepsilon,\sigma}^2}dx+|1+g(0)|\int_\Omega(e^{(4\pi-\varepsilon)u_{\varepsilon,\sigma}^2}-1)dx\\
=&o_\varepsilon(1).
\end{aligned}\tag{31}$$

Moreover, we estimate

$$\begin{aligned}
&\int_{\{x\in\Omega|u_\varepsilon>\sigma c_\varepsilon\}}((1+g(u_\varepsilon))e^{(4\pi-\varepsilon)u_\varepsilon^2}-(1+g(0))dx\\
=&\int_{\{x\in\Omega|u_\varepsilon>\sigma c_\varepsilon\}}((1+g(u_\varepsilon))e^{(4\pi-\varepsilon)u_\varepsilon^2}dx-(1+g(0))\mid\{x\in\Omega\mid u_\varepsilon>\sigma c_\varepsilon\}\mid\\
<&\frac{1}{\sigma^2}\int_{\{x\in\Omega|u_\varepsilon>\sigma c_\varepsilon\}}\frac{u_\varepsilon^2}{c_\varepsilon^2}((1+g(u_\varepsilon))e^{(4\pi-\varepsilon)u_\varepsilon^2}dx+o_\varepsilon(1)\leqslant\frac{\lambda_\varepsilon}{\sigma^2c_\varepsilon^2}+o_\varepsilon(1),
\end{aligned}\tag{32}$$

where $o_\varepsilon(1)\to 0$ as $\varepsilon\to 0$. Combining (31) and (32), it follows by letting $\sigma\to 1$,

$$\limsup_{\varepsilon\to 0}\left(\int_\Omega(1+g(u_\varepsilon))e^{(4\pi-\varepsilon)u_\varepsilon^2}dx-\mid\Omega\mid(1+g(0))\right)\leqslant\lim_{\varepsilon\to 0}\frac{\lambda_\varepsilon}{c_\varepsilon^2}. \tag{33}$$

On the other hand,

$$\begin{aligned}
&\int_\Omega(1+g(u_\varepsilon))e^{(4\pi-\varepsilon)u_\varepsilon^2}dx-\mid\Omega\mid(1+g(0))\\
\geqslant&\int_\Omega\frac{u_\varepsilon^2}{c_\varepsilon^2}\Big((1+g(u_\varepsilon))e^{(4\pi-\varepsilon)u_\varepsilon^2}-(1+g(0))\Big)dx\\
=&\frac{\lambda_\varepsilon}{c_\varepsilon^2}-\frac{1}{c_\varepsilon^2}\int_\Omega u_\varepsilon^2(1+g(0))dx-\frac{1}{c_\varepsilon^2}\int_\Omega\frac{g'(u_\varepsilon)}{2(4\pi-\varepsilon)}u_\varepsilon e^{(4\pi-\varepsilon)u_\varepsilon^2}dx.
\end{aligned}$$

In view of lemma 2 and (5), there holds

$$\lim_{\varepsilon\to 0}\frac{\lambda_\varepsilon}{c_\varepsilon^2}\leqslant\liminf_{\varepsilon\to 0}\left(\int_\Omega(1+g(u_\varepsilon))e^{(4\pi-\varepsilon)u_\varepsilon^2}dx-|\Omega|(1+g(0))\right). \tag{34}$$

Combing (33) and (34), we conclude the lemma 4. □

Letting $R \to \infty$, one can further check that

$$\lim_{\varepsilon\to 0}\frac{\lambda_\varepsilon}{c_\varepsilon^2}=\lim_{R\to\infty}\lim_{\varepsilon\to 0}\int_{B_{Rr_\varepsilon}(x_\varepsilon)}(1+g(u_\varepsilon))e^{(4\pi-\varepsilon)u_\varepsilon^2}dx. \tag{35}$$

This can be proved by adopting variables substitution, namely

$$\begin{aligned}&\lim_{R\to\infty}\lim_{\varepsilon\to 0}\int_{B_{Rr_\varepsilon}(x_\varepsilon)}(1+g(u_\varepsilon))e^{(4\pi-\varepsilon)u_\varepsilon^2}dx\\&=\lim_{R\to\infty}\lim_{\varepsilon\to 0}\int_{B_{R(0)}}(1+g(u_\varepsilon(x_\varepsilon+r_\varepsilon x)))e^{(4\pi-\varepsilon)c_\varepsilon^2}e^{(4\pi-\varepsilon)(1+\alpha_\varepsilon)\beta_\varepsilon}dx\\&=\lim_{\varepsilon\to 0}\frac{\lambda_\varepsilon}{c_\varepsilon^2}\lim_{R\to\infty}\int_{B_{R(0)}}e^{8\pi\beta}dx.\end{aligned}$$

Insert (26) into the above equation, and then (35) holds.

COROLLARY 1. *If $\theta<2$, then $\frac{\lambda_\varepsilon}{c_\varepsilon^\theta}\to\infty$ as $\varepsilon\to 0$.*

Proof. This can be an obvious consequence of lemma 4. □

LEMMA 5. *For any function $\phi\in C_0^1(\Omega)$, there holds*

$$\lim_{\varepsilon\to 0}\int_\Omega \phi(1+\omega(u_\varepsilon))\lambda_\varepsilon^{-1}c_\varepsilon u_\varepsilon e^{(4\pi-\varepsilon)u_\varepsilon^2}dx=\phi(x_0). \tag{36}$$

Proof. To see this, let $\phi\in C_0^1(\Omega)$ be fixed. Write $t_\varepsilon=(1+\omega(u_\varepsilon))\lambda_\varepsilon^{-1}c_\varepsilon u_\varepsilon e^{(4\pi-\varepsilon)u_\varepsilon^2}$ for simplicity. Clearly,

$$\int_\Omega t_\varepsilon\phi dx=\int_{\{u_\varepsilon<\beta c_\varepsilon\}}t_\varepsilon\phi dx+\int_{\{u_\varepsilon\geqslant\beta c_\varepsilon\}\setminus B_{Rr_\varepsilon}(x_\varepsilon)}t_\varepsilon\phi dx+\int_{\{u_\varepsilon\geqslant\beta c_\varepsilon\}\cap B_{Rr_\varepsilon}(x_\varepsilon)}t_\varepsilon\phi dx. \tag{37}$$

We estimate the three integrals on the right hand of (37) respectively. By lemma 3 and corollary 1, it follows that

$$\int_{\{u_\varepsilon<\beta c_\varepsilon\}}t_\varepsilon\phi dx\leqslant C\lambda_\varepsilon^{-1}c_\varepsilon\int_\Omega u_\varepsilon e^{(4\pi-\varepsilon)u_{\varepsilon,\beta}^2}\phi dx=o_\varepsilon(1). \tag{38}$$

Since $B_{Rr_\varepsilon}(x_\varepsilon)\subset\{x\in\Omega\mid u_\varepsilon\geqslant\beta c_\varepsilon\}$ for sufficiently small $\varepsilon>0$, we have by (19),

$$\begin{aligned}\int_{\{u_\varepsilon\geqslant\beta c_\varepsilon\}\cap B_{Rr_\varepsilon}(x_\varepsilon)}t_\varepsilon\phi dx&=(\phi(x_0)+o_\varepsilon(1))\int_{B_{Rr_\varepsilon}(x_\varepsilon)}(1+\omega(u_\varepsilon))\lambda_\varepsilon^{-1}c_\varepsilon u_\varepsilon e^{(4\pi-\varepsilon)u_\varepsilon^2}dx\\&=(\phi(x_0)+o_\varepsilon(1))\int_{B_R(x_0)}(1+\omega(u_\varepsilon))e^{8\pi\beta}dx\\&=\phi(x_0)+o_\varepsilon(1)+o_R(1).\end{aligned} \tag{39}$$

Noting that

$$\begin{aligned}\int_{\{u_\varepsilon \geqslant \beta c_\varepsilon\}\setminus B_{Rr_\varepsilon}(x_\varepsilon)} t_\varepsilon \phi dx &\leqslant C\int_{\{u_\varepsilon \geqslant \beta c_\varepsilon\}\setminus B_{Rr_\varepsilon}(x_\varepsilon)} \frac{u_\varepsilon^2}{\beta\lambda_\varepsilon} e^{(4\pi-\varepsilon)u_\varepsilon^2}\phi dx\\ &\leqslant \frac{C}{\beta}\int_{\Omega\setminus B_{Rr_\varepsilon}(x_\varepsilon)} \frac{u_\varepsilon^2}{\lambda_\varepsilon} e^{(4\pi-\varepsilon)u_\varepsilon^2} dx\\ &= \frac{C}{\beta}(1-\int_{B_{R(x_0)}} e^{8\pi\beta} dx + o_\varepsilon(1)),\end{aligned}$$

we derive by (19),

$$\lim_{R\to\infty}\lim_{\varepsilon\to 0}\int_{\{u_\varepsilon \geqslant \beta c_\varepsilon\}\setminus B_{Rr_\varepsilon}(x_\varepsilon)} t_\varepsilon \phi dx = 0. \tag{40}$$

Inserting (38)-(40) to (37), we conclude (36) finally. □

Let $\phi = 1$, one can further infer that

$$\frac{(1+\omega(u_\varepsilon))}{\lambda_\varepsilon}(c_\varepsilon u_\varepsilon) e^{(4\pi-\varepsilon)u_\varepsilon^2} \leqslant C \quad \text{in} \quad L^1(\Omega), \tag{41}$$

which will be used in the following proof.

LEMMA 6. *(Struwe [22]) If $f \in L^1(\Omega)$, $u \in W_0^{1,2}(\Omega)\cap C^1(\Omega)$ is a positive solution of $-\Delta u = f$. Then for any $1 < q < 2$, there have $\|\nabla u\|_q \leqslant C\|f\|_1$, for some constant C depending on q and Ω.*

By equation (12), $c_\varepsilon u_\varepsilon$ is a distributional solution to

$$-\Delta(c_\varepsilon u_\varepsilon) = t_\varepsilon \quad \text{in} \quad \Omega. \tag{42}$$

It follows from (41) that t_ε is bounded in $L^1(\Omega)$. Using lemma 6 and elliptic estimates to (42), one concludes that $c_\varepsilon u_\varepsilon$ is bounded in $W_0^{1,q}(\Omega)$ for all $1 < q < 2$.

We now prove that $c_\varepsilon u_\varepsilon$ converges to a Green function in distributional sense when $\varepsilon \to 0$, where δ_{x_0} stands for the Dirac measure centered at x_0. More precisely, we have

LEMMA 7. *$c_\varepsilon u_\varepsilon \to G$ in $C^1_{loc}(\overline{\Omega}\setminus\{x_0\})$ and weakly in $W_0^{1,q}(\Omega)$ for all $1<q<2$, where $G \in C^1(\Omega\setminus\{x_0\})$ is a distributional solution satisfying the following*

$$\begin{cases} -\Delta G = \delta_{x_0} & \text{in} \quad \Omega,\\ G = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{43}$$

Moreover, G takes the form

$$G(x) = -\frac{1}{2\pi}\log|x-x_0| + A_{x_0} + \nu(x), \tag{44}$$

where A_{x_0} is a constant depending on x_0 and $\nu(x) \in C^1(\overline{\Omega})$.

Proof. By lemma 6, we can assume for any $1<q<2$, $r>1$ that

$$\begin{aligned} c_\varepsilon u_\varepsilon &\rightharpoonup G \quad \text{weakly in} \quad W_0^{1,q}(\Omega), \\ c_\varepsilon u_\varepsilon &\to G \quad \text{strongly in} \quad L^r(\Omega). \end{aligned}$$

Test (42) by $\phi \in C_0^1(\Omega)$ and there holds

$$\int_\Omega \nabla(c_\varepsilon u_\varepsilon)\nabla\phi dx = \int_\Omega \phi(1+\omega(u_\varepsilon))\lambda_\varepsilon^{-1}c_\varepsilon u_\varepsilon e^{(4\pi-\varepsilon)u_\varepsilon^2}dx.$$

Letting $\varepsilon\to 0$, we obtain by (43)

$$\int_\Omega \nabla G\nabla\phi dx = \phi(x_0).$$

Consequently, $-\Delta G = \delta_{x_0}$ in a distributional sense. Since $\Delta(G+\frac{1}{2\pi}\log|x-x_0|)\in L^p(\Omega)$ for any $p>2$, (44) follows from elliptic solution immediately. Applying elliptic estimates to the equation (42), we arrive at the conclusion

$$c_\varepsilon u_\varepsilon \to G \quad \text{in } C^1_{loc}(\overline{\Omega}\setminus\{x_0\}). \tag{45}$$

Thus, the two assertions holds. $\square$

2.4. Upper bound estimates

Similar to [12, 25], we demand the following result belongs to [3], namely

LEMMA 8. *(Carleson-Chang) Let B be the unit disc in $\mathbb{R}^2$. Assume $v_\varepsilon\in W_0^{1,2}(B)$ satisfying $\int_B|\nabla v_\varepsilon|^2dx\leqslant 1$, and $|\nabla v_\varepsilon|^2dx\rightharpoonup\delta_{x_0}$ weakly in a sense of measure as $\varepsilon\to 0$. Then we have*

$$\limsup_{\varepsilon\to 0}\int_B (e^{4\pi v_\varepsilon^2}-1)dx\leqslant \pi e.$$

Multiplying both sides of (43) by G and integrating by parts on the domain $\Omega\setminus B_\delta(x_0)$ for some fixed $\delta>0$, we get

$$\begin{aligned} \int_{\Omega\setminus B_\delta(x_0)}|\nabla G|^2dx &= -\int_{\partial B_\delta(x_0)} G\frac{\partial G}{\partial\overrightarrow{n}}ds - \int_{\Omega\setminus B_\delta(x_0)} G\cdot\Delta G dx \\ &= -\frac{1}{2\pi}\log\delta + A_{x_0} + o_\varepsilon(1) + o_\delta(1). \end{aligned} \tag{46}$$

In view of (45), there holds

$$\begin{aligned} \int_{\Omega\setminus B_\delta(x_0)}|\nabla u_\varepsilon|^2dx &= \frac{1}{c_\varepsilon^2}\int_{\Omega\setminus B_\delta(x_0)}(|\nabla G|^2+o_\varepsilon(1))dx \\ &= \frac{1}{c_\varepsilon^2}(A_{x_0} - \frac{\log\delta}{2\pi} + o_\varepsilon(1)+o_\delta(1)). \end{aligned} \tag{47}$$

Define $s_\varepsilon = \sup_{\partial B_\delta(x_0)} u_\varepsilon$, $\overline{u}_\varepsilon = (u_\varepsilon - s_\varepsilon)^+$, the positive part of $u_\varepsilon - s_\varepsilon$. It can easily be verified that $\overline{u}_\varepsilon \in W_0^{1,2}(B_\delta(x_0))$. Together with (47) and the fact that $\int_{B_\delta(x_0)} |\nabla \overline{u}_\varepsilon|^2 dx = 1 - \int_{\Omega\setminus B_\delta(x_0)} |\nabla u_\varepsilon|^2 dx$ gives

$$\int_{B_\delta(x_0)} |\nabla \overline{u}_\varepsilon|^2 dx \leqslant \eta_\varepsilon = 1 - \int_{\Omega\setminus B_\delta(x_0)} |\nabla u_\varepsilon|^2 dx = 1 - \frac{1}{c_\varepsilon^2}\left(A_{x_0} - \frac{\log\delta}{2\pi} + o_\varepsilon(1) + o_\delta(1)\right).$$

And then by lemma 8, there have

$$\limsup_{\varepsilon\to 0} \int_{B_\delta(x_0)} \left(e^{4\pi \frac{\overline{u}_\varepsilon}{\eta_\varepsilon}^2} - 1\right) dx \leqslant \pi\delta^2 e. \tag{48}$$

In $B_{Rr_\varepsilon}(x_\varepsilon) \subset B_\delta(x_0)$, note that

$$s_\varepsilon = \sup_{\partial B_\delta(x_0)} \frac{1}{c_\varepsilon}(G + o_\varepsilon(1)) = \frac{1}{c_\varepsilon}\left(A_{x_0} - \frac{1}{2\pi}\log\delta + o_\varepsilon(1) + o_\delta(1)\right),$$

and

$$u_\varepsilon^2 \leqslant (s_\varepsilon + \overline{u}_\varepsilon)^2.$$

Thus, we derive

$$(4\pi - \varepsilon)u_\varepsilon^2 \leqslant 4\pi \frac{\overline{u}_\varepsilon^2}{\eta_\varepsilon} + 2\log\frac{1}{\delta} + 4\pi A_{x_0} + o(1).$$

In $B_{Rr_\varepsilon}(x_\varepsilon) \subset B_\delta(x_0)$, together with (3), we estimate

$$\begin{aligned}\int_{B_{Rr_\varepsilon}(x_\varepsilon)} (1+g(u_\varepsilon))e^{(4\pi-\varepsilon)u_\varepsilon^2} dx &\leqslant \delta^{-2} e^{4\pi A_{x_0} + o(1)} \int_{B_{Rr_\varepsilon}(x_\varepsilon)} (1+g(u_\varepsilon))e^{4\pi \frac{\overline{u}_\varepsilon}{\eta_\varepsilon}} dx \\ &\leqslant \delta^{-2} e^{4\pi A_{x_0} + o(1)} \int_{B_\delta(x_0)} \left(e^{4\pi \frac{\overline{u}_\varepsilon}{\eta_\varepsilon}} - 1\right) dx,\end{aligned} \tag{49}$$

where $o(1) \to 0$ as $\varepsilon \to 0$. In view of (48) and (49), one gets

$$\limsup_{\varepsilon\to 0} \int_{B_{Rr_\varepsilon}(x_\varepsilon)} (1+g(u_\varepsilon))e^{(4\pi-\varepsilon)u_\varepsilon^2} dx \leqslant \pi e^{1+4\pi A_{x_0}}. \tag{50}$$

Passing to the limit $R \to \infty$, adapted from (30) and (35), there holds on

$$\lim_{\varepsilon\to 0} \int_\Omega (1+g(u_\varepsilon))e^{(4\pi-\varepsilon)u_\varepsilon^2} dx = (1+g(0))|\Omega| + \lim_{R\to\infty}\lim_{\varepsilon\to 0} \int_{B_{Rr_\varepsilon}(x_\varepsilon)} (1+g(u_\varepsilon))e^{(4\pi-\varepsilon)u_\varepsilon^2} dx. \tag{51}$$

Combining (50) and (51), in view of (3), we arrive at

$$\sup_{u\in W_0^{1,2}(\Omega), \|\nabla u\|_2 = 1} \int_\Omega (1+g(u))e^{4\pi u^2} dx \leqslant (1+g(0))|\Omega| + \pi e^{1+4\pi A_{x_0}}. \tag{52}$$

2.5. Existence of extremal function

In this subsection, we will construct a blow-up sequence $\phi_\varepsilon \in W_0^{1,2}(\Omega)$ with $\|\nabla\phi_\varepsilon\|_2 = 1$. While for sufficiently small $\varepsilon > 0$, there exists

$$\int_\Omega (1+g(\phi_\varepsilon))e^{4\pi\phi_\varepsilon^2}dx > (1+g(0))|\Omega| + \pi e^{1+4\pi A_{x_0}}. \tag{53}$$

Then, we will find (52) is a contradiction to (53), so that c_ε has to be bounded, which means the blow-up cannot take place. Furthermore, the theorem follows immediately from what we have proved according to elliptic estimates.

To prove (53), as we did in [26], we set $v = G + \frac{1}{2\pi}\log r - A_{x_0}$, where x_0 is the concentration point as before, $r(x) = |x - x_0|$, and $v = O(r)$. Define

$$\phi_\varepsilon = \begin{cases} c + \frac{1}{c}(-\frac{1}{4\pi}\log(1+\frac{\pi r^2}{\varepsilon^2}) + b), & \text{for } r \leqslant R\varepsilon, \\ \frac{G-\xi v}{c}, & \text{for } R\varepsilon < r < 2R\varepsilon, \\ \frac{G}{c}, & \text{for } R \geqslant 2R\varepsilon, \end{cases} \tag{54}$$

where $\xi \in C_0^\infty(B_{2R\varepsilon}(x_0))$ is a cut-off function satisfying $\xi = 1$ on $B_{R\varepsilon}(x_0)$, and $\|\nabla\xi\|_{L^\infty} = O(\frac{1}{R\varepsilon})$.

Above all, b and c are constants which depending only on ε to be determined later. Obviously, $B_{2R\varepsilon}(x_0) \subset \Omega$ provided that ε is sufficiently small. In order to assure $\phi_\varepsilon \in W_0^{1,2}(\Omega)$, we set

$$c + \frac{1}{c}(-\frac{1}{4\pi}\log(1+\frac{\pi(R\varepsilon)^2}{\varepsilon^2}) + b) = \frac{1}{c}(-\frac{1}{2\pi}\log R\varepsilon + A_{x_0}),$$

which gives

$$2\pi c^2 = -\log\varepsilon - 2\pi b + 2\pi A_{x_0} + \frac{1}{2}\log\pi + O(\frac{1}{R^2}). \tag{55}$$

Noting that $v(x) = O(|x|)$ as $x \to 0$, we have $|\nabla(\xi v)| = O(1)$ as $\varepsilon \to 0$. Now we calculate

$$\int_{B_{R\varepsilon}(x_0)} |\nabla\phi_\varepsilon|^2 dx = \int_0^{R\varepsilon} \frac{\pi r^2}{4c^2(\varepsilon^2+\pi r^2)^2} dr^2 = \frac{1}{4\pi c^2}(\log\pi - 1 + O(\frac{1}{R^2})). \tag{56}$$

Set $\delta = R\varepsilon$. One can check from (46) that

$$\begin{aligned} \int_{\Omega\setminus B_{R\varepsilon}(x_0)} |\nabla\phi_\varepsilon|^2\,dx &= \frac{1}{c^2}\int_{\Omega\setminus 2B_{R\varepsilon}(x_0)} |\nabla G|^2\,dx + \int_{2B_{R\varepsilon}(x_0)\setminus B_{R\varepsilon}(x_0)} \frac{|\nabla(G-\xi v)|^2}{c^2}dx \\ &= \frac{1}{c^2}\int_{\Omega\setminus B_{R\varepsilon}(x_0)} |\nabla G|^2\,dx \\ &= \frac{1}{c^2}(\frac{1}{2\pi}\log\frac{1}{R\varepsilon} + A_{x_0} + O(R\varepsilon\log(R\varepsilon))). \end{aligned} \tag{57}$$

Combing (56) and (57), we can obtain

$$\int_{\Omega} |\nabla \phi_\varepsilon|^2 dx = \frac{1}{4\pi c^2}(2\log\frac{1}{\varepsilon} + \log\pi - 1 + 4\pi A_{x_0} + O(\frac{1}{R^2}) + O(R\varepsilon\log(R\varepsilon))). \quad (58)$$

Put $\|\nabla \phi_\varepsilon\|_2 = 1$. We know from (58) that

$$c^2 = A_{x_0} - \frac{1}{2\pi}\log\varepsilon + \frac{1}{4\pi}\log\pi - \frac{1}{4\pi} + O(\frac{1}{R^2}) + O(R\varepsilon\log(R\varepsilon)). \quad (59)$$

Together with (55) and (59) gives

$$b = \frac{1}{4\pi} + O(\frac{1}{R^2}) + O(R\varepsilon\log(R\varepsilon)). \quad (60)$$

When $R = -\log\varepsilon$, in view of (59) and (60), there holds on $B_{R\varepsilon}(x_0)$,

$$\begin{aligned} 4\pi\phi_\varepsilon^2 &\geqslant 4\pi c^2 - 2\log(1+\frac{\pi r^2}{\varepsilon^2}) + 8\pi b \\ &= 4\pi A_{x_0} + \log\pi - 2\log\varepsilon - 2\log(1+\frac{\pi r^2}{\varepsilon^2}) + 1 + O(R\varepsilon\log(R\varepsilon)). \end{aligned} \quad (61)$$

Besides, we derive $\|\frac{\phi_\varepsilon(x)}{c}\|_{L^\infty(B_{R\varepsilon})} \to 1$ by passing to the limit $\varepsilon \to 0$. When $r \leqslant R\varepsilon$, there exists

$$|\frac{\phi_\varepsilon(x)}{c}| = |1 + \frac{-\log(1+\pi\frac{r^2}{\varepsilon^2}) + b}{c^2}| \to 1.$$

Since $\phi_\varepsilon(x) \sim c$ in $B_{R\varepsilon}(x_\varepsilon)$ and $g(c) = o(\frac{1}{c^2})$, we conclude $g(\phi_\varepsilon) = o(\frac{1}{c^2})$ as $\varepsilon \to 0$, which together with (61) leads to

$$\int_{B_{R\varepsilon}(x_0)} (1+g(\phi_\varepsilon))e^{4\pi\phi_\varepsilon^2} dx \geqslant \pi e^{1+4\pi A_{x_0}} + O(R\varepsilon\log(R\varepsilon)) + o(\frac{1}{c^2}). \quad (62)$$

On the other hand, in view of lemma 4 and (3), there exists $g(\phi_\varepsilon) \geqslant g(0)$ in $\Omega \setminus 2B_{R\varepsilon}(x_0)$. Thus, by using the inequality $e^t \geqslant t+1$, $\forall t \geqslant 0$, we estimate

$$\begin{aligned} \int_{\Omega\setminus B_{R\varepsilon}(x_0)} (1+g(\phi_\varepsilon))e^{4\pi\phi_\varepsilon^2} dx &\geqslant \int_{\Omega\setminus 2B_{R\varepsilon}(x_0)} (1+g(\phi_\varepsilon))(1+4\pi\frac{G^2}{c^2}) \\ &\geqslant (1+g(0))|\Omega| + (1+g(0))\frac{4\pi}{c^2}\|G\|_2^2 + O(\frac{1}{R^2}). \end{aligned} \quad (63)$$

Since $R = -\log\varepsilon$, it is clear that $\frac{1}{R} = o(\frac{1}{c^2})$. In view of (62) and (63), we conclude

$$\int_{\Omega} (1+g(\phi_\varepsilon))e^{4\pi\phi_\varepsilon^2} dx > (1+g(0))|\Omega| + \pi e^{1+4\pi A_{x_0}},$$

provided that $\varepsilon > 0$ is chosen sufficiently small.

2.6. Completion of proof of theorem 1

Comparing (52) with (53), we arrive at the final conclusion that c_ε must be bounded. Then applying elliptic estimates to (12), we can get the desired extremal function. This ends the proof of theorem 1. $\square$

In next section, we will confirm the results still remain correct if we add nonnegative weights.

3. Proof of theorem 2

3.1. The subcritical functions

Using the same argument as in the proof of theorem 1, we can easily carry out the proof of this theorem. In the beginning, we shall prove the existence of the maximizers for the subcritical functionals. Here we also adopt the method of variations during calculation.

LEMMA 9. *For any $0<\varepsilon<4\pi$, there exists $u_\varepsilon \in W_0^{1,2}(\Omega)\cap C^1(\overline{\Omega})$ with $\|\nabla u_\varepsilon\|_2 = 1$ such that*

$$\int_\Omega h(1+g(u_\varepsilon))e^{(4\pi-\varepsilon)u_\varepsilon^2}dx = \sup_{u\in W_0^{1,2}(\Omega),\,\|\nabla u\|_2\leqslant 1}\int_\Omega h(1+g(u))e^{(4\pi-\varepsilon)u^2}dx. \tag{64}$$

Proof. For $0<\varepsilon<4\pi$, take a function sequence $u_j\in W_0^{1,2}(\Omega)$ satisfying $\|\nabla u_j\|_2 \leqslant 1$, and

$$\int_\Omega h(1+g(u_j))e^{(4\pi-\varepsilon)u_j^2}dx \to \sup_{u\in W_0^{1,2}(\Omega),\,\|\nabla u\|_2\leqslant 1}\int_\Omega h(1+g(u))e^{(4\pi-\varepsilon)u^2}dx, \tag{65}$$

as $j\to\infty$. Note that there exists some $u_\varepsilon \in W_0^{1,2}(\Omega)$ such that up to a subsequence, which is similar to (16). And we can verify that $h(1+g(u))e^{(4\pi-\varepsilon)u_\varepsilon^2}$ is bounded in $L^p(\Omega)$ $(p>1)$. Then

$$h(1+g(u_j))e^{(4\pi-\varepsilon)u_j^2}dx \to h(1+g(u))e^{(4\pi-\varepsilon)u^2}\text{strongly in } L^1(\Omega). \tag{66}$$

Together with (65) and (66), it yields (64). $\square$

Moreover, the Euler-Lagrange equation of u_ε follows after a straightforward calculation:

$$\begin{cases} -\Delta u_\varepsilon = \frac{h}{\lambda_\varepsilon}(1+\omega(u_\varepsilon))u_\varepsilon e^{(4\pi-\varepsilon)u_\varepsilon^2}, & \text{in } \Omega,\\ u_\varepsilon>0,\ \|\nabla u_\varepsilon\|_2=1, & \text{in } \Omega,\\ \lambda_\varepsilon = \int_\Omega h(1+\omega(u_\varepsilon))u_\varepsilon^2 e^{(4\pi-\varepsilon)u_\varepsilon^2}dx, \end{cases} \tag{67}$$

where ω is defined as in (5). Using elliptic estimates, we get $u_\varepsilon\in C^1(\overline{\Omega})$. Set $c_\varepsilon = u_\varepsilon(x_\varepsilon)=\max_\Omega u_\varepsilon$. If c_ε is bounded, the existence of the extremal function is trivial by the standard elliptic estimates. Hence we discuss the opposite circumstance $c_\varepsilon\to\infty$

and $x_\varepsilon \to x_0 \in \overline{\Omega}$. We get $x_0 \notin \partial\Omega$ by using the result of Gidas, Ni and Nirenberg in [8].

We now proceed as the proof in theorem 1 so as to analysis the energy concentration. And lemma 2 still holds as before. Furthermore, $u_\varepsilon \to 0$ in $C^1_{loc}(\overline{\Omega}\setminus\{x_0\})$ by using elliptic estimates to (67).

Then we prove that h is positive at the blow-up point x_0. This property plays an important role in our next analysis.

LEMMA 10. *There exists* $h(x_0) > 0$.

Proof. For otherwise, if $h(x_0) = 0$. Note that up to a sequence

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_\Omega h(1+g(u_\varepsilon))(e^{(4\pi-\varepsilon)u_\varepsilon^2}-1)dx\\ =&\sup_{u\in W_0^{1,2}(\Omega),\|\nabla u\|_2\leqslant 1}\int_\Omega h(1+g(u))(e^{(4\pi-\varepsilon)u^2}-1)dx \geqslant \eta,\end{aligned}$$

where η is a positive constant. Choose sufficiently small ε so that

$$\int_\Omega h(1+g(u_\varepsilon))(e^{(4\pi-\varepsilon)u_\varepsilon^2}-1)dx > \frac{\eta}{2}. \tag{68}$$

To divide the whole domain Ω into two parts, we select proper radius $r > 0$, satisfying $B_r(x_0) \subset \Omega$. Thus

$$\begin{aligned}&\int_\Omega h(1+g(u_\varepsilon))(e^{(4\pi-\varepsilon)u_\varepsilon^2}-1)dx\\ =&o_r(1)\int_{B_r(x_0)}(1+g(u_\varepsilon))(e^{(4\pi-\varepsilon)u_\varepsilon^2}-1)dx+\int_{\Omega\setminus B_r(x_0)} h(1+g(u_\varepsilon))(e^{(4\pi-\varepsilon)u_\varepsilon^2}-1)dx,\end{aligned} \tag{69}$$

where $o_r(1)\to 0$ as $r\to 0$. There have $|g|<1$ from (3). Thus, we pick r small enough to get

$$o_r(1)\int_{B_r(x_0)}(1+g(u_\varepsilon))(e^{(4\pi-\varepsilon)u_\varepsilon^2}-1)dx \leqslant \frac{\eta}{4}. \tag{70}$$

On the other hand,

$$\int_{\Omega\setminus B_r(x_0)} h(1+g(u_\varepsilon))(e^{(4\pi-\varepsilon)u_\varepsilon^2}-1)dx = o(\varepsilon). \tag{71}$$

Letting $\varepsilon \to 0$, we deduce that by combing (69), (70) and (71),

$$\int_\Omega h(1+g(u_\varepsilon))(e^{(4\pi-\varepsilon)u_\varepsilon^2}-1)dx < \frac{2}{\eta}. \tag{72}$$

There exists a contradiction between (68) and (72). Hence $h(x_0) > 0$. $\square$

3.2. Blow-up analysis

Let

$$r_\varepsilon = \sqrt{\lambda_\varepsilon}[h(x_0)]^{-1/2} c_\varepsilon^{-1} e^{-(2\pi-\varepsilon/2)c_\varepsilon^2}.$$

Similar to (20), for $\forall 0 < \mu < 4\pi$, we have

$$r_\varepsilon^2 \leqslant C[h(x_0)]^{-1} c_\varepsilon^{-2} e^{-(4\pi-\varepsilon-\mu)c_\varepsilon^2} \to 0,$$

as $\varepsilon \to 0$. Define two sequences of functions on $\Omega_\varepsilon = \{x \in \mathbb{R}^2 : x_\varepsilon + r_\varepsilon x \in \Omega\}$:

$$\psi_\varepsilon(x) = c_\varepsilon^{-1} u_\varepsilon(x_\varepsilon + \omega_\varepsilon), \quad \varphi_\varepsilon(x) = c_\varepsilon(u_\varepsilon(x_\varepsilon + \omega_\varepsilon) - c_\varepsilon).$$

Through a straightforward calculation, they satisfy the following:

$$\begin{cases} -\Delta\psi_\varepsilon(x) = \frac{h\psi_\varepsilon}{c_\varepsilon^2 h(x_0)}(1+\omega_\varepsilon(u_\varepsilon))e^{(4\pi-\varepsilon)(u_\varepsilon^2-c_\varepsilon^2)} & \text{in} \quad \Omega_\varepsilon, \\ -\Delta\varphi_\varepsilon(x) = \frac{h\psi_\varepsilon}{h(x_0)}(1+\omega_\varepsilon(u_\varepsilon))e^{(4\pi-\varepsilon)(1+\psi_\varepsilon(x))\varphi_\varepsilon(x)} & \text{in} \quad \Omega_\varepsilon, \end{cases}$$

where ω is defined as in (5). Then the research to the blow-up functions $\psi_\varepsilon(x)$ and $\varphi_\varepsilon(x)$ can be completed by the method analogous to that used in section 2. And we also use the results in Chen and Li [4].

3.3. Upper bound estimates

Choose δ small enough so that $B_\delta(x_0) \subset \Omega$. Recall $s_\varepsilon = \sup_{\partial B_\delta(x_0)} u_\varepsilon$ and $\overline{u}_\varepsilon = (u_\varepsilon - s_\varepsilon)^+$. Let $\eta_\varepsilon = 1 - \frac{1}{c_\varepsilon^2}(A_{x_0} + \frac{1}{2\pi}\log\frac{1}{\delta} + o_\varepsilon(1) + o_\delta(1))$. It follows from lemma 8, thus

$$\limsup_{\varepsilon\to 0} \int_{B_\delta(x_0)} (e^{4\pi\frac{\overline{u}_\varepsilon^2}{\eta_\varepsilon}} - 1)dx \leqslant \pi\delta^2 e. \tag{73}$$

Since

$$(4\pi-\varepsilon)u_\varepsilon^2 \leqslant 4\pi\frac{\overline{u}_\varepsilon^2}{\eta_\varepsilon} + 2\log\frac{1}{\delta} + 4\pi A_{x_0} + o(1),$$

we estimate

$$\begin{aligned} \int_{B_{Rr_\varepsilon}(x_\varepsilon)} h(1+g(u_\varepsilon))e^{(4\pi-\varepsilon)u_\varepsilon^2}dx &\leqslant \delta^{-2}e^{4\pi A_{x_0}+o_\varepsilon(1)} \int_{B_{Rr_\varepsilon}(x_\varepsilon)} h(1+g(u_\varepsilon))e^{4\pi\frac{\overline{u}_\varepsilon^2}{\eta_\varepsilon}}dx \\ &\leqslant \delta^{-2}h(x_0)e^{4\pi A_{x_0}+o_\varepsilon(1)} \int_{B_\delta(x_0)} (e^{4\pi\frac{\overline{u}_\varepsilon^2}{\eta_\varepsilon}} - 1)dx, \end{aligned}$$

where $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$. In view of (73), there holds

$$\limsup_{\varepsilon\to 0} \int_{B_{Rr_\varepsilon}(x_\varepsilon)} h(1+g(u_\varepsilon))e^{(4\pi-\varepsilon)u_\varepsilon^2}dx \leqslant \pi h(x_0)e^{1+4\pi A_{x_0}}.$$

Passing to the limit $R \to \infty$, we obtain by the argument in the proof of lemma 3.3 in [16],

$$\lim_{\varepsilon\to 0} \int_\Omega h(1+g(u_\varepsilon))e^{(4\pi-\varepsilon)u_\varepsilon^2}dx \leqslant (1+g(0))\gamma + \pi h(x_0)e^{1+4\pi A_{x_0}},$$

where $\gamma = \int_\Omega h dx$. This together with (64) implies that

$$\sup_{u\in W_0^{1,2}(\Omega),\|\nabla u\|_2\leqslant 1}\int_\Omega h(1+g(u))e^{4\pi u^2}dx \leqslant (1+g(0))\gamma+\pi h(x_0)e^{1+4\pi A_{x_0}}. \tag{74}$$

3.4. Completion of the proof of theorem 2

Let ϕ_ε be defined as in (54). By a straightforward calculation, we conclude

$$\int_{B_{R\varepsilon}(x_0)} h(1+g(\phi_\varepsilon))e^{4\pi\phi_\varepsilon^2}dx \geqslant \pi h(x_0)e^{1+4\pi A_{x_0}}+O(\frac{1}{R^2}), \tag{75}$$

and

$$\begin{aligned}\int_{\Omega\setminus B_{R\varepsilon}(x_0)} h(1+g(\phi_\varepsilon))e^{4\pi\phi_\varepsilon^2}dx &\geqslant \int_{\Omega\setminus B_{2R\varepsilon}(x_0)} h(1+4\pi\frac{G^2}{C^2})(1+g(\phi_\varepsilon))dx \\ &\geqslant (1+g(0))\int_\Omega hdx+4\pi(1+g(0))\frac{\|\sqrt{h}G\|_2^2}{c^2}+o(\frac{1}{c^2}).\end{aligned} \tag{76}$$

Combining (75) and (76), we obtain

$$\int_\Omega h(1+g(\phi_\varepsilon))e^{4\pi\phi_\varepsilon^2}dx > (1+g(0))\gamma+\pi h(x_0)e^{1+4\pi A_{x_0}},$$

which contradicts (74), and implies that c_ε must be bounded. As a consequence, elliptic estimates lead to (67). This completes the proof of the theorem 2. □

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