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TWO TRACE INEQUALITIES FOR OPERATOR FUNCTIONS

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Abstract. In this paper we show that for a non-negative operator monotone function f on $[0,\infty)$ such that $f(0)=0$ and for any positive semidefinite matrices A and B,

$$\operatorname{Tr}((A-B)(f(A)-f(B))) \leqslant \operatorname{Tr}(|A-B|f(|A-B|)).$$

When the function f is operator convex on $[0,\infty)$, the inequality is reversed.

1. Introduction

For arbitrary nonnegative numbers $a \geqslant b$ and $p \geqslant 2$,

$$a^{p-1}-b^{p-1} \geqslant (a-b)^{p-1}.$$

Multiplying both sides of this inequality by $(a-b)$ we get

$$(a-b)(a^{p-1}-b^{p-1}) \geqslant (a-b)^p. \tag{1.1}$$

For $p \in [1,2]$, the inequality (1.1) is reversed. Namely, we have

$$(a-b)(a^{p-1}-b^{p-1}) \leqslant (a-b)^p. \tag{1.2}$$

From (1.1) it implies that for $f,g \in L_p(\Omega,\mu)$ (where (Ω,μ) is some measure space),

$$\int_\Omega (f(x)-g(x))(f(x)^{p-1}-g(x)^{p-1})d\mu \geqslant \int_\Omega (f(x)-g(x))^p d\mu. \tag{1.3}$$

In [4], Mustapha Mokhtar-Kharroubi pointed out that this inequality may be used to get contractivity on the positive cone of $L_p(\Omega,\mu)$. Recently, Ricard [5] proved a non-commutative version of (1.3) for von Neumann algebras. His result if translated into the language of matrices states that for $p \geqslant 2$ and for any $A,B \geqslant 0$,

$$\operatorname{Tr}((A-B)(A^{p-1}-B^{p-1})) \geqslant \operatorname{Tr}(|A-B|^p). \tag{1.4}$$

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Although the last inequality holds true, it is not obvious that the left hand side part is non-negative for any positive semidefinite matrices A and B. That fact can be proved by using the Klein inequality [3] which states that for a differentiable convex function f on $(0,\infty)$,

$$\mathrm{Tr}\,(f(A)-f(B)) \geqslant \mathrm{Tr}\,((A-B)f'(B)).$$

Applying the Klein inequality for t^p with $p \geqslant 1$ we obtain

$$\mathrm{Tr}\,(A^p - B^p) \geqslant p\mathrm{Tr}\,((A-B)B^{p-1}) \quad \text{and} \quad \mathrm{Tr}\,(B^p - A^p) \geqslant p\mathrm{Tr}\,((B-A)A^{p-1}).$$

From the last two inequalities, we get

$$\mathrm{Tr}\,(((A-B)(A^{p-1}-B^{p-1})) \geqslant 0.$$

Recall a famous inequality for unitarily invariant norm due to Ando [1]: For $p \geqslant 1$ and for any unitarily invariant norm $|||\cdot|||$,

$$|||A^p - B^p||| \geqslant ||||A-B|^p|||.$$

Applying the above inequality for the trace norm $||A||_1 = \mathrm{Tr}\,(|A|)$, we obtain

$$\mathrm{Tr}\,(|A^p - B^p|) \geqslant \mathrm{Tr}\,(|A-B|^p). \tag{1.5}$$

The inequality (1.4) attracts our attention because of the following reason: it provides an interpolation of the mentioned above Ando inequality.

PROPOSITION 1.1. *Let $p \geqslant 2$. Then for any positive semidefinite matrices A and B,*

$$\mathrm{Tr}\,(|A^p - B^p|) \geqslant \mathrm{Tr}\,((A-B)(A^{p-1}-B^{p-1})) \geqslant \mathrm{Tr}\,(|A-B|^p). \tag{1.6}$$

Proof. We prove the first inequality in (1.6) for any $p \geqslant 1$. In order to do that, let us recall the famous Powers-Størmer inequality in quantum hypothesis testing theory [2]: For any $A,B \geqslant 0$ and for any $s \in [0,1]$,

$$\mathrm{Tr}\,(A+B-|A-B|) \leqslant 2\mathrm{Tr}\,(A^s B^{1-s}). \tag{1.7}$$

Applying (1.7) for A^p and B^p and for $s = 1/p$, we have

$$\mathrm{Tr}\,(A^p + B^p - |A^p - B^p|) \leqslant 2\mathrm{Tr}\,(AB^{p-1}). \tag{1.8}$$

Since A and B play the same role in the Powers-Størmer inequality, we also have

$$\mathrm{Tr}\,(A^p + B^p - |A^p - B^p|) \leqslant 2\mathrm{Tr}\,(BA^{p-1}). \tag{1.9}$$

From (1.8) and (1.9) we have

$$\mathrm{Tr}\,(A^p + B^p - |A^p - B^p|) \leqslant \mathrm{Tr}\,(AB^{p-1}) + \mathrm{Tr}\,(BA^{p-1}),$$

or,

$$\mathrm{Tr}\,(|A^p - B^p|) \geqslant \mathrm{Tr}\,((A-B)(A^{p-1}-B^{p-1})). \quad \square$$

REMARK 1.1. During the preparation of this paper, we received a comment from Dr. Ricard via a private communication. He provided a nice proof for the inequality (1.6) as follows. Recall that the Schatten p-norm is defined as $||A||_p = (\mathrm{Tr}\,(|A|^p))^{1/p}$. For $p \geqslant 1$, using Ando's inequality with $\theta = 1/p$ and $\theta = (p-1)/p$, we get

$$||A-B||_p \leqslant ||A^p - B^p||_1^{1/p}, \quad ||A^{p-1} - B^{p-1}||_{p/(p-1)} \leqslant ||A^p - B^p||_1^{p/(p-1)},$$

where A and B are assumed to be positive semidefinite. Consequently,

$$\mathrm{Tr}\,((A-B)(A^{p-1} - B^{p-1})) \leqslant ||A-B||_p \cdot ||A^{p-1} - B^{p-1}||_{p/(p-1)} \leqslant ||A^p - B^p||_1.$$

Now we should mention that the inequality (1.1) is reversed when $p \in [1,2]$. Therefore, it is natural to ask whether the corresponding inequality holds for matrices.

At the same time, Ricard also gave us a short proof of the following inequality: For $p \in [1,2]$ and for any positive semidefinite matrices A and B,

$$\mathrm{Tr}\,(|A-B|^p) \geqslant \mathrm{Tr}\,((A-B)(A^{p-1} - B^{p-1})). \tag{1.10}$$

Indeed,

$$\begin{aligned} \mathrm{Tr}\,((A-B)(A^{p-1} - B^{p-1})) &\leqslant ||A-B||_p \cdot ||A^{p-1} - B^{p-1}||_{p/(p-1)} \\ &\leqslant ||A-B||_p \cdot ||A-B||_p^{p-1} = \mathrm{Tr}\,(|A-B|^p), \end{aligned}$$

where we used the Hölder inequality in the first inequality, and the Ando inequality for $0 < \theta = p-1 < 1$ and $q \geqslant \theta$ as $||A^\theta - B^\theta||_{q/\theta} \leqslant ||A-B||_q^\theta$.

In this paper we establish a generalization of (1.10) for operator monotone functions. Also the inequality (1.6) holds for operator convex functions instead of power functions t^p.

2. Main inequalities

We should mention that for $p \in [1,2]$ the function t^{p-1} is operator monotone on $[0,\infty)$. Therefore, from the inequality (1.10) it is interesting to know whether the following inequality is true

$$\mathrm{Tr}\,((A-B)(f(A) - f(B))) \leqslant \mathrm{Tr}\,(|A-B| f(|A-B|))$$

for some operator monotone function f under some conditions.

Based on the integral representation of operator monotone functions and operator convex functions we can establish a direct generalization of (1.10) for operator monotone functions on $[0,\infty)$.

THEOREM 2.1. *Let f be a non-negative operator monotone function on $[0,\infty)$ such that $f(0) = 0$. Then for any positive semidefinite matrices A and B,*

$$\mathrm{Tr}\,((A-B)(f(A) - f(B))) \leqslant \mathrm{Tr}\,(|A-B| f(|A-B|)). \tag{2.1}$$

Proof. It is well-known ([6]) that for any operator monotone function f on $[0,\infty)$ there exists a positive measure μ on $[0,\infty)$ such that $f(t) = \alpha + \beta t + \int_0^\infty \frac{st}{s+t} d\mu(s)$, where $\alpha = f(0)$ and $\beta \geqslant 0$. By the assumption of the theorem, $\alpha = 0$. Now, suppose that $A \geqslant B$ and put $C = A - B$. First we mention that

$$(B+s)^{-1} - (B+C+s)^{-1} = (B+s)^{-1}C(B+C+s)^{-1}. \tag{2.2}$$

Therefore, we have

$$\begin{aligned}\mathrm{Tr}\,(A-B)(f(A)-f(B)) &= \mathrm{Tr}\,(\beta C^2) + \int_0^\infty \mathrm{Tr}\,(s^2C((B+s)^{-1} - (B+C+s)^{-1}))d\mu(s)\\ &= \mathrm{Tr}\,(\beta C^2) + \int_0^\infty \mathrm{Tr}\,(s^2C((B+s)^{-1}C(B+C+s)^{-1}))d\mu(s)\\ &\leqslant \mathrm{Tr}\,(\beta C^2) + \int_0^\infty \mathrm{Tr}\,(sC^2(C+s)^{-1})d\mu(s) = \mathrm{Tr}\,(Cf(C)),\end{aligned}$$

where the inequality follows from the fact that for any $s > 0$, $(B+s)^{-1} \leqslant s^{-1}$ and $(B+C+s)^{-1} \leqslant (C+s)^{-1}$, and the positivity of $\mathrm{Tr}\,(XY) \geqslant 0$ for positive semidefinite matrices X and Y.

In general, denote by C_- and C_+ the negative and positive parts of C, respectively. Then we have $|A-B| = C_- + C_+$, and $A - B = C_+ - C_-$. Put $Z = A + C_- = B + C_+$. Then we have

$$\begin{aligned}\mathrm{Tr}\,((A-B)(f(A)-f(B))) &= \mathrm{Tr}\,((A-Z)(f(A)-f(Z))) + \mathrm{Tr}\,((A-Z)(f(Z)-f(B)))\\ &\quad + \mathrm{Tr}\,((Z-B)(f(Z)-f(B))) + \mathrm{Tr}\,((Z-B)(f(A)-f(Z))).\end{aligned}$$

Using the fact that the function f is operator monotone and $A, B \leqslant Z$, one can see the second and the fourth terms in the last identity are negative. According to the previous case, we have

$$\begin{aligned}\mathrm{Tr}\,((A-Z)(f(A)-f(Z))) + \mathrm{Tr}\,((Z-B)(f(A)-f(Z))) &\leqslant \mathrm{Tr}\,(C_-f(C_-)) + \mathrm{Tr}\,(C_+f(C_+))\\ &= \mathrm{Tr}\,(|C|f(|C|)). \quad \square\end{aligned}$$

REMARK 2.1. Combining inequality (1.10) with Ando's inequality, we have

$$\mathrm{Tr}\,(|A^p - B^p|) \geqslant \mathrm{Tr}\,(|A-B|^p) \geqslant \mathrm{Tr}\,((A-B)(A^{p-1} - B^{p-1})).$$

If we compare the last inequality with the inequality (1.7) it turns out that the last one is an interpolation of the Powers-Størmer inequality for the power s in $[1/2, 1]$.

Now let us give a generalization of Ricard's result for operator convex functions.

THEOREM 2.2. *Let f be a non-negative operator convex function on $[0,\infty)$ such that $f(0) = 0$. Then for any positive semidefinite matrices A and B,*

$$\mathrm{Tr}\,((A-B)(f(A)-f(B))) \geqslant \mathrm{Tr}\,(|A-B|f(|A-B|)). \tag{2.3}$$

Proof. It is well-known ([6]) that for any operator convex function f on $[0,\infty)$ there exists a positive measure μ on $[0,\infty)$ such that $f(t) = \alpha + \beta t + \gamma t^2 + \int_0^\infty \frac{st^2}{s+t} d\mu(s)$, where α and β are real and $\gamma \geqslant 0$. By the assumption of the theorem, $\alpha = 0$. Now, suppose that $A \geqslant B$ and put $C = A - B$. Therefore,

$$\begin{aligned}
&\operatorname{Tr}((A-B)(f(A)-f(B))) \\
=&\operatorname{Tr}(\beta C^2 + \gamma C((B+C)^2 - B^2)) + \int_0^\infty s\operatorname{Tr}(C(C + s^2(B+C+s)^{-1} - s^2(B+s)^{-1}))d\mu(s) \\
=&\operatorname{Tr}(\beta C^2 + \gamma C(C^2 + BC + CB)) + \int_0^\infty s\operatorname{Tr}(C^2 - s^2 C(B+C+s)^{-1} C(B+s)^{-1}))d\mu(s) \\
\geqslant&\operatorname{Tr}(\beta C^2 + \gamma C^3) + \int_0^\infty s\operatorname{Tr}(C^2 - sC^2(C+s)^{-1}))d\mu(s) \\
=&\operatorname{Tr}\left(C\left(\beta C + \gamma C^2 + \int_0^\infty s(C - s + s^2(C+s)^{-1})d\mu(s)\right)\right) = \operatorname{Tr}(Cf(C)),
\end{aligned}$$

where the inequality follows from the fact that for any $s > 0$, $(B+s)^{-1} \leqslant s^{-1}$ and $(B+C+s)^{-1} \leqslant (C+s)^{-1}$, and the positivity of $\operatorname{Tr}(XY) \geqslant 0$ for positive semidefinite matrices X and Y.

In general, denote by C_- and C_+ the negative and positive parts of C, respectively. Then we have $|A-B| = C_- + C_+$, and $A - B = C_+ - C_-$. Put $Z = A + C_- = B + C_+$. Then we have

$$\begin{aligned}
\operatorname{Tr}((A-B)(f(A)-f(B))) =& \operatorname{Tr}((A-Z)(f(A)-f(Z))) + \operatorname{Tr}((A-Z)(f(Z)-f(B))) \\
&+ \operatorname{Tr}((Z-B)(f(Z)-f(B))) + \operatorname{Tr}((Z-B)(f(A)-f(Z))).
\end{aligned}$$

According to the previous case, we have

$$\begin{aligned}
\operatorname{Tr}((A-Z)(f(A)-f(Z))) + \operatorname{Tr}((Z-B)(f(A)-f(Z))) \geqslant& \operatorname{Tr}(C_- f(C_-)) + \operatorname{Tr}(C_+ f(C_+)) \\
=& \operatorname{Tr}(|C| f(|C|)).
\end{aligned}$$

To finish the proof, we need to show that the second and the fourth terms are positive. We again use the integral representation of operator convex functions and the fact that $C_- C_+ = 0$. We have

$$\begin{aligned}
\operatorname{Tr}((A-Z)(f(Z)-f(A))) =& -\operatorname{Tr}(C_-(f(B+C_+) - f(B))) \\
=& -\operatorname{Tr}(\beta C_- C_+ + \gamma C_-((B+C_+)^2 - B^2)) \\
&- \int_0^\infty s\operatorname{Tr}(C_-(C_+ + s^2(B+C_+ + s)^{-1} - s^2(B+s)^{-1}))d\mu(s) \\
=& \int_0^\infty s^3 \operatorname{Tr}(C_-((B+s)^{-1} - (B+C_+ + s)^{-1}))d\mu(s) \geqslant 0.
\end{aligned}$$

Similarly, we also have that the fourth term is positive. Thus, we finish the proof. □

To finish the paper, we would like to mention that the Ando inequality [1] was proved for general operator monotone functions and operator convex functions. Therefore, the following conjecture is natural.

CONJECTURE 2.1. Let $|||\cdot|||$ be an arbitrary unitarily invariant norm and f an operator monotone function on $[0,\infty)$ such that $f(0)=0$. Then for any positive matrices A and B,

$$|||(A-B)(f(A)-f(B))||| \leqslant |||\,|A-B|f(|A-B|)|||.$$

Also, the above inequality is reversed for an operator convex function f on $[0,\infty)$ such that $f(0)=0$.

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