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INEQUALITIES FOR CERTAIN POWERS OF SEVERAL POSITIVE DEFINITE MATRICES

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Abstract. Let $A_i, i = 1,\dots,m$, and X be $n\times n$ matrices such that each A_i is positive definite with $0 < a_i \leqslant s_n(A_i)$ and X is Hermitian. Then it is shown that

$$\left|\left|\left|\left(\sum_{i=1}^{m} A_i^{a_{m+1-i}}\right)X + X\left(\sum_{i=1}^{m} A_{m+1-i}^{a_i}\right)\right|\right|\right| \geqslant m(1+l^2)\,|||X|||,$$

for every unitarily invariant norm, where $l = \min\limits_{1\leqslant i\leqslant m} a_i$.

1. Introduction

Let $\mathbb{M}_n(\mathbb{C})$ be the algebra of all $n\times n$ complex matrices. The identity matrix $I_n \in \mathbb{M}_n(\mathbb{C})$ is a square matrix that has 1's along the main diagonal and 0's for all other entries. For a matrix $A \in \mathbb{M}_n(\mathbb{C})$, let $\lambda_1(A),\dots,\lambda_n(A)$ be the eigenvalues of A repeated according to multiplicity. The singular values of A, denoted by $s_1(A),\dots,s_n(A)$, are the eigenvalues of the positive semidefinite matrix $|A| = (A^*A)^{1/2}$ arranged in decreasing order and repeated according to multiplicity. A Hermitian matrix $A \in \mathbb{M}_n(\mathbb{C})$ is said to be positive semidefinite if $x^*Ax \geqslant 0$ for all $x \in \mathbb{C}^n$ and it is called positive definite if $x^*Ax > 0$ for all $x \in \mathbb{C}^n$ with $x \neq 0$.

The spectral norm $||\cdot||$ is the norms defined on $\mathbb{M}_n(\mathbb{C})$ by $||A|| = \max\{||Ax|| : x \in \mathbb{C}^n, ||x|| = 1\}$. It is known that (see, e.g., [3, p. 76]) for every $A \in \mathbb{M}_n(\mathbb{C})$, we have

$$||A|| = s_1(A) \tag{1.1}$$

and for each $k = 1,\dots,n$, we have

$$||A||_{(k)} = \max\left|\sum_{j=1}^{k} y_j^* A x_j\right|, \tag{1.2}$$

where the maximum is taken over all choices of orthonormal $k-$tuples $x_1,\dots,x_k$ and $y_1,\dots,y_k$. In fact, replacing each y_j by $z_j y_j$ for some suitable complex number z_j of

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modulus 1 for which $\bar{z}_j y_j^* A x_j = \left| y_j^* A x_j \right|$, implies that the $k-$tuple $z_1 y_1, \ldots, z_k y_k$ is still orthonormal, and so an identity equivalent to identity (1.2) can be seen as follows:

$$||A||_{(k)} = \max \sum_{j=1}^{k} \left| y_j^* A x_j \right|, \tag{1.3}$$

where the maximum is taken over all choices of orthonormal $k-$tuples $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$.

A unitarily invariant norm $|||\cdot|||$ is a norm defined on $\mathbb{M}_n(\mathbb{C})$ that satisfies the invariance property $|||UAV||| = |||A|||$ for every $A \in \mathbb{M}_n(\mathbb{C})$ and every unitary matrices $U, V \in \mathbb{M}_n(\mathbb{C})$.

An elementary inequality (see, e.g., [7, p. 281]) for positive real numbers a, b, asserts that

$$a^b + b^a > 1. \tag{1.4}$$

It can be easily seen that the inequality (1.4) implies that if a and b are real numbers such that a is positive and b is nonnegative, then

$$a^b + b^a \geqslant 1 \tag{1.5}$$

with equality if and only if $b = 0$.

It has been shown in [Lemma 2.12, 1] that if a and b are two positive real numbers, then

$$a^a + b^b \geqslant 2e^{-e^{-1}}. \tag{1.6}$$

A generalization of the inequality (1.4) has been shown in [Lemma 3.1, 1],

$$\sum_{i=1}^{m} a_i^{a_{m+1-i}} > \frac{m}{2}, \tag{1.7}$$

where $a_1, a_2, \ldots, a_m$ are positive real numbers. Also in [1], many inequalities and applications has been given depending on this inequality.

In this paper, we give further related inequalities for scalars and we extend them into matrices. In Section 2, we give a refinement of the inequality (1.7). In Section 3, we give matrix versions for our scalar inequalities. In Section 4, we apply our results that we obtained to some known results for convex functions.

2. Preliminary results

In this section, we give a refinement of the inequality (1.7). First, we need the following lemma (see, [Theorem 2.2, 2]).

LEMMA 2.1 *Let a and b be positive real numbers. Then*

$$a^b + b^a \geqslant 1 + \min(a^2, b^2), \tag{2.1}$$

with equality if and only if $a=b=1$. *In particular,*

$$a^a \geqslant \frac{1+a^2}{2}, \tag{2.2}$$

with equality if and only if $a=1$.

The following theorem is our main result in this section.

THEOREM 2.2. *Let* $a_1, a_2, \ldots, a_m$ *be positive real numbers. Then*

$$\sum_{i=1}^{m} a_i^{a_{m+1-i}} \geqslant \frac{m}{2}(1+l^2), \tag{2.3}$$

where $l = \min\limits_{1\leqslant i\leqslant m} a_i$.

Proof. We have two cases for m.
Case 1. If m is even, then

$$\begin{aligned}\sum_{i=1}^{m} a_i^{a_{m+1-i}} &= \sum_{i=1}^{m/2}\left(a_i^{a_{m+1-i}} + a_{a_{m+1-i}}^{i}\right)\\ &\geqslant \sum_{i=1}^{m/2}\left(1+\min(a_i^2, a_{a_{m+1-i}}^2)\right) \quad \text{(by the inequality (2.1))}\\ &\geqslant \sum_{i=1}^{m/2}(1+l^2) = \frac{m}{2}(1+l^2).\end{aligned}$$

Case 2. If m is odd, then

$$\begin{aligned}\sum_{i=1}^{m} a_i^{a_{m+1-i}} &= a_{\frac{m+1}{2}}^{\frac{m+1}{2}} + \sum_{i=1}^{\frac{m-1}{2}}\left(a_i^{a_{m+1-i}} + a_{a_{m+1-i}}^{i}\right)\\ &\geqslant \frac{1}{2}\left(1+a_{\frac{m+1}{2}}^2\right) + \sum_{i=1}^{\frac{m-1}{2}}\left(1+\min(a_i^2, a_{a_{m+1-i}}^2)\right) \quad \text{(by the inequality (2.1))}\\ &\geqslant \frac{1}{2}(1+l^2) + \frac{m-1}{2}(1+l^2),\end{aligned}$$

this completes the proof. $\square$

The following corollary can be considered as a generalization of our result in the previous theorem.

COROLLARY 2.3 *Let* $a_1, a_2, \ldots, a_n$ *be positive real numbers and let* σ *be a permutation of the set* $\{1, \ldots, m\}$. *Then*

$$\sum_{i=1}^{n} a_i^{a_{\sigma_i}} \geqslant \frac{m}{2}(1+l^2), \tag{2.4}$$

in particular

$$\left(\sum_{i=1}^{m-1} a_i^{a_{i+1}}\right) + a_m^{a_1} \geqslant \frac{m}{2}(1+l^2),$$

where $l = \min_{1\leqslant i\leqslant m} a_i$.

Proof. The proof follows from the fact that $a_i^{a_j} \geqslant \min(a_i^{a_i}, a_j^{a_j})$ where $i,j = 1,\ldots,m$, and the inequality (2.2). $\square$

3. Matrix versions of the inequality (2.3)

In this section, we derive inequalities for matrices that present generalizations of the inequality (2.3). Our results in this section can be considered as refinements of some results given in [1]. First, we need the following lemma (see, e.g., [3, p. 62]).

LEMMA 3.1 *Let* $A, B \in \mathbb{M}_n(\mathbb{C})$ *such that* A *and* B *are positive semidefinite. Then*

$$s_j(A+B) \geqslant s_k(A) + s_{j-k+n}(B), \tag{3.1}$$

for $j,k = 1,\ldots,n$ *with* $k \geqslant j$.

The following lemma is a direct consequence of the Weyl's Monotonicity Theorem (see, e.g., [3, p. 63]).

LEMMA 3.2 *Let* $A, X \in \mathbb{M}_n(\mathbb{C})$ *such that* A *is positive semidefinite. Then*

$$s_j(X^*AX) \geqslant s_j^2(X)\, s_n(A), \tag{3.2}$$

for $j = 1,\ldots,n$.

Based on Theorem 2.2, Lemma 3.1, and Lemma 3.2, we have the following result. This result can be considered as a generalization of the inequality (2.3) in the setting of the singular values of matrices.

THEOREM 3.3. *Let* $A_i, X_i \in \mathbb{M}_n(\mathbb{C}), i = 1,2,\ldots,m,$ *such that each* A_i *is positive definite with* $0 < a_i \leqslant s_n(A_i)$. *Then*

$$s_j\left(\sum_{i=1}^{m} X_i^* A_i^{a_{m+1-i}} X_i\right) \geqslant \frac{c_j m}{2}(1+l^2), \tag{3.3}$$

for $j = 1,\ldots,n,$ *where* $c_j = \min\{s_j(X_1), s_n(X_2), \ldots, s_n(X_m)\}$ *and* $l = \min_{1\leqslant i\leqslant m} a_i$.

Proof. Since $A_1,\ldots,A_m$ are positive definite, we have

$$\begin{aligned} s_j\left(\sum_{i=1}^{m} X_i^* A_i^{a_{m+1-i}} X_i\right) &= s_j\left(X_1^* A_1^{a_m} X_1 + \sum_{i=2}^{m} X_i^* A_i^{a_{m+1-i}} X_i\right) \\ &\qquad \text{(by Lemma 3.1)} \\ &\geqslant s_j\left(X_1^* A_1^{a_m} X_1\right) + s_n\left(\sum_{i=2}^{m} X_i^* A_i^{a_{m+1-i}} X_i\right) \\ &\qquad \text{(by Lemma 3.2)} \\ &\geqslant s_j^2(X_1) s_n(A_1^{a_m}) + \sum_{i=2}^{m} s_n\left(X_i^* A_i^{a_{m+1-i}} X_i\right) \\ &\geqslant s_j^2(X_1) s_n(A_1^{a_m}) + \sum_{i=2}^{m} s_n^2(X_i) s_n(A_i^{a_{m+1-i}}) \\ &\geqslant \min\{s_j(X_1), s_n(X_2),\ldots,s_n(X_m)\} \sum_{i=1}^{m} a_i^{a_{m+1-i}} \\ &\geqslant \frac{c_j m}{2}(1+l^2), \qquad \text{(by Theorem 2.2)} \end{aligned}$$

for $j=1,\ldots,n$. $\square$

Applications of Theorem 3.3 can be seen in the following two results.

COROLLARY 3.4 *Let $A_i, X_i \in \mathbb{M}_n(\mathbb{C}), i=1,2,\ldots,m$, such that each A_i is positive definite with $0 < a_i \leqslant s_n(A_i)$. Then*

$$\sum_{i=1}^{m} X_i^* A_i^{a_{m+1-i}} X_i \geqslant \frac{c_n m}{2}(1+l^2) I_n, \tag{3.4}$$

and

$$\sum_{i=1}^{m} A_i^{a_{m+1-i}} \geqslant \frac{m}{2}(1+l^2) I_n, \tag{3.5}$$

where $c_n = \min\{s_n(X_1), s_n(X_2),\ldots,s_n(X_m)\}$ and $l = \min_{1\leqslant i\leqslant m} a_i$. In particular,

$$\sum_{i=1}^{m} A_i^{s_n(A_{m+1-i})} \geqslant \frac{m}{2}(1+\min_{1\leqslant i\leqslant m} s_n^2(A_i)) I_n, \tag{3.6}$$

with equality if and only if $A_i = I_n, i=1,2,\ldots,m$.

Proof. Since $\sum_{i=1}^{m} X_i^* A_i^{a_{m+1-i}} X_i$ is positive semidefinite, we have

$$\begin{aligned} \sum_{i=1}^{m} X_i^* A_i^{a_{m+1-i}} X_i &\geqslant s_n\left(\sum_{i=1}^{m} X_i^* A_i^{a_{m+1-i}} X_i\right) I_n \\ &\geqslant \frac{c_n m}{2}(1+l^2) I_n \ \text{(by the inequality (3.3))}. \quad \square \end{aligned}$$

COROLLARY 3.5 *Let $A_i, X_i \in \mathbb{M}_n(\mathbb{C}), i = 1,2,...,m,$ such that each A_i is positive definite with $a_i \geqslant \|A_i\|$ and $l = \min_{1 \leqslant i \leqslant m} a_i$. Then*

$$s_j\left(\sum_{i=1}^{m} X_i^* A_i^{-a_{m+1-i}^{-1}} X_i\right) \geqslant \frac{c_j m}{2}(1+l^2), \tag{3.7}$$

for $j = 1,...,n$, and

$$\sum_{i=1}^{m} X_i^* A_i^{-a_{m+1-i}^{-1}} X_i \geqslant \frac{c_n m}{2}(1+l^2) I_n, \tag{3.8}$$

where $c_j = \min\{s_j(X_1), s_n(X_2), ..., s_n(X_m)\}$, and

$$\sum_{i=1}^{m} A_i^{-a_{m+1-i}^{-1}} \geqslant \frac{m}{2}(1+l^2) I_n. \tag{3.9}$$

In particular

$$\sum_{i=1}^{m} A_i^{-\|A_{m+1-i}\|^{-1}} \geqslant \frac{m}{2}(1+\min_{1 \leqslant i \leqslant m} \|A_i\|^{-2}) I_n, \tag{3.10}$$

with equality if and only if $A_i = I_n, i = 1,2,...,m$.

Proof. Since $A_i, i = 1,2,...,m$ are positive definite, the matrices $A_i^{-1}, i = 1,2,...,m$ are positive definite matrices. Also the conditions $a_i \geqslant \|A_i\|, i = 1,...,m$ are equivalent to the conditions $0 < a_i^{-1} \leqslant s_n(A_i^{-1}), i = 1,2,...,m.$ So the desired inequalities follow from Theorem 3.3 and Corollary 3.4 by replacing each A_i and each a_i by A_i^{-1} and a_i^{-1}, respectively. $\square$

THEOREM 3.6. *Let $A_i, X \in \mathbb{M}_n(\mathbb{C}), i = 1,2,...,m,$ such that each A_i is positive definite with $0 < a_i \leqslant s_n(A_i)$ and X is Hermitian. Then*

$$\left|\left|\left|\left(\sum_{i=1}^{m} A_i^{a_{m+1-i}}\right) X + X \left(\sum_{i=1}^{m} A_{m+1-i}^{a_i}\right)\right|\right|\right| \geqslant m(1+l^2)\,|||X||| \tag{3.11}$$

for every unitarily invariant norm, where $l = \min_{1 \leqslant i \leqslant m} a_i$.

Proof. Since X is Hermitian, it follows that there is an orthonormal basis $\{e_j\}$ of $\mathbb{C}^n$ consisting of eigenvectors corresponding to the eigenvalues $\{\lambda_j(X)\}$ arranged in such a way that $|\lambda_1(X)| \geqslant \cdots \geqslant |\lambda_n(X)|$. Since $s_j(X) = \left|\lambda_j(X)\right|$ for $j = 1,...,n,$ we have

$$
\begin{aligned}
&\left\|\left(\sum_{i=1}^{m} A_i^{a_{m+1-i}}\right)X + X\left(\sum_{i=1}^{m} A_{m+1-i}^{a_i}\right)\right\|_{(k)} \\
&\geqslant \sum_{j=1}^{k}\left|e_j^*\left(\left(\sum_{i=1}^{m} A_i^{a_{m+1-i}}\right)X + X\left(\sum_{i=1}^{m} A_{m+1-i}^{a_i}\right)\right)e_j\right| \quad \text{(by the identity (1.3))} \\
&= \sum_{j=1}^{k}\left|e_j^*\left(\sum_{i=1}^{m} A_i^{a_{m+1-i}}\right)Xe_j + e_j^*X\left(\sum_{i=1}^{m} A_{m+1-i}^{a_i}\right)e_j\right| \\
&= \sum_{j=1}^{k}\left|e_j^*\left(\sum_{i=1}^{m} A_i^{a_{m+1-i}}\right)Xe_j + (Xe_j)^*\left(\sum_{i=1}^{m} A_{m+1-i}^{a_i}\right)e_j\right| \\
&= \sum_{j=1}^{k}\left|\lambda_j(X)e_j^*\left(\sum_{i=1}^{m} A_i^{a_{m+1-i}} + \sum_{i=1}^{m} A_{m+1-i}^{a_i}\right)e_j\right| \\
&= \sum_{j=1}^{k}\left|\lambda_j(X)\right|\left(e_j^*\left(\sum_{i=1}^{m} A_i^{a_{m+1-i}} + \sum_{i=1}^{m} A_{m+1-i}^{a_i}\right)e_j\right) \\
&= \sum_{j=1}^{k}s_j(X)\left(e_j^*\left(\sum_{i=1}^{m} A_i^{a_{m+1-i}} + \sum_{i=1}^{m} A_{m+1-i}^{a_i}\right)e_j\right) \\
&\geqslant m(1+l^2)\sum_{j=1}^{k}s_j(X) \ \text{(by the inequality (3.5))} \ = m(1+l^2)\left\|X\right\|_{(k)},
\end{aligned}
$$

for $k = 1,\dots,n$. Now the inequality (3.11) follows by the Fan Dominance Theorem (see, e.g., [3, p. 93]). $\square$

We close this section by the following remark.

REMARK 3.7 *Using the inequality (2.4), other matrix-type inequalities related to the inequalities (3.3), (3.4), (3.5), (3.6), (3.7), (3.8), (3.9), (3.10), and (3.11) can be obtained.*

4. Results related to concave functions

In this section we apply our results that we obtained in section three to some known results for convex functions. First, we need the following lemma [Theorem 3.2, 4]. Other related results can be found in [5] and [6]. Also, all convex functions here are assumed to be continuous.

LEMMA 4.1 *Let $A_i, X_i \in \mathbb{M}_n(\mathbb{C}), i = 1,2,\dots,m,$ such that each A_i is Hermitian and*

$\sum_{i=1}^{m} X_i^* X_i = I_n$. *If f is monotone convex function, then*

$$s_j\left(\sum_{i=1}^{m} X_i^* f(A_i) X_i\right) \geqslant s_j\left(f\left(\sum_{i=1}^{m} X_i^* A_i X_i\right)\right), \tag{4.1}$$

for $j = 1, \ldots, n$.

Our first main result in this section can be stated as follows.

THEOREM 4.2. *Let $A_i, X_i \in \mathbb{M}_n(\mathbb{C}), i = 1, 2, \ldots, m,$ such that each A_i is positive definite with $0 < a_i \leqslant s_n(A_i)$ and $\sum_{i=1}^{m} X_i^* X_i = I_n$. If f is monotone convex function on $[0, \infty)$, then*

$$s_j\left(\sum_{i=1}^{m} X_i^* f(A_i^{a_{m+1-i}}) X_i\right) \geqslant f\left(\frac{c_j m}{2}(1+l^2)\right), \tag{4.2}$$

for $j = 1, \ldots, n$, where $c_j = \min\{s_j(X_1), s_n(X_2), \ldots, s_n(X_m)\}$ and $l = \min_{1 \leqslant i \leqslant m} a_i$.

Proof.

$$\begin{aligned} s_j\left(\sum_{i=1}^{m} X_i^* f(A_i^{a_{m+1-i}}) X_i\right) &\geqslant s_j\left(f\left(\sum_{i=1}^{m} X_i^* A_i^{a_{m+1-i}} X_i\right)\right) \qquad \text{(by Lemma 4.1)} \\ &= f\left(s_j\left(\sum_{i=1}^{m} X_i^* A_i^{a_{m+1-i}} X_i\right)\right) \\ &\geqslant f\left(\frac{c_j m}{2}(1+l^2)\right) \qquad \text{(by Theorem 3.3)}. \quad \square \end{aligned}$$

Applications on Theorem 4.2 can be seen in the following results.

COROLLARY 4.3 *Let $A_i, X_i \in \mathbb{M}_n(\mathbb{C}), i = 1, 2, \ldots, m,$ such that each A_i is positive definite with $0 < a_i \leqslant s_n(A_i)$ and $\sum_{i=1}^{m} X_i^* X_i = I_n$. If f is monotone convex function on $[0, \infty)$, then*

$$\sum_{i=1}^{m} X_i^* f(A_i^{a_{m+1-i}}) X_i \geqslant f\left(\frac{c_n m}{2}(1+l^2)\right) I_n, \tag{4.3}$$

where $c_n = \min\{s_n(X_1), s_n(X_2), \ldots, s_n(X_m)\}$ and $l = \min_{1 \leqslant i \leqslant m} a_i$.

COROLLARY 4.4 *Let $A_i \in \mathbb{M}_n(\mathbb{C}), i = 1, 2, \ldots, m,$ such that each A_i is positive definite with $0 < a_i \leqslant s_n(A_i)$. If f is monotone convex function on $[0, \infty)$, then*

$$\sum_{i=1}^{m} f(A_i^{a_{m+1-i}}) \geqslant m f\left(\frac{1+l^2}{2}\right) I_n, \tag{4.4}$$

where $l = \min_{1 \leqslant i \leqslant m} a_i$.

COROLLARY 4.5 *Let $A_i, X_i \in \mathbb{M}_n(\mathbb{C}), i = 1,2,...,m,$ such that each A_i is positive definite with $0 < a_i \leqslant s_n(A_i)$ and $\sum_{i=1}^{m} X_i^* X_i = I_n$. Then*

$$s_j\left(\sum_{i=1}^{m} X_i^* e^{A_i^{a_{m+1-i}}} X_i\right) \geqslant e^{\frac{c_j m}{2}(1+l^2)}, \tag{4.5}$$

for $j = 1,...,n$, where $c_j = \min\{s_j(X_1), s_n(X_2),...,s_n(X_m)\}$ and $l = \min_{1\leqslant i\leqslant m} a_i$.

Proof. The result follows by Theorem 4.2 by taking $f(t) = e^t$. □

COROLLARY 4.6 *Let $A_i, X_i \in \mathbb{M}_n(\mathbb{C}), i = 1,2,...,m,$ such that each A_i is positive definite with $0 < a_i \leqslant s_n(A_i)$ and $\sum_{i=1}^{m} X_i^* X_i = I_n$. Then*

$$\sum_{i=1}^{m} X_i^* e^{A_i^{a_{m+1-i}}} X_i \geqslant e^{\frac{c_n m}{2}(1+l^2)} I_n, \tag{4.6}$$

where $c_n = \min\{s_n(X_1), s_n(X_2),...,s_n(X_m)\}$ and $l = \min_{1\leqslant i\leqslant m} a_i$.

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