GENERALIZED WEIGHTED COMPOSITION OPERATORS ON WEIGHTED BERGMAN SPACES, II

XIANGLING ZHU

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Abstract. The boundedness, compactness, essential norm, Hilbert-Schmidt class and order boundedness of generalized weighted composition operators on weighted Bergman spaces are investigated in this paper.

1. Introduction

Let \( \mathbb{D} = \{ z : |z| < 1 \} \) be the unit disk of complex plane \( \mathbb{C} \) and let \( \partial \mathbb{D} \) be the boundary of \( \mathbb{D} \). Denote by \( H(\mathbb{D}) \) the class of functions analytic in \( \mathbb{D} \). For \( a \in \mathbb{D} \), \( \sigma_a(z) = \frac{a-z}{1-ar{a}z} \) is the Möbius transformation of \( \mathbb{D} \).

For a subarc \( I \subseteq \partial \mathbb{D} \), let \( S(I) \) be the Carleson box based on \( I \) with

\[
S(I) = \{ z \in \mathbb{D} : 1 - |I| \leq |z| < 1 \text{ and } \frac{z}{|z|} \in I \}.
\]

If \( I = \partial \mathbb{D} \), let \( S(I) = \mathbb{D} \). Let \( \mu \) denote a positive Borel measure on \( \mathbb{D} \). For \( 0 < \alpha < \infty \), we say that \( \mu \) is an \( \alpha \)-Carleson measure on \( \mathbb{D} \) if (see [1])

\[
\sup_{I \subseteq \partial \mathbb{D}} \mu(S(I))/|I|^{\alpha} < \infty.
\]

Here and henceforth \( \sup_{I \subseteq \partial \mathbb{D}} \) indicates the supremum taken over all subarcs \( I \) of \( \partial \mathbb{D} \). \( |I| = (2\pi)^{-1} \int_I |d\xi| \) is the normalized length of the subarc \( I \). Note that \( \alpha = 1 \) gives the classical Carleson measure.

For \( 0 < p < \infty \) and \( \gamma > -1 \), the weighted Bergman space, denoted by \( A^p_\gamma \), is the set of all functions \( f \in H(\mathbb{D}) \) satisfying

\[
\|f\|_{A^p_\gamma} = \int_{\mathbb{D}} |f(z)|^p dA_\gamma(z) < \infty,
\]

where \( dA_\gamma(z) = (\gamma+1)(1-|z|^2)^\gamma dA(z) \) and \( dA \) is the normalized Lebesgue area measure in \( \mathbb{D} \) such that \( A(\mathbb{D}) = 1 \). This means that \( A^p_\gamma = H(\mathbb{D}) \cap L^p(\mathbb{D}, dA_\gamma) \). When \( p = 2 \), \( A^2_\gamma \) is a Hilbert space.


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We denote the set of nonnegative integers by \( \mathbb{Z} \). Let \( \varphi \) be an analytic self-map of \( \mathbb{D} \), \( u \in H(\mathbb{D}) \) and \( n \in \mathbb{Z} \). The generalized weighted composition operator \( D_{\varphi,u}^n \) is defined as follows (see [34, 36]).

\[
(D_{\varphi,u}^n)(z) = u(z) \cdot f^{(n)}(\varphi(z)), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.
\]

If \( n = 0 \), then \( D_{\varphi,u}^n \) is just the weighted composition operator, which is frequently denoted by \( uC_{\varphi} \) in the literature. When \( n = 0 \) and \( u(z) = 1 \), then \( D_{\varphi,u}^n \) is just the composition operator \( C_{\varphi} \), which is defined by

\[
(C_{\varphi}f)(z) = f(\varphi(z)), \quad f \in H(\mathbb{D}).
\]

See [2, 32] for more information about the theory of composition operators. When \( u(z) = 1 \), \( D_{\varphi,u}^n = C_{\varphi}D^n \). See, for example, [4, 7, 8, 9, 12, 18, 19, 21, 25, 31] for the study of the operator \( C_{\varphi}D^n \). See, for example, [5, 6, 10, 11, 13, 20, 22, 23, 24, 26, 27, 33, 34, 35, 36, 37] and the references therein for the study of the operator \( D_{\varphi,u}^n \). For some other product-type operators see, for example [16, 28, 29].

In [19], Stević studied the operator \( C_{\varphi}D^n \) on weighted Bergman spaces. In [30], Ueki studied the order boundedness of the operator \( uC_{\varphi} : A_{\alpha}^n \to A_{\beta}^n \). In [34], the author studied the operator \( D_{\varphi,u}^n : A_{\alpha}^n \to A_{\beta}^n \). Among others, we prove that, under the assumption that \( u \in A_{\beta}^2 \), \( D_{\varphi,u}^n : A_{\alpha}^2 \to A_{\beta}^2 \) is bounded if and only if

\[
\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |u(z)|^2 \frac{(1 - |a|^2)^{\alpha+2}}{|1 - \alpha \varphi(z)|^{2\alpha+4+2n}} dA_{\beta}(z) < \infty.
\]

\( D_{\varphi,u}^n : A_{\alpha}^2 \to A_{\beta}^2 \) is compact if and only if \( D_{\varphi,u}^n : A_{\alpha}^2 \to A_{\beta}^2 \) is bounded and

\[
\lim_{|a| \to 1} \int_{\mathbb{D}} |u(z)|^2 \frac{(1 - |a|^2)^{\alpha+2}}{|1 - \alpha \varphi(z)|^{2\alpha+4+2n}} dA_{\beta}(z) = 0.
\]

Motivated by results in [19, 30, 34], in this work we give another characterization of the boundedness, compactness and essential norm of the operator \( D_{\varphi,u}^n : A_{\alpha}^2 \to A_{\beta}^2 \). Moreover, we study the order boundedness and the Hilbert-Schmidt class of the operator \( D_{\varphi,u}^n : A_{\alpha}^2 \to A_{\beta}^2 \).

Recall that the linear operator \( T : X \to Y \) is order bounded if \( T \) maps the unit ball of \( X \) into an order interval of \( Z \), namely there exists a nonnegative element \( P \) in \( Z \) such that \( |T(f)| \leq P \) for all \( f \) belongs to the unit ball of \( X \). Here \( X \) is a quasi-Banach space and \( Y \) a subspace of quasi-Banach Lattice \( Z \).

Throughout the paper, we denote by \( C \) a positive constant which may differ from one occurrence to the next. In addition, we say that \( A \lesssim B \) if there exists a constant \( C \) such that \( A \leq CB \). The symbol \( A \approx B \) means that \( A \lesssim B \lesssim A \).

2. Boundedness and essential norm of \( D_{\varphi,u}^n : A_{\alpha}^2 \to A_{\beta}^2 \)

In this section, we give another characterization for the boundedness, compactness and essential norm of the operator \( D_{\varphi,u}^n : A_{\alpha}^2 \to A_{\beta}^2 \). Hence, we first state some lemmas which will be used in the proofs of the main results in this section.
LEMMA 2.1. [14] Let $\mu$ be a positive measure on $\mathbb{D}$, $m \in \mathbb{Z}$ and $-1 < \alpha < \infty$. Then $\mu$ is a bounded $2 + \alpha + 2m$-Carleson measure if and only if there is a positive constant $C$, depending only on $\alpha$ and $m$ such that

$$\int_\mathbb{D} |f^{(m)}(z)|^2 d\mu(z) \leq C \|f\|^2_{A^2_\alpha}$$

for all $f \in A^2_\alpha$. Moreover, if $\mu$ is a bounded $2 + \alpha + 2m$-Carleson measure, then $C = C_1C_2$, where $C_1 > 0$ depends only on $\alpha$ and $m$ and

$$C_2 = \sup_I \frac{\mu(S(I))}{|I|^{2+\alpha+2m}}.$$

Let $0 < s < \infty$. The bounded $s$-Carleson measure can be characterized by a global integral condition (see [1]), namely,

$$\sup_I \frac{\mu(S(I))}{|I|^s} \approx \sup_{|b| \geq \rho} \int_\mathbb{D} |\sigma'_b(z)|^s d\mu(z),$$

(1)

LEMMA 2.2. [15] Let $0 < \rho < 1$, $1 \leq s < \infty$ and let $\mu$ be a positive Borel measure on $\mathbb{D}$. Then

$$\sup_I \frac{\mu(S(I) \setminus \Delta(0, \rho))}{|I|^s} \lesssim \sup_{|b| \geq \rho} \int_\mathbb{D} |\sigma'_b(z)|^s d\mu(z),$$

where $\Delta(0, \rho) := \{z : |z| < \rho\}$.

LEMMA 2.3. [2] Let $g$ and $u$ be positive measurable functions on $\mathbb{D}$, and let $\varphi$ be an analytic self-map of $\mathbb{D}$. Then

$$\int_\mathbb{D} (g \circ \varphi)(z)|\varphi'(z)|^2 u(z) d\Lambda(z) = \int_\mathbb{D} g(w) U(\varphi, w) d\Lambda(z),$$

where $U(\varphi, w) = \sum_{z \in \varphi^{-1}(w)} u(z)$ for $w \in \mathbb{D} \setminus \{\varphi(0)\}$.

For an $f(z) = \sum_{k=0}^{\infty} a_k z^k \in H(\mathbb{D})$, define

$$T_j f(z) = \sum_{k=0}^{j} a_k z^k, \quad R_j f(z) = \sum_{k=j+1}^{\infty} a_k z^k.$$

LEMMA 2.4. Let $m \in \mathbb{Z}$ and $-1 < \alpha < \infty$. For each $w \in \mathbb{D}$, positive integer $j$ and $f \in A^2_\alpha$,

$$\left| (R_j f(w))^{(m)} \right| \lesssim \|f\|_{A^2_\alpha} \sum_{k=j+1}^{\infty} \frac{\Gamma(k + \alpha + 2 + m)}{k! \Gamma(\alpha + 2 + m)} |w|^k,$$

where $\Gamma$ denotes the Gamma function.

Proof. Since $f \in A^2_\alpha$, it is clear that $R_j f \in A^2_\alpha$. Hence

$$(R_j f)(w) = \int_\mathbb{D} (R_j f)(z) K_\alpha(w, z) d\Lambda_\alpha(z),$$
Lemma 2.3 we have

\[(R_j f)^{(m)}(w) = \frac{\Gamma(\alpha + 2 + m)}{\Gamma(\alpha + 2)} \int_\mathbb{D} R_j f(z) \frac{\bar{z}^m}{(1 - \bar{z} w)^{\alpha + 2 + m}} dA_\alpha(z) \]

Using Hölder’s inequality, we get

\[\left| (R_j f)^{(m)}(w) \right| \leq \frac{\Gamma(\alpha + 2 + m)}{\Gamma(\alpha + 2)} \int_\mathbb{D} |f(z)| R_j \left( \frac{\bar{z}^m}{(1 - \bar{z} w)^{\alpha + 2 + m}} \right) dA_\alpha(z)\]

\[\approx \int_\mathbb{D} |f(z)| \left( \sum_{k=0}^{\infty} \frac{\Gamma(k + m + \alpha + 2)}{k! \Gamma(m + \alpha + 2)} |w|^k |z|^{k+m} \right)^2 dA_\alpha(z)\]

\[\leq \|f\|_{A_\alpha^2} \sum_{k=0}^{\infty} \frac{\Gamma(k + m + \alpha + 2)}{k! \Gamma(m + \alpha + 2)} |w|^k. \quad \Box\]

**Theorem 2.1.** Let \( \phi \) be an analytic self-map of \( \mathbb{D} \), \(-1 < \alpha, \beta < \infty, n \in \mathbb{Z} \) and \( u \in H(\mathbb{D}) \). Then \( D^n_{\phi, u} : A_\alpha^2 \to A_\beta^2 \) is bounded if and only if

\[\sup_{b \in \mathbb{D}} \int_\mathbb{D} |\sigma'_b(\phi(z))|^{2+\alpha+2n} |u(z)|^2 dA_\beta(z) < \infty. \quad (2)\]

**Proof.** First we assume that (2) holds. Let \( d\mu(z) = |u(z)|^2 dA_\beta(z) \). By (1) and Lemma 2.3 we have

\[\sup_{b \in \mathbb{D}} \frac{\mu \circ \phi^{-1}(S(I))}{|I|^{2+\alpha+2n}} \approx \sup_{b \in \mathbb{D}} \int_\mathbb{D} |\sigma'_b(w)|^{2+\alpha+2n} d\mu \circ \phi^{-1}(w) \]

\[= \sup_{b \in \mathbb{D}} \int_\mathbb{D} |\sigma'_b(\phi(z))|^{2+\alpha+2n} |u(z)|^2 dA_\beta(z) < \infty. \quad (3)\]

For any \( f \in A_\alpha^2 \), by Lemmas 2.1 and 2.3, and (3) we have

\[\|D^n_{\phi, u} f\|_{A_\beta^2}^2 \approx \int_\mathbb{D} |D^n_{\phi, u} f(z)|^2 dA_\beta(z) = \int_\mathbb{D} |f^{(n)}(w)|^2 d\mu \circ \phi^{-1} \]

\[\leq \sup_I \frac{\mu \circ \phi^{-1}(S(I))}{|I|^{2+\alpha+2n}} \|f\|_{A_\alpha^2}^2 < \infty.\]

Thus \( D^n_{\phi, u} : A_\alpha^2 \to A_\beta^2 \) is bounded.

Conversely, assume that \( D^n_{\phi, u} : A_\alpha^2 \to A_\beta^2 \) is bounded. For any \( a \in \mathbb{D} \), set

\[f_a(z) = \frac{1 - |a|^2}{(1 - \overline{a} z)^{\frac{\alpha+2}{2}}} \], \quad z \in \mathbb{D}.\]
Then \( \|f_a\|_{A^2_\alpha} \approx 1 \). Let \( I \subset \partial \mathbb{D} \), and let \( \zeta \in \partial \mathbb{D} \) be the center of arc \( I \) and \( b = (1 - |I|) \zeta \in \mathbb{D} \). Then
\[
f_b^{(n)}(z) = \frac{\Gamma(2 + \alpha + n)}{\Gamma(2 + \alpha)} \frac{(1 - |b|^2)^{\alpha + 2}}{(1 - b z)^{2 + \alpha + n}}
\]
and
\[
|f_b^{(n)}(z)|^2 \geq \frac{1}{(1 - |b|)^{2 + \alpha + 2n}}, \quad z \in S(I).
\]
Thus, by the boundedness of \( D^n_{\phi, u} : A^2_\alpha \to A^2_\beta \), we get
\[
\infty > \|D^n_{\phi, u}\|^2 \|f_b\|^2_{A^2_\alpha} \geq \|D^n_{\phi, u}f_b\|^2_{A^2_\beta} = \int_{\mathbb{D}} |f_b^{(n)}(\phi(z))|^2 |u(z)|^2 dA_\beta(z)
\]
\[
= \int_{\mathbb{D}} |f_b^{(n)}(w)|^2 d\mu \circ \phi^{-1}(w) \geq \int_{S(I)} \frac{1}{(1 - |b|)^{2 + \alpha + 2n}} d\mu \circ \phi^{-1}(w) \approx \frac{\mu \circ \phi^{-1}(S(I))}{|I|^{2 + \alpha + 2n}},
\]
for all \( I \subset \partial \mathbb{D} \). By (1) and Lemma 2.3 we have
\[
\sup_{b \in \mathbb{D}} \int_{\mathbb{D}} |\sigma'_b(\phi(z))|^{2 + \alpha + 2n} |u(z)|^2 dA_\beta(z) = \sup_{b \in \mathbb{D}} \int_{\mathbb{D}} |\sigma'_b(w)|^{2 + \alpha + 2n} d\mu \circ \phi^{-1}(w)
\]
\[
\approx \sup_I \frac{\mu \circ \phi^{-1}(S(I))}{|I|^{2 + \alpha + 2n}} < \infty.
\]
This completes the proof of this theorem. \( \square \)

**Theorem 2.2.** Let \( \phi \) be an analytic self-map of \( \mathbb{D} \), \(-1 < \alpha, \beta < \infty, n \in \mathbb{Z} \) and \( u \in H(\mathbb{D}) \). Suppose that \( D^n_{\phi, u} : A^2_\alpha \to A^2_\beta \) is bounded. Then
\[
\|D^n_{\phi, u}\|^2_{e_{A^2_\alpha} \to A^2_\beta} \approx T,
\]
where
\[
T := \limsup_{|b| \to 1} \int_{\mathbb{D}} |\sigma'_b(\phi(z))|^{2 + \alpha + 2n} |u(z)|^2 dA_\beta(z).
\]

**Proof.** First we prove that \( \|D^n_{\phi, u}\|^2_{e_{A^2_\alpha} \to A^2_\beta} \gtrsim T \). Let \( b \in \mathbb{D} \). Set
\[
f_b(z) = \left( \frac{1 - |b|^2}{(1 - b z)^2} \right)^{\alpha + 2}, \quad z \in \mathbb{D}.
\]
We have \( \|f_b\|_{A^2_\alpha} \approx 1 \) and \( f_b \to 0 \) weakly in \( A^2_\alpha \) as \( |b| \to 1 \). Thus \( \|K(f_b)\|_{A^2_\beta} \to 0 \) as \( |b| \to 1 \) for every compact operator \( K : A^2_\alpha \to A^2_\beta \). Thus,
\[
\|D^n_{\phi, u} - K\|^2_{A^2_\alpha \to A^2_\beta} \geq \limsup_{|b| \to 1} \|D^n_{\phi, u}(f_b) - K(f_b)\|^2_{A^2_\beta}
\]
\[
\geq \limsup_{|b| \to 1} \|D^n_{\phi, u}(f_b)\|^2_{A^2_\beta} - \limsup_{|b| \to 1} \|K(f_b)\|^2_{A^2_\beta} = \limsup_{|b| \to 1} \|D^n_{\phi, u}(f_b)\|^2_{A^2_\beta}
\]

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for every compact operator \( K : A^2_\alpha \rightarrow A^2_\beta \). By Lemma 2.3 we have

\[
\limsup_{|b| \to 1} \left\| D^n_{\varphi,u}(fb) \right\|_{A^2_\beta}^2 \approx \limsup_{|b| \to 1} \int_{|b|} \left| f^{(n)}(w) \right|^2 d\mu \circ \varphi^{-1} \\
= \limsup_{|b| \to 1} \int_{|b|} \left| \frac{\Gamma(2 + \alpha + n)(1 - |b|)^{\frac{\alpha + n}{2}}}{\Gamma(2 + \alpha)(1 - b)^{2 + \alpha + n}} \right|^2 d\mu \circ \varphi^{-1} \\
\geq \limsup_{|b| \to 1} \int_{|b|} \left| \sigma_n'(w) \right|^{2 + \alpha + 2n} d\mu \circ \varphi^{-1} = T.
\]

Therefore, from the definition of the essential norm, we obtain

\[
\left\| D^n_{\varphi,u} \right\|_{e, A^2_\alpha \rightarrow A^2_\beta}^2 = \inf \left\{ \left\| D^n_{\varphi,u} - J \right\|_{A^2_\alpha \rightarrow A^2_\beta}^2 \geq T \right\}.
\]

Next, we prove that \( \left\| D^n_{\varphi,u} \right\|_{e, A^2_\alpha \rightarrow A^2_\beta}^2 \lesssim T \). It is clear that

\[
\left\| D^n_{\varphi,u} \right\|_{e, A^2_\alpha \rightarrow A^2_\beta}^2 = \left\| D^n_{\varphi,u} (T_j + R_j) \right\|_{e, A^2_\alpha \rightarrow A^2_\beta} \leq \left\| D^n_{\varphi,u} T_j \right\|_{e, A^2_\alpha \rightarrow A^2_\beta} + \left\| D^n_{\varphi,u} R_j \right\|_{e, A^2_\alpha \rightarrow A^2_\beta}
\]

Here we used the fact that \( T_j \) is compact on \( A^2_\alpha \). Hence

\[
\left\| D^n_{\varphi,u} \right\|_{e, A^2_\alpha \rightarrow A^2_\beta} \leq \liminf_{j \to \infty} \left\| D^n_{\varphi,u} R_j \right\|_{A^2_\alpha \rightarrow A^2_\beta}.
\]

For an \( f(z) = \sum_{k=0}^\infty a_k z^k \in H(\mathbb{D}) \), by Lemma 2.3 we have

\[
\left\| D^n_{\varphi,u} \right\|_{A^2_\alpha \rightarrow A^2_\beta}^2 \leq \liminf_{j \to \infty} \left\| D^n_{\varphi,u} R_j \right\|_{A^2_\alpha \rightarrow A^2_\beta} \leq \liminf_{j \to \infty} \sup_{\left\| f \right\|_{A^2_\alpha} \leq 1} \left\| D^n_{\varphi,u} (R_j f) \right\|_{A^2_\beta}^2 \\
\approx \liminf_{j \to \infty} \sup_{\left\| f \right\|_{A^2_\alpha} \leq 1} \int_{\mathbb{D}} \left| (R_j f)^{(n)}(\varphi(z)) \right|^2 u(z)^2 \, dA_{\beta}(z)
\]

\[
= \liminf_{j \to \infty} \sup_{\left\| f \right\|_{A^2_\alpha} \leq 1} \int_{\mathbb{D}} \left| (R_j f)^{(n)}(w) \right|^2 d\mu \circ \varphi^{-1}.
\]

Let \( r \in (0, 1) \). For each \( f \in A^2_\alpha \), by Lemma 2.4 we have

\[
\int_{|w| \leq r} \left| (R_j f)^{(n)}(w) \right|^2 d\mu \circ \varphi^{-1} \\
\leq \left\| f \right\|_{A^2_\alpha}^2 \int_{|w| \leq r} \left( \sum_{k=j+1}^\infty \frac{\Gamma(k + \alpha + 2 + n)}{k! \Gamma(\alpha + 2 + n)} |w|^k \right)^2 d\mu \circ \varphi^{-1}(w)
\]

\[
\leq \left\| f \right\|_{A^2_\alpha}^2 \left( \sum_{k=j+1}^\infty \frac{\Gamma(k + \alpha + 2 + n)}{k! \Gamma(\alpha + 2 + n)} r^k \right)^2 \int_{|w| \leq r} d\mu \circ \varphi^{-1}.
\]

By the boundedness of \( D^n_{\varphi,u} : A^2_\alpha \rightarrow A^2_\beta \) is bounded, we have \( u \in A^2_\beta \). Hence by Lemma 2.3 we have

\[
\int_{|w| \leq r} d\mu \circ \varphi^{-1} = \int_{|\varphi(z)| \leq r} |u(z)|^2 \, dA_{\beta}(z) < \infty.
\]
Hence
\[
\liminf_{j \to \infty} \sup_{\|f\|_{A^2_{\alpha}} \leq 1} \int_{|w| \leq r} |(R_j f)^{(n)}(w)|^2 d\mu \circ \varphi^{-1} = 0. \tag{5}
\]

We now estimate \(\int_{|w| > r} |(R_j f)^{(n)}(w)|^2 d\mu \circ \varphi^{-1}\). By Lemmas 2.1, 2.2 and 2.3 we obtain
\[
\int_{|w| > r} |(R_j f)^{(n)}(w)|^2 d\mu \circ \varphi^{-1} \lesssim \|R_j f\|_{A^2_{\alpha}}^2 \sup_{|b| \geq r} \int_{|\Delta(0, r)|} \|\sigma_b'(w)\|^{2+\alpha+2n} d\mu \circ \varphi^{-1}
\]
\[
\lesssim \|R_j f\|_{A^2_{\alpha}}^2 \sup_{|b| \geq r} \int_{\mathbb{D}} |\sigma_b'(\varphi(z))|^{2+\alpha+2n} |u(z)|^2 dA_{\beta}(z).
\tag{6}
\]

Using (4), (5) and (6), for any \(r \in (0, 1)\) we get
\[
\|D^n_{\varphi, u}\|_{c_{A^2_{\alpha}} \to A^2_{\beta}} \leq \liminf_{j \to \infty} \sup_{\|f\|_{A^2_{\alpha}} \leq 1} \int_{|w| > r} |(R_j f)^{(n)}(w)|^2 d\mu \circ \varphi^{-1}
\]
\[
\leq \liminf_{j \to \infty} \sup_{\|f\|_{A^2_{\alpha}} \leq 1} \|R_j f\|_{A^2_{\alpha}}^2 \sup_{|b| \geq r} \int_{\mathbb{D}} |\sigma_b'(\varphi(z))|^{2+\alpha+2n} |u(z)|^2 dA_{\beta}(z)
\]
\[
\leq \sup_{|b| \geq r} \int_{\mathbb{D}} |\sigma_b'(\varphi(z))|^{2+\alpha+2n} |u(z)|^2 dA_{\beta}(z).
\]

Taking the limit as \(r \to 1\), we get the desired result. The proof is complete. \(\square\)

From Theorem 2.2, we immediately get the following result.

**Theorem 2.3.** Let \(\varphi\) be an analytic self-map of \(\mathbb{D}\), \(-1 < \alpha, \beta < \infty, n \in \mathbb{Z}\) and \(u \in H(\mathbb{D})\). Suppose that \(D^n_{\varphi, u} : A^2_{\alpha} \to A^2_{\beta}\) is bounded. Then \(D^n_{\varphi, u} : A^2_{\alpha} \to A^2_{\beta}\) is compact if and only if
\[
\limsup_{|b| \to 1} \int_{\mathbb{D}} |\sigma_b'(\varphi(z))|^{2+\alpha+2n} |u(z)|^2 dA_{\beta}(z) = 0.
\]

3. **Hilbert-Schmidt operator** \(D^n_{\varphi, u} : A^2_{\alpha} \to A^2_{\beta}\)

When \(\alpha = \beta = 0\), \(\acute{C}u\v{c}kovi\v{c} and Zhao\) [3] proved that \(uC_\varphi : A^2 \to A^2\) is a Hilbert-Schmidt operator if and only if
\[
\int_{\mathbb{D}} \frac{|u(z)|^2}{(1 - |\varphi(z)|^2)^2} dA(z) < \infty.
\]

In this section, we generalize the above result and study the Hilbert-Schmidt operator \(D^n_{\varphi, u} : A^2_{\alpha} \to A^2_{\beta}\). For the case \(u = 1\) see also [2]. The following result was essentially proved in [24], but since there are some minor differences and for the completeness we present a proof of it.
THEOREM 3.1. Let \( \phi \) be an analytic self-map of \( \mathbb{D} \), \(-1 < \alpha, \beta < \infty, n \in \mathbb{Z} \) and \( u \in H(\mathbb{D}) \). Assume that \( D_{\phi,u}^n : A_2^\alpha \to A_2^\beta \) is bounded. Then \( D_{\phi,u}^n : A_2^\alpha \to A_2^\beta \) is a Hilbert-Schmidt operator if and only if

\[
\int_{\mathbb{D}} \frac{|u(z)|^2}{(1 - |\phi(z)|^2)^{\alpha + 2n}} dA_\beta(z) < \infty.
\]

Proof. Let \( e_m^\alpha(z) = \sqrt{\frac{\Gamma(m + \alpha + 2)}{m! \Gamma(\alpha + 2)}} z^m \). Then \( \{e_m^\alpha\}_{m=0}^\infty \) is an orthonormal basis for \( A_2^\alpha \). We have

\[
D_{\phi,u}^n : A_2^\alpha \to A_2^\beta \quad \text{is Hilbert-Schmidt}
\]

\[
\Leftrightarrow \sum_{m=0}^\infty \|D_{\phi,u}^n(e_m^\alpha)\|_{A_2^\beta}^2 < \infty
\]

\[
\Leftrightarrow \sum_{m=0}^\infty \int_{\mathbb{D}} |u(z)|^2 |(e_m^\alpha)^{(n)}(\phi(z))|^2 dA_\beta(z) < \infty
\]

\[
\Leftrightarrow \int_{\mathbb{D}} |u(z)|^2 \sum_{m=n}^\infty \frac{\Gamma(m + \alpha + 2)}{m! \Gamma(\alpha + 2)} \left( \frac{n-1}{m-j} \right)^2 |\phi(z)|^{2m-2n} dA_\beta(z) < \infty
\]

\[
\Leftrightarrow \int_{\mathbb{D}} \frac{|u(z)|^2}{(1 - |\phi(z)|^2)^{\alpha + 2n}} dA_\beta(z) < \infty. \quad \square
\]

From the last theorem, we easily get the following result.

COROLLARY 3.1. Let \( \phi \) be an analytic self-map of \( \mathbb{D} \) such that \( \|\phi\|_\infty < 1, -1 < \alpha, \beta < \infty \) and \( n \in \mathbb{Z} \). Then for any \( u \in A_2^\alpha \), \( D_{\phi,u}^n : A_2^\alpha \to A_2^\beta \) is a Hilbert-Schmidt operator.

THEOREM 3.2. Let \( \phi \) be an analytic self-map of \( \mathbb{D} \) and \( u \in H(\mathbb{D}) \). Let \(-1 < \alpha, \beta < \infty, n \in \mathbb{Z} \) such that \( \beta > 2n - 1 + \alpha \). If

\[
\int_{\mathbb{D}} |u(z)|^2 (1 - |z|^2)^{\beta - \alpha - 2n} dA(z) < \infty,
\]

then \( D_{\phi,u}^n : A_2^\alpha \to A_2^\beta \) is a Hilbert-Schmidt operator.

Proof. From page 41 of [2], we have

\[
\frac{1 - |z|^2}{|1 - \phi(z)|^2} \leq \frac{2}{1 - |\phi(0)|} \frac{1 + |\phi(0)|}{1 - |\phi(0)|} \frac{1 + |\phi(0)|}{1 - |\phi(0)|},
\]

which implies that

\[
\int_{\mathbb{D}} \frac{|u(z)|^2 (1 - |z|^2)^\beta}{(1 - |\phi(z)|^2)^{\alpha + 2n}} dA(z)
\]

\[
\leq 2^{\alpha + 2n} \int_{\mathbb{D}} \frac{|u(z)|^2 (1 - |z|^2)^\beta}{(1 - |\phi(z)|^2)^{\alpha + 2n}} \left( \frac{1 + |\phi(0)|}{1 - |\phi(0)|} \right)^{\alpha + 2n} dA(z)
\]

\[
\lesssim \int_{\mathbb{D}} |u(z)|^2 (1 - |z|^2)^{\beta - \alpha - 2n} dA(z) < \infty.
\]
Here we use the fact that \( \frac{1+|\varphi(0)|}{1-|\varphi(0)|} \) is a constant. By Theorem 3.1, we see that \( D^n_{\varphi,u} : A^2_\alpha \to A^2_\beta \) is a Hilbert-Schmidt operator.

The above theorem gives a sufficient condition for \( D^n_{\varphi,u} : A^2_\alpha \to A^2_\beta \) to be a Hilbert-Schmidt operator for any \( \varphi \). However, when \( \varphi \) is an automorphism of \( \mathbb{D} \), we prove that this condition is also a necessary condition.

**Theorem 3.3.** Let \( u \in H(\mathbb{D}) \), \(-1 < \alpha, \beta < \infty\) and \( n \) be a nonnegative integer such that \( \beta > 2n - 1 + \alpha \). Assume that \( \varphi \) is an automorphism of \( \mathbb{D} \). Then \( D^n_{\varphi,u} : A^2_\alpha \to A^2_\beta \) is a Hilbert-Schmidt operator if and only if

\[
\int_{\mathbb{D}} |u(z)|^2 (1 - |z|^2)^{2n-\alpha-2}dA(z) < \infty.
\]

**Proof.** We only need to prove the necessary part. Suppose that \( D^n_{\varphi,u} : A^2_\alpha \to A^2_\beta \) is a Hilbert-Schmidt operator. For \( a \in \mathbb{D} \), let \( \varphi(z) = \lambda \frac{a-z}{1-\bar{a}z} \) where \( |\lambda| = 1 \). After some calculation, we have

\[
(1 - |z|^2) \geq \frac{1 - |a|}{1 + |a|} (1 - |\varphi(z)|^2).
\]

Hence by Theorem 3.1 and the fact that \( \frac{1+|a|}{1-|a|} \) is a constant, we get

\[
\int_{\mathbb{D}} |u(z)|^2 (1 - |z|^2)^{\beta-\alpha-2}dA(z) \lesssim \left( \frac{1 + |a|}{1 - |a|} \right)^{\alpha+2+2n} \int_{\mathbb{D}} \frac{|u(z)|^2 (1 - |z|^2)^{\beta}}{(1 - |\varphi(z)|^2)^{\alpha+2+2n}}dA(z) < \infty. \quad \square
\]

### 4. Order boundedness of \( D^n_{\varphi,u} : A^2_\alpha \to A^2_\beta \)

In this section, we investigate the order boundedness of \( D^n_{\varphi,u} : A^2_\alpha \to A^2_\beta \).

**Theorem 4.1.** Let \( \varphi \) be an analytic self-map of \( \mathbb{D} \), \(-1 < \alpha, \beta < \infty\), \( n \in \mathbb{Z} \) and \( u \in H(\mathbb{D}) \). The operator \( D^n_{\varphi,u} : A^2_\alpha \to A^2_\beta \) is order bounded if and only if

\[
\int_{\mathbb{D}} \frac{|u(z)|^2}{(1 - |\varphi(z)|^2)^{2+\alpha+2n}}dA_\beta(z) < \infty. \quad (7)
\]

**Proof.** First we assume that \( D^n_{\varphi,u} : A^2_\alpha \to A^2_\beta \) is order bounded. Then, for any \( f \in A^2_\alpha \) with \( ||f||_{A^2_\alpha} \leq 1 \), there exists a nonnegative function \( g \in L^2(\mathbb{D}, dA_\beta) \) such that

\[
|D^n_{\varphi,u}f(z)| \leq g(z)
\]

for almost every \( z \in \mathbb{D} \). For any \( z \in \mathbb{D} \), set

\[
h_z(a) = \left( \frac{1 - |\varphi(z)|^2}{(1 - a\varphi(z))^2} \right)^{\frac{\alpha+2}{2}}, \quad a \in \mathbb{D}.
\]
A simple computation shows that $h_z \in A^2_\alpha$ with $\|h_z\|_{A^2_\alpha} \leq 1$. So

$$\frac{|\varphi(z)|^n|u(z)|}{(1 - |\varphi(z)|^2)^{\frac{\alpha + 2}{2} + n}} \lesssim |D^n_{\varphi,u}h_z(z)| \leq g(z).$$

Since $g \in L^2(\mathbb{D},dA_\beta)$, the above inequality implies

$$\int_{|\varphi(z)| > 1/2} \frac{|u(z)|^2}{(1 - |\varphi(z)|^2)^{\alpha + 2 + 2n}}dA_\beta(z) \lesssim \int_{\mathbb{D}} |g(z)|^2dA_\beta(z) < \infty. \quad (8)$$

On the other hand, set

$$k(z) = \frac{z^n}{\|z^n\|_{A^2_\alpha}}, \quad z \in \mathbb{D}.$$ 

Here

$$\|z^n\|_{A^2_\alpha} = \frac{(\alpha + 1)\Gamma(n + 1)\Gamma(\alpha + 1)}{\Gamma(n + 2 + \alpha)}.$$ 

It is clear that $k \in A^2_\alpha$ with $\|k\|_{A^2_\alpha} = 1$. So,

$$|u(z)| \lesssim |D^n_{\varphi,u}k(z)| \leq g(z), \quad z \in \mathbb{D}.$$ 

Since $g \in L^2(\mathbb{D},dA_\beta)$, the above inequality implies $u \in A^2_\beta$. Hence

$$\int_{|\varphi(z)| \leq 1/2} \frac{|u(z)|^2}{(1 - |\varphi(z)|^2)^{\alpha + 2 + 2n}}dA_\beta(z) \lesssim \int_{|\varphi(z)| \leq 1/2} |u(z)|^2dA_\beta(z) < \infty. \quad (9)$$

From (8) and (9), we get

$$\int_{\mathbb{D}} \frac{|u(z)|^2}{(1 - |\varphi(z)|^2)^{\alpha + 2 + 2n}}dA_\beta(z) < \infty.$$ 

Conversely, assume that (7) holds. By a classical estimate (see, e.g., a general point-value estimation in Lemma 5 of [17]), for any $f \in A^2_\alpha$, we have

$$|D^n_{\varphi,u}f(z)| = |u(z)| \cdot |f^{(n)}(\varphi(z))| \leq c_{n,\alpha} \frac{|u(z)|}{(1 - |\varphi(z)|^2)^{\frac{\alpha + 2}{2} + n}}\|f\|_{A^2_\alpha}, \quad z \in \mathbb{D}, \quad (10)$$

and so

$$\|D^n_{\varphi,u}f\|_{A^2_\beta}^2 \leq c_{n,\alpha} \int_{\mathbb{D}} \frac{|u(z)|^2}{(1 - |\varphi(z)|^2)^{\alpha + 2 + 2n}}dA_\beta(z) \cdot \|f\|_{A^2_\alpha}^2 < \infty.$$ 

Here $c_{n,\alpha}$ is a constant depending only on $n$ and $\alpha$. Therefore $D^n_{\varphi,u} : A^2_\alpha \rightarrow A^2_\beta$ is bounded.

Now take a function $f \in A^2_\alpha$ with $\|f\|_{A^2_\alpha} \leq 1$. From (10),

$$|D^n_{\varphi,u}f(z)| \leq \frac{c_{n,\alpha}|u(z)|}{(1 - |\varphi(z)|^2)^{\frac{\alpha + 2}{2} + n}},$$
for any $z \in \mathbb{D}$. Set

$$
g = c_{n,\alpha}|u|(1 - |\varphi|^2)^{-\frac{\alpha+2}{2}-n}.
$$

Then the assumed condition implies $g \in L^2(\mathbb{D}, dA_B)$ and $g \geq 0$. Moreover, $|D_{\varphi,u}^n f| \leq g$. That is, $D_{\varphi,u}^n : A^2_\alpha \to A^2_\beta$ is order bounded. This completes the proof. \square

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(Received December 22, 2018) Xiangling Zhu
Zhongshan Institute
University of Electronic Science and Technology of China
528402, Zhongshan, Guangdong, P. R. China
e-mail: jyuzxl@163.com