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ON STEVIĆ–SHARMA TYPE OPERATOR FROM THE BESOV SPACES INTO THE WEIGHTED–TYPE SPACE H_μ^∞

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Abstract. We completely describe the boundedness and compactness of Stević-Sharma type operator $T_{\psi_1,\psi_2,\varphi}$ from the Besov spaces B_p $(1 < p < \infty)$ into the weighted-type space H_μ^∞ or the little weighted-type space $H_{\mu,0}^\infty$.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disc in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$ and $S(\mathbb{D})$ the set of all analytic self-maps on $\mathbb{D}$.

For $\psi \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$, the multiplication, composition, differentiation, and weighted composition operator on $H(\mathbb{D})$ are defined respectively as follows:

$$(M_\psi f)(z) = \psi(z)f(z);$$
$$(C_\varphi f)(z) = (f \circ \varphi)(z) = f(\varphi(z));$$
$$Df(z) = f'(z);$$
$$W_{\psi,\varphi}f(z) = (\psi C_\varphi)f(z) = \psi(z)f(\varphi(z)),$$

for $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$. The differentiation operator is a typical example of an unbounded linear operator on many spaces of functions. This is even true on the space of differentiable functions with the max-norm of $C[a;b]$, so no analyticity. Weighted composition operators have been extensively studied recently. As a combination of composition operators and multiplication operators, weighted composition operators arise naturally. For example, surjective isometries on Hardy spaces H^p and Bergman spaces A^p, $1 < p < \infty$, $p \neq 2$, are given by weighted composition operators.

Having studied above mentioned operators, some experts proposed studying their products, nowadays called product-type operators. Attention of the experts seems has been focused first on some product type-operators including the differentiation operator (see, e.g., [4, 7, 8, 9, 12, 22, 25, 28, 29, 30, 34, 35, 37, 39]). For some later results, see, e.g. [5, 14, 15, 47]. After these investigations, it was a natural question to generalize

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many of these product-type operators. For $\psi_1, \psi_2 \in H(\mathbb{D})$ and φ denotes an analytic self-map of $\mathbb{D}$, let

$$T_{\psi_1,\psi_2,\varphi} f(z) = \psi_1(z) f(\varphi(z)) + \psi_2(z) f'(\varphi(z)), \quad f \in H(\mathbb{D}).$$

The operator $T_{\psi_1,\psi_2,\varphi}$ was studied by S. Stević and co-workers for the first time in [40]. The boundedness and compactness of $T_{\psi_1,\psi_2,\varphi}$ have been extensively studied in many spaces of analytic functions in the unit disc for example, in [16, 39, 44]. This operator is related to the various products of multiplication, composition, and differentiation operators. For example, all the products of composition, multiplication, and differentiation operator can be obtained from the operator $T_{\psi_1,\psi_2,\varphi}$ by some suitable choices of functions ψ_1, ψ_2. More specifically we have

$$\begin{aligned} &M_\psi C_\varphi D = T_{0,\psi,\varphi};\ \ M_\psi D C_\varphi = T_{0,\psi\varphi',\varphi};\ C_\varphi M_\psi D = T_{0,\psi\circ\varphi,\varphi};\\ &D M_\psi C_\varphi = T_{\psi',\psi\varphi,\varphi}; C_\varphi D M_\psi = T_{\psi'\circ\varphi,\psi\varphi,\varphi};\ D C_\varphi M_\psi = T_{\psi'\circ\varphi\varphi',(\psi\circ\varphi)\varphi',\varphi}. \end{aligned} \tag{1}$$

Furthermore, by using this operator all possible difference operators of product-type operators in (1) can also be obtained. For example

$$\begin{aligned} M_{\psi_4} C_\varphi D - M_{\psi_5} D C_\varphi &= T_{0,\psi_4,-\psi_5\varphi',\varphi},\\ C_\varphi M_{\psi_4} D - C_\varphi D M_{\psi_5} &= T_{-\psi_5'\circ\varphi,(\psi_4-\psi_5)\circ\varphi,\varphi},\\ D M_{\psi_4} C_\varphi - D C_\varphi M_{\psi_5} &= T_{\psi_4'-\varphi'\psi_5\circ\varphi,\varphi'(\psi_4-\psi_5\circ\varphi),\varphi}, \end{aligned}$$

etc., where $\psi_4, \psi_5 \in H(\mathbb{D})$.

For a fixed positive continuous function μ on $\mathbb{D}$, the weighted-type space H_μ^∞ (see, for example, [24]) consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{H_\mu^\infty} = \sup\{\mu(z)|f(z)| : z \in \mathbb{D}\} < \infty.$$

The little weighted-type space $H_{\mu,0}^\infty$ is a subspace of H_μ^∞ consisting of all $f \in H(\mathbb{D})$ such that

$$\lim_{|z|\to 1} \mu(z)|f(z)| = 0.$$

Let A denote the area measure on $\mathbb{D}$ normalized by the condition $dA(\mathbb{D}) = 1$ and let $1 < p < \infty$. The analytic Besov space B_p is the Banach space consisting of the analytic functions f on $\mathbb{D}$ such that

$$(b_p(f))^p := \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} dA(z) < \infty,$$

with Besov norm $\|f\|_{B_p} = |f(0)| + b_p(f)$. For $p = 2$, B_p is the classical Dirichlet space $\mathscr{D}$. An equivalent norm, called the Dirichlet norm, is defined as

$$\|f\|_{\mathscr{D}} = \left(|f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z)\right)^{1/2}.$$

The Besov spaces are Möbius invariant and the Dirichlet space is the unique Möbius invariant Hilbert space that is continuously embedded in the Bloch space. It is well known (see, e.g., [41]) that if $1 < p < q < \infty$, then $B_p \subset B_q \subset VMOA \subset BMOA \subset \mathscr{B}$, where $\mathscr{B}$ is the Bloch space defined as the set of analytic functions f on $\mathbb{D}$ such that

$$\|f\| = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

The Bloch space is widely regarded as the limit of B_p as $p \to \infty$, since for $1 < p < \infty$, an analytic function f belongs to B_p if and only if the function $z \mapsto (1-|z|^2)|f'(z)|$ is in $L^p(d\lambda)$, where $d\lambda$ is the conformally invariant area measure

$$d\lambda(z) = \frac{dA(z)}{(1-|z|^2)^2}.$$

Moreover, B_p is continuously embedded in B_q and $BMOA$ is continuously embedded in the Bloch space $\mathscr{B}$. The Besov spaces B_p are contained in the Hardy space H^2. Another noteworthy property of the Besov spaces is that the polynomials are dense in B_p. We recommend to the interested reader [49] for an in-depth study on the spaces $BMOA$, $VMOA$, and the analytic Besov spaces B_p.

Product-type operators on some spaces of analytic functions on the unit disk or the unit ball have become a subject of increasing interest in the last fifteen years (see, e.g., the following representative papers:[6, 10, 13, 17, 18, 19, 23, 26, 27, 31, 32, 36, 38, 42, 45, 46], and the related references therein).

Our work is motivated by the above work. We investigate the boundedness and compactness of the operator $T_{\psi_1,\psi_2,\varphi}$ from the Besov spaces B_p $(1 < p < \infty)$ into the weighted-type space H_μ^∞ or the little weighted-type space $H_{\mu,0}^\infty$. The paper is partially motivated by paper [2] where the weighted composition operators are studied between the related spaces. The paper is also partially motivated by paper [11] where the integral operators are studied between the Besov space and the Bloch-type space. Throughout the paper, constants are often given without computing their exact values, and the value of a constant C may change from one occurrence to the next.

2. Background

In this section we introduce some notation and recall some well-known results that will be used throughout the paper. Note that by Theorem 9 in [49], the functions in B_p satisfy the following Lipschitz-type condition.

LEMMA 1. *There is a constant $C > 0$ only dependent on p such that for all $f \in B_p$,*

$$|f(z) - f(w)| \leqslant C\|f\|_{B_p}(\rho(z,w))^{1-1/p}, \text{ for } z,\, w \in \mathbb{D},$$

where $\rho(z,w)$ denotes the hyperbolic distance between z and w. In particular,

$$|f(z) - f(0)| \leqslant C\|f\|_{B_p}\left(\frac{1}{2}\log\frac{1+|z|}{1-|z|}\right)^{1-1/p}, \text{ for } z \in \mathbb{D}.$$

By Proposition 4.3.8 in [48] it is easy to get that the following lemma.

LEMMA 2. *There is a constant $C>0$ only dependent on p such that for all $f \in B_p$,*

$$|f'(z)| \leqslant \frac{C}{1-|z|^2}\|f\|_{B_p}, \text{ for } z \in \mathbb{D}.$$

The following lemma can be proved by using Lemma 4.2.2 in [48].

LEMMA 3. *(1) For $1/2<|w|<1$, put*

$$f_w(z)=\left(\log\frac{2}{1-|w|^2}\right)^{-1/p}\left(\log\frac{2}{1-\overline{w}z}\right), z \in \mathbb{D},$$

then $f_w \in B_p$, moreover there is a positive constant C such that

$$\sup_{1/2<|w|<1} \|f_w\|_{B_p} \leqslant C.$$

(2) For $w \in \mathbb{D}$, put

$$g_w(z)=\frac{(1-|w|^2)^2 z}{1-\overline{w}z}+w|w|^2, z \in \mathbb{D},$$

then $g_w \in B_p$, moreover there is a positive constant C such that

$$\sup_{w\in\mathbb{D}} \|g_w\|_{B_p} \leqslant C.$$

(3) For $w \in \mathbb{D}$, put

$$h_w(z)=\frac{(1-|w|^2)^2 z}{(1-\overline{w}z)^2}, z \in \mathbb{D},$$

then $h_w \in B_p$, moreover there is a positive constant C such that

$$\sup_{w\in\mathbb{D}} \|h_w\|_{B_p} \leqslant C.$$

Proof. The test function f_w in (1) comes from [11]. We only prove that (2) holds. The proof of (3) is very similar to that of (2). Since for $w \in \mathbb{D}$, $g_w'(z)=\frac{(1-|w|^2)^2}{(1-\overline{w}z)^2}$, we have by using Lemma 4.2.2 in [48] for $t=p-2$, $c=p$

$$\int_{\mathbb{D}} |g_w'(z)|^p(1-|z|^2)^{p-2}dA(z)=\int_{\mathbb{D}}\frac{(1-|w|^2)^{2p}}{|1-\overline{w}z|^{2p}}(1-|z|^2)^{p-2}dA(z)$$

$$=(1-|w|^2)^{2p}\int_{\mathbb{D}}\frac{(1-|z|^2)^{p-2}}{|1-\overline{w}z|^{2p}}dA(z) \leqslant C(1-|w|^2)^{2p}\frac{1}{(1-|w|^2)^p} \leqslant C.$$

Thus $g_w \in B_p$,

$$\sup_{w\in\mathbb{D}} \|g_w\|_{B_p} \leqslant 1+C^{1/p}. \quad \square$$

The following criterion for the compactness follows by standard arguments (see, *e.g.*, the proofs of the corresponding lemmas in [41, Lemma 2.10]). The details will not be pursued here.

LEMMA 4. *Suppose* $\psi_1, \psi_2 \in H(\mathbb{D})$ *and* $\varphi \in S(\mathbb{D})$. *Then* $T_{\psi_1,\psi_2,\varphi} : B_p \to H_\mu^\infty$ *is compact if and only if* $T_{\psi_1,\psi_2,\varphi} : B_p \to H_\mu^\infty$ *is bounded and for any bounded sequence* $\{f_n\}$ *in* B_p *which converges to zero uniformly on compact subsets of* $\mathbb{D}$ *as* $n \to \infty$, *we have* $\|T_{\psi_1,\psi_2,\varphi} f_n\|_{H_\mu^\infty} \to 0$ *as* $n \to \infty$.

The following lemma can be proved similar to Lemma 1 in [20] (see, also [21]). The details are omitted.

LEMMA 5. *A closed set* K *in* $H_{\mu,0}^\infty$ *is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in K} \mu(z)|f(z)| = 0.$$

3. The boundedness of the operator $T_{\psi_1,\psi_2,\varphi} : B_p \to H_\mu^\infty$ (or $H_{\mu,0}^\infty$)

First we consider the boundedness of the operator $T_{\psi_1,\psi_2,\varphi} : B_p \to H_\mu^\infty$.

THEOREM 1. *Suppose* $\psi_1, \psi_2 \in H(\mathbb{D})$ *and* $\varphi \in S(\mathbb{D})$. *Then the following statements are equivalent:*

(a) $T_{\psi_1,\psi_2,\varphi} : B_p \to H_\mu^\infty$ *is bounded;*

(b) $\psi_1 \in H_\mu^\infty$,

$$\sup_{z\in\mathbb{D}} \mu(z)|\psi_1(z)| \left(\frac{1}{2}\log\frac{1+|\varphi(z)|}{1-|\varphi(z)|}\right)^{1-1/p} < \infty, \tag{2}$$

and

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|\psi_2(z)|}{1-|\varphi(z)|^2} < \infty. \tag{3}$$

Proof. $(b) \Rightarrow (a)$. First assume that $\psi_1 \in H_\mu^\infty$, (2), and (3) hold. Then for every $z \in \mathbb{D}$, $f \in B_p$, by Lemmas 1 and 2 we have

$$\begin{aligned} \mu(z)\left|T_{\psi_1,\psi_2,\varphi} f(z)\right| &= \mu(z)|\psi_1(z)f(\varphi(z)) + \psi_2(z)f'(\varphi(z))| \\ &\leqslant \mu(z)|\psi_1(z)||f(\varphi(z))| + \mu(z)|\psi_2(z)||f'(\varphi(z))| \\ &\leqslant C\|f\|_{B_p}\mu(z)|\psi_1(z)|\left(\frac{1}{2}\log\frac{1+|\varphi(z)|}{1-|\varphi(z)|}\right)^{1-1/p} + \frac{C\mu(z)|\psi_2(z)|}{1-|\varphi(z)|^2}\|f\|_{B_p} \\ &\leqslant C\|f\|_{B_p}. \end{aligned} \tag{4}$$

On the other hand, by Lemmas 1 and 2 we have

$$\begin{aligned} &\left|\left(T_{\psi_1,\psi_2,\varphi} f\right)(0)\right| = \left|\psi_1(0)f(\varphi(0)) + \psi_2(0)f'(\varphi(0))\right| \\ &\leqslant C\left(|\psi_1(0)|\left(\frac{1}{2}\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{1-1/p} + \frac{|\psi_2(0)|}{1-|\varphi(0)|^2}\right)\|f\|_{B_p}. \end{aligned} \tag{5}$$

Applying conditions (4) and (5), we deduce that the operator $T_{\psi_1,\psi_2,\varphi} : B_p \to H_\mu^\infty$ is bounded.

$(a) \Rightarrow (b)$. Now assume that $T_{\psi_1,\psi_2,\varphi} : B_p \to H_\mu^\infty$ is bounded. That means that there exists a constant C such that

$$\|T_{\psi_1,\psi_2,\varphi} f\|_{H_\mu^\infty} \leqslant C\|f\|_{B_p},$$

for all $f \in B_p$. For $f(z) = 1 \in B_p$, we have

$$K_1 := \sup_{z\in\mathbb{D}} \mu(z)|\psi_1(z)| < \infty, \tag{6}$$

that is, $\psi_1 \in H_\mu^\infty$. For $f(z) = z \in B_p$, we have

$$\sup_{z\in\mathbb{D}} \mu(z)|\psi_1(z)\varphi(z) + \psi_2(z)| < \infty. \tag{7}$$

From (6), the triangle inequality, and the boundedness of the function $\varphi(z)$, we have

$$K_2 := \sup_{z\in\mathbb{D}} \mu(z)|\psi_2(z)| < \infty. \tag{8}$$

By taking the function f_w defined in Lemma 3, we get

$$f_w'(z) = \left(\log \frac{2}{1-|w|^2}\right)^{-1/p} \frac{\overline{w}}{1-\overline{w}z}, z \in \mathbb{D}.$$

For $a \in \mathbb{D}$ such that $1/2 < |\varphi(a)|$, we have

$$f_{\varphi(a)}(\varphi(a)) = \left(\log \frac{2}{1-|\varphi(a)|^2}\right)^{1-1/p}.$$

Hence we obtain that

$$\begin{aligned} \sup_{1/2<|\varphi(a)|} \mu(a) \Bigg| \psi_1(a) \left(\log \frac{2}{1-|\varphi(a)|^2}\right)^{1-1/p} \\ + \psi_2(a) \left(\log \frac{2}{1-|\varphi(a)|^2}\right)^{-1/p} \frac{\overline{\varphi(a)}}{1-|\varphi(a)|^2} \Bigg| \\ \leqslant \sup_{1/2<|\varphi(a)|} \|T_{\psi_1,\psi_2,\varphi} f_{\varphi(a)}\|_{H_\mu^\infty} \leqslant C < \infty, \end{aligned} \tag{9}$$

and

$$\begin{aligned} \sup_{|\varphi(a)|\leqslant 1/2} \mu(a) \Bigg| \psi_1(a) \left(\log \frac{2}{1-|\varphi(a)|^2}\right)^{1-1/p} \\ + \psi_2(a) \left(\log \frac{2}{1-|\varphi(a)|^2}\right)^{-1/p} \frac{\overline{\varphi(a)}}{1-|\varphi(a)|^2} \Bigg| \end{aligned}$$

$$\leqslant \sup_{|\varphi(a)|\leqslant 1/2}\left(\log\frac{8}{3}\right)^{1-1/p}|\mu(a)\psi_1(a)|+\sup_{|\varphi(a)|\leqslant 1/2}\frac{4\mu(a)|\psi_2(a)|}{3}(\log 2)^{1/p}$$

$$\leqslant \left(\log\frac{8}{3}\right)^{1-1/p}K_1+\frac{4K_2}{3}(\log 2)^{1/p}<\infty. \tag{10}$$

We use the triangle inequality, and the fact that $\|\varphi\|_\infty \leqslant 1$ to get

$$\begin{aligned}&\sup_{a\in\mathbb{D}}\mu(a)|\psi_1(a)|\left(\log\frac{2}{1-|\varphi(a)|^2}\right)^{1-1/p}\\ &\leqslant C+\sup_{a\in\mathbb{D}}\left|\mu(a)\psi_2(a)\left(\log\frac{2}{1-|\varphi(a)|^2}\right)^{-1/p}\frac{\overline{\varphi(a)}}{1-|\varphi(a)|^2}\right|\\ &\leqslant C+C\sup_{a\in\mathbb{D}}\left|\frac{\mu(a)\psi_2(a)}{1-|\varphi(a)|^2}\right|.\end{aligned} \tag{11}$$

Take the functions g_w, h_w defined in Lemma 3, $j_w(z)=g_w(z)-h_w(z)\in B_p$, then

$$\sup_{w\in\mathbb{D}}\|j_w\|_{B_p}\leqslant C<\infty,$$

and

$$j_w'(z)=\frac{(1-|w|^2)^2}{(1-\overline{w}z)^2}-\frac{(1-|w|^2)^2(1+\overline{w}z)}{(1-\overline{w}z)^3}=\frac{-2(1-|w|^2)^2\overline{w}z}{(1-\overline{w}z)^3}, z\in\mathbb{D},$$

so

$$j_w(w)=0,\ j_w'(w)=\frac{-2|w|^2}{1-|w|^2}, w\in\mathbb{D}.$$

Thus by Lemma 3

$$\sup_{a\in\mathbb{D}}\frac{2\mu(a)|\psi_2(a)||\varphi(a)|^2}{1-|\varphi(a)|^2}\leqslant\|T_{\psi_1,\psi_2,\varphi}j_{\varphi(a)}\|_{H_\mu^\infty}\leqslant C<\infty. \tag{12}$$

From (8) and (12), we have

$$\sup_{a\in\mathbb{D}}\frac{\mu(a)|\psi_2(a)|}{1-|\varphi(a)|^2}\leqslant C<\infty,$$

that is (3) holds.

By (3) and (11) we obtain

$$\sup_{a\in\mathbb{D}}\mu(a)|\psi_1(a)|\left(\log\frac{2}{1-|\varphi(a)|^2}\right)^{1-1/p}\leqslant C<\infty.$$

Since $\log\frac{1+|w|}{1-|w|}\asymp\log\frac{2}{1-|w|^2}$ as $|w|$ approaches 1, for $r\in(0,1)$ large enough we have

$$\sup_{r<|\varphi(a)|}\mu(a)|\psi_1(a)|\left(\frac{1}{2}\log\frac{1+|\varphi(a)|}{1-|\varphi(a)|}\right)^{1-1/p}$$

$$\leqslant C \sup_{r<|\varphi(a)|} \mu(a)|\psi_1(a)| \left(\log \frac{2}{1-|\varphi(a)|^2}\right)^{1-1/p} \leqslant C < \infty, \tag{13}$$

and

$$\sup_{|\varphi(a)|\leqslant r} \mu(a)|\psi_1(a)| \left(\frac{1}{2}\log \frac{1+|\varphi(a)|}{1-|\varphi(a)|}\right)^{1-1/p}$$
$$\leqslant C \sup_{|\varphi(a)|\leqslant r} \mu(a)|\psi_1(a)| \left(\frac{1}{2}\log \frac{1+r}{1-r}\right)^{1-1/p} \leqslant CK_1 < \infty. \tag{14}$$

It from (13) and (14) follows that (2) holds. That ends the proof of Theorem 1. □

The following corollary follows by setting $\psi_1(z) = \psi(z)$ and $\psi_2(z) = 0$ in the Theorem 1 at once.

COROLLARY 1. ([2, Theorem 3],[3, Theorem 4.5]) *Suppose* $\psi \in H(\mathbb{D})$ *and* $\varphi \in S(\mathbb{D})$. *Then the weighted composition operator* $W_{\psi,\varphi} : B_p \to H^\infty_\mu$ *is a bounded operator if and only if* $\psi \in H^\infty_\mu$ *and*

$$\sup_{z\in\mathbb{D}} \mu(z)|\psi(z)| \left(\frac{1}{2}\log \frac{1+|\varphi(z)|}{1-|\varphi(z)|}\right)^{1-1/p} < \infty.$$

The following corollary follows by setting $\psi_1(z) = \psi'(z)$ and $\psi_2(z) = \psi(z)\varphi(z)$ in the Theorem 1 at once.

COROLLARY 2. *Suppose* $\psi \in H(\mathbb{D})$ *and* $\varphi \in S(\mathbb{D})$. *Then the weighted composition followed by differentiation* $DW_{\psi,\varphi} : B_p \to H^\infty_\mu$ *is a bounded operator if and only if* $\psi' \in H^\infty_\mu$,

$$\sup_{z\in\mathbb{D}} \mu(z)|\psi'(z)| \left(\frac{1}{2}\log \frac{1+|\varphi(z)|}{1-|\varphi(z)|}\right)^{1-1/p} < \infty,$$

and

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|\psi(z)\varphi(z)|}{1-|\varphi(z)|^2} < \infty.$$

The following corollary follows by setting $\psi_1(z) = 0$ and $\psi_2(z) = \psi(z)$ in the Theorem 1 at once.

COROLLARY 3. *Suppose* $\psi \in H(\mathbb{D})$ *and* φ *denotes an analytic self-map of* $\mathbb{D}$. *Then the weighted composition followed by differentiation* $W_{\psi,\varphi}D : B_p \to H^\infty_\mu$ *is a bounded operator if and only if*

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|\psi(z)|}{1-|\varphi(z)|^2} < \infty.$$

The following theorem characterizes the boundedness of the operator $T_{\psi_1,\psi_2,\varphi} : B_p \to H^\infty_{\mu,0}$.

THEOREM 2. *Suppose* $\psi_1, \psi_2 \in H(\mathbb{D})$ *and* $\varphi \in S(\mathbb{D})$. *Then* $T_{\psi_1,\psi_2,\varphi} : B_p \to H^\infty_{\mu,0}$ *is bounded if and only if* $T_{\psi_1,\psi_2,\varphi} : B_p \to H^\infty_\mu$ *is bounded,* $\psi_1, \psi_2 \in H^\infty_{\mu,0}$.

Proof. $\Rightarrow$: Suppose first that $T_{\psi_1,\psi_2,\varphi} : B_p \to H^\infty_{\mu,0}$ is bounded. Then $T_{\psi_1,\psi_2,\varphi} : B_p \to H^\infty_\mu$ is bounded and for $f \in B_p$, $T_{\psi_1,\psi_2,\varphi} f \in H^\infty_{\mu,0}$. Taking $f(z) = 1 \in B_p$, we have

$$\lim_{|z|\to 1} \mu(z)|\psi_1'(z)| = 0,$$

that is, $\psi_1 \in H^\infty_{\mu,0}$. Taking $f(z) = z \in B_p$, we get

$$\lim_{|z|\to 1} \mu(z)|\psi_1(z)\varphi(z) + \psi_2(z)| = 0. \tag{15}$$

By $\psi_1 \in H^\infty_{\mu,0}$, the triangle inequality, and $\|\varphi\|_\infty \leqslant 1$ we have

$$\lim_{|z|\to 1} \mu(z)|\psi_2(z)| = 0,$$

that is, $\psi_2 \in H^\infty_{\mu,0}$.

$\Leftarrow$: Suppose now that $T_{\psi_1,\psi_2,\varphi} : B_p \to H^\infty_\mu$ is bounded, $\psi_1, \psi_2 \in H^\infty_{\mu,0}$. Since for each polynomial L, one has

$$\begin{aligned} \mu(z)\left|T_{\psi_1,\psi_2,\varphi} L(z)\right| &= \mu(z)|\psi_1(z)L(\varphi(z)) + \psi_2(z)L'(\varphi(z))| \\ &\leqslant \mu(z)|\psi_1(z)||L(\varphi(z))| + \mu(z)|\psi_2(z)||L'(\varphi(z))| \\ &\leqslant \mu(z)|\psi_1(z)|\|L\|_\infty + \mu(z)|\psi_2(z)|\|L'\|_\infty \to 0 \text{ as } |z| \to 1, \end{aligned}$$

from which it follows that $T_{\psi_1,\psi_2,\varphi} L \in H^\infty_{\mu,0}$. Since the set of all polynomials is dense in B_p ([1]), thus for each $f \in B_p$, there is a sequence of polynomials $\{L_k\}_{k\in\mathbb{N}}$, such that

$$\lim_{k\to\infty} \|L_k - f\|_{B_p} = 0. \tag{16}$$

Since the operator $T_{\psi_1,\psi_2,\varphi} : B_p \to H^\infty_\mu$ is bounded, we have

$$\|T_{\psi_1,\psi_2,\varphi} L_k - T_{\psi_1,\psi_2,\varphi} f\|_{H^\infty_\mu} \leqslant \|T_{\psi_1,\psi_2,\varphi}\|\|L_k - f\|_{B_p} \to 0 \text{ as } k \to \infty.$$

Since $H^\infty_{\mu,0}$ is the closed subset of H^∞_μ, we see that $T_{\psi_1,\psi_2,\varphi} f \in H^\infty_{\mu,0}$, and consequently $T_{\psi_1,\psi_2,\varphi}(B_p) \subseteqq H^\infty_{\mu,0}$. The boundedness of the operator $T_{\psi_1,\psi_2,\varphi} : B_p \to H^\infty_\mu$ implies that $T_{\psi_1,\psi_2,\varphi} : B_p \to H^\infty_{\mu,0}$ is bounded. This ends the proof of Theorem 2. □

According to Theorem 2 we immediately get the following.

COROLLARY 4. *Suppose* $\psi \in H(\mathbb{D})$ *and* $\varphi \in S(\mathbb{D})$. *Then the weighted composition operator* $W_{\psi,\varphi} : B_p \to H^\infty_{\mu,0}$ *is bounded if and only if* $W_{\psi,\varphi} : B_p \to H^\infty_\mu$ *is bounded and* $\psi \in H^\infty_{\mu,0}$.

COROLLARY 5. *Suppose* $\psi \in H(\mathbb{D})$ *and* $\varphi \in S(\mathbb{D})$. *Then the weighted composition followed by differentiation* $DW_{\psi,\varphi} : B_p \to H^\infty_{\mu,0}$ *is a bounded operator if and only if* $DW_{\psi,\varphi} : B_p \to H^\infty_{\mu,0}$ *is bounded,* $\psi' \in H^\infty_{\mu,0}$, *and* $\psi\varphi \in H^\infty_{\mu,0}$.

COROLLARY 6. *Suppose* $\psi \in H(\mathbb{D})$ *and* $\varphi \in S(\mathbb{D})$. *Then the differentiation followed by the weighted composition operator* $W_{\psi,\varphi}D : B_p \to H^\infty_{\mu,0}$ *is a bounded operator if and only if* $W_{\psi,\varphi}D : B_p \to H^\infty_{\mu,0}$ *is bounded and* $\psi \in H^\infty_{\mu,0}$.

4. The compactness of the operator $T_{\psi_1,\psi_2,\varphi} : B_p \to H^\infty_\mu$ (or $H^\infty_{\mu,0}$)

Now we are ready to state and prove the results on the compactness of the operator $T_{\psi_1,\psi_2,\varphi} : B_p \to H^\infty_\mu$.

THEOREM 3. *Suppose* $\psi_1, \psi_2 \in H(\mathbb{D})$ *and* $\varphi \in S(\mathbb{D})$. *Then the following statements are equivalent:*

(a) $T_{\psi_1,\psi_2,\varphi} : B_p \to H^\infty_\mu$ *is compact;*

(b) $T_{\psi_1,\psi_2,\varphi} : B_p \to H^\infty_\mu$ *is bounded,*

$$\lim_{s\to 1} \sup_{|\varphi(z)|>s} \mu(z)|\psi_1(z)| \left(\frac{1}{2} \log \frac{1+|\varphi(z)|}{1-|\varphi(z)|} \right)^{1-1/p} = 0, \tag{17}$$

and

$$\lim_{s\to 1} \sup_{|\varphi(z)|>s} \frac{\mu(z)|\psi_2(z)|}{1-|\varphi(z)|^2} = 0. \tag{18}$$

Proof. $(b) \Rightarrow (a)$. Suppose that $T_{\psi_1,\psi_2,\varphi} : B_p \to H^\infty_\mu$ is bounded, (17), and (18) hold. To prove that $T_{\psi_1,\psi_2,\varphi} : B_p \to H^\infty_\mu$ is compact, for any bounded sequence $\{f_k\}$ in B_p with $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$, let $\|f_k\|_{B_p} \leqslant 1$, it suffices, in view of Lemma 4, to show that

$$\|T_{\psi_1,\psi_2,\varphi} f_k\|_{H^\infty_\mu} \to 0 \text{ as } k \to \infty.$$

By (17) and (18), we have for any $\varepsilon > 0$, there exists $\rho \in (0,1)$ such that

$$\sup_{|\varphi(z)|>s} \mu(z)|\psi_1(z)| \left(\frac{1}{2} \log \frac{1+|\varphi(z)|}{1-|\varphi(z)|} \right)^{1-1/p} < \varepsilon, \tag{19}$$

and

$$\sup_{|\varphi(z)|>s} \frac{\mu(z)|\psi_2(z)|}{1-|\varphi(z)|^2} < \varepsilon, \tag{20}$$

for $\rho < s < 1$. From the boundedness of the operator $T_{\psi_1,\psi_2,\varphi} : B_p \to H^\infty_\mu$ and the proof of Theorem 1, (6) and (8) hold. Since $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$, Cauchy's estimate shows that f_k' converges to 0 uniformly on compact subsets of $\mathbb{D}$, there exists a $K_0 \in \mathbb{N}$ such that $k > K_0$ implies that

$$|(T_{\psi_1,\psi_2,\varphi} f_k)(0)| + \sup_{|\varphi(z)|\leqslant s} \mu(z)|T_{\psi_1,\psi_2,\varphi} f_k(z)|$$

$$
\begin{aligned}
&\leqslant |\psi_1(0)f_k(\varphi(0))+\psi_2(0)f_k'(\varphi(0))| + \sup_{|\varphi(z)|\leqslant s}\mu(z)|\psi_1(z)||f_k(\varphi(z))| \\
&\quad + \sup_{|\varphi(z)|\leqslant s}\mu(z)|\psi_2(z)||f_k'(\varphi(z))| \\
&\leqslant |\psi_1(0)||f_k(\varphi(0))| + |\psi_2(0)f_k'(\varphi(0))| + K_1\sup_{|\varphi(z)|\leqslant s}|f_k(\varphi(z))| + K_2\sup_{|\varphi(z)|\leqslant s}|f_k'(\varphi(z))| \\
&< C\varepsilon. \qquad (21)
\end{aligned}
$$

When $k>K_0$, from (19), (20), (21) and Lemma 1, one has

$$
\begin{aligned}
&\|T_{\psi_1,\psi_2,\varphi}f_k\|_{H_\mu^\infty} = |\left(T_{\psi_1,\psi_2,\varphi}f_k\right)(0)| + \sup_{z\in\mathbb{D}}\mu(z)\left|T_{\psi_1,\psi_2,\varphi}f_k(z)\right| \\
&\leqslant \left(|\left(T_{\psi_1,\psi_2,\varphi}f_k\right)(0)| + \sup_{|\varphi(z)|\leqslant s}\mu(z)\left|T_{\psi_1,\psi_2,\varphi}f_k(z)\right|\right) + \sup_{s<|\varphi(z)|<1}\mu(z)\left|T_{\psi_1,\psi_2,\varphi}f_k(z)\right| \\
&< C\varepsilon + 2C\sup_{s<|\varphi(z)|<1}\mu(z)|\psi_1(z)|\left(\frac{1}{2}\log\frac{1+|\varphi(z)|}{1-|\varphi(z)|}\right)^{1-1/p}\|f_k\|_{B_p} \\
&\quad + C\sup_{s<|\varphi(z)|<1}\frac{\mu(z)\,|\psi_2(z)|}{1-|\varphi(z)|^2}\|f_k\|_{B_p} \\
&< C\varepsilon + 2C\sup_{s<|\varphi(z)|<1}\mu(z)|\psi_1(z)|\left(\frac{1}{2}\log\frac{1+|\varphi(z)|}{1-|\varphi(z)|}\right)^{1-1/p} + C\sup_{s<|\varphi(z)|<1}\frac{\mu(z)\,|\psi_2(z)|}{1-|\varphi(z)|^2} \\
&< 4C\varepsilon,
\end{aligned}
$$

it follows that the operator $T_{\psi_1,\psi_2,\varphi}: B_p \to H_\mu^\infty$ is compact.

$(a)\Rightarrow(b)$. It is clear that the compactness of $T_{\psi_1,\psi_2,\varphi}: B_p \to H_\mu^\infty$ implies the boundedness of $T_{\psi_1,\psi_2,\varphi}: B_p \to H_\mu^\infty$. If $\|\varphi\|_\infty < 1$, it is clear that the limit in (17) and (18) is automatically equal to zero. Hence, assume that $\|\varphi\|_\infty = 1$, let $\{z_k\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)|\to 1$ as $k\to\infty$. We can use the test functions

$$
j_k(z) = j_{\varphi(z_k)}(z),
$$

then

$$
\sup_{k\in\mathbb{N}}\|j_k\|_{B_p} \leqslant C,
$$

$$
j_k(\varphi(z_k)) = 0, \text{ and } j_k'(\varphi(z_k)) = \frac{-2|\varphi(z_k)|^2}{1-|\varphi(z_k)|^2}.
$$

It is easy to see that j_k converges to 0 uniformly on compact subsets of $\mathbb{D}$. By Lemma 4 we obtain

$$
\lim_{k\to\infty}\|T_{\psi_1,\psi_2,\varphi}j_k\|_{H_\mu^\infty} = 0.
$$

Thus

$$
\frac{2|\mu(z_k)\,|\psi_2(z_k)|\,|\varphi(z_k)|^2}{1-|\varphi(z_k)|^2} \leqslant \|T_{\psi_1,\psi_2,\varphi}j_k\|_{H_\mu^\infty} \to 0 \text{ as } k\to\infty. \qquad (22)
$$

By (22) and $|\varphi(z_k)| \to 1$ we have

$$\lim_{k\to\infty} \frac{|\mu(z_k)|\psi_2(z_k)|}{1-|\varphi(z_k)|^2} = 0,$$

it implies that (17) holds.

From (9), we have

$$\begin{aligned} &\mu(z_k)|\psi_1(z_k)| \left(\log \frac{2}{1-|\varphi(z_k)|^2}\right)^{1-1/p} \\ &\leqslant \|T_{\psi_1,\psi_2,\varphi} f_{\varphi(z_k)}\|_{H_\mu^\infty} + \mu(z_k)|\psi_2(z_k)| \left(\log \frac{2}{1-|\varphi(z_k)|^2}\right)^{-1/p} \frac{|\overline{\varphi(z_k)}|}{1-|\varphi(z_k)|^2} \\ &\leqslant \|T_{\psi_1,\psi_2,\varphi} f_{\varphi(z_k)}\|_{H_\mu^\infty} + (\log 2)^{1/p} \frac{\mu(z_k)|\psi_2(z_k)|}{1-|\varphi(z_k)|^2} \to 0 \text{ as } k\to\infty. \end{aligned} \tag{23}$$

(18) follows. This finishes the proof of Theorem 3. □

From Theorem 3 we can get the characterization of the compactness of the weighted composition operator $W_{\psi,\varphi} : B_p \to H_\mu^\infty$, the operator $DW_{\psi,\varphi} : B_p \to H_\mu^\infty$ and the operator $W_{\psi,\varphi}D : B_p \to H_\mu^\infty$.

COROLLARY 7. ([2, Corollary 2]) *Suppose $\psi \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then the weighted composition operator $W_{\psi,\varphi} : B_p \to H_\mu^\infty$ is compact if and only if $W_{\psi,\varphi} : B_p \to H_\mu^\infty$ is bounded,*

$$\lim_{s\to 1} \sup_{|\varphi(z)|>s} \mu(z)|\psi(z)| \left(\frac{1}{2}\log\frac{1+|\varphi(z)|}{1-|\varphi(z)|}\right)^{1-1/p} = 0.$$

COROLLARY 8. *Suppose $\psi \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then the weighted composition followed by differentiation $DW_{\psi,\varphi} : B_p \to H_\mu^\infty$ is a compact operator if and only if $DW_{\psi,\varphi} : B_p \to H_\mu^\infty$ is bounded,*

$$\lim_{s\to 1} \sup_{|\varphi(z)|>s} \mu(z)|\psi'(z)| \left(\frac{1}{2}\log\frac{1+|\varphi(z)|}{1-|\varphi(z)|}\right)^{1-1/p} = 0,$$

and

$$\lim_{s\to 1} \sup_{|\varphi(z)|>s} \frac{\mu(z)|\psi(z)\varphi(z)|}{1-|\varphi(z)|^2} = 0.$$

COROLLARY 9. *Suppose $\psi \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then the differentiation followed by the weighted composition operator $W_{\psi,\varphi}D : B_p \to H_\mu^\infty$ is a compact operator if and only if $W_{\psi,\varphi}D : B_p \to H_\mu^\infty$ is bounded,*

$$\lim_{s\to 1} \sup_{|\varphi(z)|>s} \frac{\mu(z)|\psi(z)|}{1-|\varphi(z)|^2} = 0.$$

Next we are ready for the description of the compactness of $T_{\psi_1,\psi_2,\varphi}: B_p \to H^\infty_{\mu,0}$. The compactness of operators whose range is in $H^\infty_{\mu,0}$ has a close relation with Lemma 5.

THEOREM 4. *Suppose $\psi_1, \psi_2 \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then the following statements are equivalent:*

(a) $T_{\psi_1,\psi_2,\varphi}: B_p \to H^\infty_{\mu,0}$ is compact;

(b) $\psi_1 \in H^\infty_{\mu,0}$,

$$\lim_{|z|\to 1} \mu(z)|\psi_1(z)| \left(\frac{1}{2}\log\frac{1+|\varphi(z)|}{1-|\varphi(z)|}\right)^{1-1/p} = 0, \tag{24}$$

and

$$\lim_{|z|\to 1} \frac{\mu(z)|\psi_2(z)|}{1-|\varphi(z)|^2} = 0. \tag{25}$$

Proof. $(b) \Rightarrow (a)$. Suppose that $\psi_1 \in H^\infty_{\mu,0}$, (24), and (25) hold. By Theorem 2, it is clear that $T_{\psi_1,\psi_2,\varphi}: B_p \to H^\infty_{\mu,0}$ is bounded. Taking the supremum in inequality (4) over all $f \in B_p$ such that $\|f\|_{B_p} \leqslant 1$ and letting $|z| \to 1$, yields

$$\lim_{|z|\to 1} \sup_{\|f\|_{B_p}\leqslant 1} \mu(z)\left|T_{\psi_1,\psi_2,\varphi} f(z)\right| = 0.$$

From this and Lemma 5 we have that $T_{\psi_1,\psi_2,\varphi}: B_p \to H^\infty_{\mu,0}$ is compact.

$(a) \Rightarrow (b)$. Assume that $T_{\psi_1,\psi_2,\varphi}: B_p \to H^\infty_{\mu,0}$ is compact. Firstly, it is obvious $T_{\psi_1,\psi_2,\varphi}: B_p \to H^\infty_{\mu}$ is compact. By Theorem 3, ψ_1, ψ_2, and φ satisfy conditions (17) and (18). It follows that for every $\varepsilon > 0$, there exists $\rho \in (0,1)$ such that (19) and (20) hold for $\rho < s < 1$. On the other hand, since $T_{\psi_1,\psi_2,\varphi}: B_p \to H^\infty_{\mu,0}$ is compact, then $T_{\psi_1,\psi_2,\varphi}: B_p \to H^\infty_{\mu,0}$ is bounded. By Theorem 2, $\psi_1, \psi_2 \in H^\infty_{\mu,0}$. Thus for $\varepsilon > 0$, there exists $\gamma \in (0,1)$ such that

$$\mu(z)|\psi_1(z)| < \left(\frac{1}{2}\log\frac{1+s}{1-s}\right)^{1/p-1} \varepsilon \tag{26}$$

and

$$\mu(z)|\psi_2(z)| < (1-s^2)\varepsilon, \tag{27}$$

for $\gamma < |z| < 1$. Next, we prove that (19) and (26) imply (24). The proof of (25) is similar, hence it will be omitted.

From (19) one has, when $\gamma < |z| < 1$ and $\rho < s < 1$,

$$\sup_{|\varphi(z)|>s} \mu(z)|\psi_1(z)| \left(\frac{1}{2}\log\frac{1+|\varphi(z)|}{1-|\varphi(z)|}\right)^{1-1/p} < \varepsilon. \tag{28}$$

By (26) we get, when $\gamma < |z| < 1$ and s,

$$\sup_{|\varphi(z)|\leqslant s} \mu(z)|\psi_1(z)| \left(\frac{1}{2}\log\frac{1+|\varphi(z)|}{1-|\varphi(z)|}\right)^{1-1/p} \leqslant \sup_{|\varphi(z)|\leqslant s} \mu(z)|\psi_1(z)| \left(\frac{1}{2}\log\frac{1+s}{1-s}\right)^{1-1/p} < \varepsilon. \tag{29}$$

Having in mind (28) and (29) we conclude that (24) holds completing the proof of the theorem. □

Due to Theorem 4, the characterization of the compactness of the weighted composition operator $W_{\psi,\varphi} : B_p \to H^\infty_{\mu,0}$, the operator $DW_{\psi,\varphi} : B_p \to H^\infty_{\mu,0}$ and the operator $W_{\psi,\varphi}D : B_p \to H^\infty_{\mu,0}$ are now obvious, which to the best of our knowledge, have not appeared in the literature.

COROLLARY 10. *Suppose $\psi \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then the weighted composition operator $W_{\psi,\varphi} : B_p \to H^\infty_{\mu,0}$ is compact if and only if and*

$$\lim_{|z|\to 1} \mu(z)|\psi(z)| \left(\frac{1}{2}\log\frac{1+|\varphi(z)|}{1-|\varphi(z)|}\right)^{1-1/p} = 0.$$

COROLLARY 11. *Suppose $\psi \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then the weighted composition followed by differentiation $DW_{\psi,\varphi} : B_p \to H^\infty_{\mu,0}$ is a compact operator if and only if*

$$\lim_{|z|\to 1} \mu(z)|\psi'(z)| \left(\frac{1}{2}\log\frac{1+|\varphi(z)|}{1-|\varphi(z)|}\right)^{1-1/p} = 0,$$

and

$$\lim_{|z|\to 1} \frac{\mu(z)|\psi(z)\varphi(z)|}{1-|\varphi(z)|^2} = 0.$$

COROLLARY 12. *Suppose $\psi \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then the differentiation followed by the weighted composition operator $W_{\psi,\varphi}D : B_p \to H^\infty_{\mu,0}$ is a compact operator if and only if*

$$\lim_{|z|\to 1} \frac{\mu(z)|\psi(z)|}{1-|\varphi(z)|^2} = 0.$$

Finally, we deduce the following compactness characterization of the differentiation followed by the weighted composition operator $W_{\psi,\varphi}D : B_p \to H^\infty_{\mu,0}$.

THEOREM 5. *Suppose $\psi \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then $W_{\psi,\varphi}D : B_p \to H^\infty_{\mu,0}$ is compact if and only if $W_{\psi,\varphi}D : B_p \to H^\infty_{\mu,0}$ is bounded.*

Proof. We only need to prove sufficiency. Assume that $W_{\psi,\varphi}D : B_p \to H^\infty_{\mu,0}$ is bounded. Then we have $\psi \in H^\infty_{\mu,0}$. Taking $f(z) = z^n \in B_p$, we have

$$\mu(z)\left|W_{\psi,\varphi}Df(z)\right| = \mu(z)|\psi(z)f'(\varphi(z))| = \mu(z)\left|n\psi(z)(\varphi(z))^{n-1}\right| \to 0 \text{ as } |z| \to 1.$$

Thus,

$$\lim_{|z|\to 1}\left(n\mu(z)|\psi(z)||\varphi(z)|^{n-1}\right)=0. \tag{30}$$

For $N\in\mathbb{N}$ and $n\geqslant N$, define the sets

$$E_N=\left\{z\in\mathbb{D}:|\varphi(z)|\leqslant 1-\frac{1}{N}\right\}$$

and

$$\Delta_n=\left\{z\in\mathbb{D}:1-\frac{1}{n-1}\leqslant|\varphi(z)|\leqslant 1-\frac{1}{n}\right\}.$$

Fix an integer $N>2$ and $z\in\mathbb{D}$. If $z\in E_N$, we have

$$\frac{\mu(z)|\psi(z)|}{1-|\varphi(z)|^2}\leqslant\frac{\mu(z)|\psi(z)|}{1-|1-\frac{1}{N}|^2}. \tag{31}$$

If z is not in E_N, that is $|\varphi(z)|>1-\frac{1}{N}$, there exists $n>N$ such that $z\in\Delta_n$. Since ([43])

$$\inf_{z\in\Delta_n} n(1-|\varphi(z)|)|\varphi(z)|^{n-1}\geqslant\frac{1}{e},$$

we obtain

$$\frac{\mu(z)|\psi(z)|}{1-|\varphi(z)|^2}\leqslant\frac{n\mu(z)|\psi(z)||\varphi(z)|^{n-1}}{n(1-|\varphi(z)|)|\varphi(z)|^{n-1}}\leqslant e\left(n\mu(z)|\psi(z)||\varphi(z)|^{n-1}\right). \tag{32}$$

Using (30), (31), and (32), we get

$$\lim_{|z|\to 1}\frac{\mu(z)|\psi(z)|}{1-|\varphi(z)|^2}=0.$$

Corollary 12 gives the operator $W_{\psi,\varphi}D:B_p\to H_{\mu,0}^{\infty}$ is compact. □

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