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GENERALIZED WEIGHTED COMPOSITION OPERATORS ON WEIGHTED BERGMAN SPACES, II

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Abstract. The boundedness, compactness, essential norm, Hilbert-Schmidt class and order boundedness of generalized weighted composition operators on weighted Bergman spaces are investigated in this paper.

1. Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disk of complex plane $\mathbb{C}$ and let $\partial\mathbb{D}$ be the boundary of $\mathbb{D}$. Denote by $H(\mathbb{D})$ the class of functions analytic in $\mathbb{D}$. For $a \in \mathbb{D}$, $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$ is the Möbius transformation of $\mathbb{D}$.

For a subarc $I \subseteq \partial\mathbb{D}$, let $S(I)$ be the Carleson box based on I with

$$S(I) = \{z \in \mathbb{D} : 1 - |I| \leqslant |z| < 1 \ \text{ and } \frac{z}{|z|} \in I\}.$$

If $I = \partial\mathbb{D}$, let $S(I) = \mathbb{D}$. Let μ denote a positive Borel measure on $\mathbb{D}$. For $0 < \alpha < \infty$, we say that μ is an α-Carleson measure on $\mathbb{D}$ if (see [1])

$$\sup_{I \subset \partial\mathbb{D}} \mu(S(I))/|I|^{\alpha} < \infty.$$

Here and henceforth $\sup_{I\subset\partial\mathbb{D}}$ indicates the supremum taken over all subarcs I of $\partial\mathbb{D}$. $|I| = (2\pi)^{-1}\int_I |d\xi|$ is the normalized length of the subarc I. Note that $\alpha = 1$ gives the classical Carleson measure.

For $0 < p < \infty$ and $\gamma > -1$, the weighted Bergman space, denoted by A^p_γ, is the set of all functions $f \in H(\mathbb{D})$ satisfying

$$\|f\|^p_{A^p_\gamma} = \int_{\mathbb{D}} |f(z)|^p dA_\gamma(z) < \infty,$$

where $dA_\gamma(z) = (\gamma+1)(1-|z|^2)^\gamma dA(z)$ and dA is the normalized Lebesgue area measure in $\mathbb{D}$ such that $A(\mathbb{D}) = 1$. This means that $A^p_\gamma = H(\mathbb{D}) \bigcap L^p(\mathbb{D}, dA_\gamma)$. When $p = 2$, A^2_γ is a Hilbert space.

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We denote the set of nonnegative integers by $\mathbb{Z}$. Let φ be an analytic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$ and $n \in \mathbb{Z}$. The generalized weighted composition operator $D^n_{\varphi,u}$ is defined as follows (see [34, 36]).

$$(D^n_{\varphi,u}f)(z) = u(z)\cdot f^{(n)}(\varphi(z)), \;\; f \in H(\mathbb{D}), \;\; z \in \mathbb{D}.$$

If $n = 0$, then $D^n_{\varphi,u}$ is just the weighted composition operator, which is frequently denoted by uC_φ in the literature. When $n = 0$ and $u(z) = 1$, then $D^n_{\varphi,u}$ is just the composition operator C_φ, which is defined by

$$(C_\varphi f)(z) = f(\varphi(z)), \; f \in H(\mathbb{D}).$$

See [2, 32] for more information about the theory of composition operators. When $u(z) = 1$, $D^n_{\varphi,u} = C_\varphi D^n$. See, for example, [4, 7, 8, 9, 12, 18, 19, 21, 25, 31] for the study of the operator $C_\varphi D^n$. See, for example, [5, 6, 10, 11, 13, 20, 22, 23, 24, 26, 27, 33, 34, 35, 36, 37] and the references therein for the study of the operator $D^n_{\varphi,u}$. For some other product-type operators see, for example [16, 28, 29].

In [19], Stević studied the operator $C_\varphi D^n$ on weighted Bergman spaces. In [30], Ueki studied the order boundedness of the operator $uC_\varphi : A^p_\alpha \to A^q_\beta$. In [34], the author studied the operator $D^n_{\varphi,u} : A^p_\alpha \to A^q_\beta$. Among others, we prove that, under the assumption that $u \in A^2_\beta$, $D^n_{\varphi,u} : A^2_\alpha \to A^2_\beta$ is bounded if and only if

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |u(z)|^2 \frac{(1-|a|^2)^{\alpha+2}}{|1-\overline{a}\varphi(z)|^{2\alpha+4+2n}} dA_\beta(z) < \infty.$$

$D^n_{\varphi,u} : A^2_\alpha \to A^2_\beta$ is compact if and only if $D^n_{\varphi,u} : A^2_\alpha \to A^2_\beta$ is bounded and

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |u(z)|^2 \frac{(1-|a|^2)^{\alpha+2}}{|1-\overline{a}\varphi(z)|^{2\alpha+4+2n}} dA_\beta(z) = 0.$$

Motivated by results in [19, 30, 34], in this work we give another characterization of the boundedness, compactness and essential norm of the operator $D^n_{\varphi,u} : A^2_\alpha \to A^2_\beta$. Moreover, we study the order boundedness and the Hilbert-Schmidt class of the operator $D^n_{\varphi,u} : A^2_\alpha \to A^2_\beta$.

Recall that the linear operator $T : X \to Y$ is order bounded if T maps the unit ball of X into an order interval of Z, namely there exists a nonnegative element P in Z such that $|T(f)| \leqslant P$ for all f belongs to the unit ball of X. Here X is a quasi-Banach space and Y a subspace of quasi-Banach Lattice Z.

Throughout the paper, we denote by C a positive constant which may differ from one occurrence to the next. In addition, we say that $A \lesssim B$ if there exists a constant C such that $A \leqslant CB$. The symbol $A \approx B$ means that $A \lesssim B \lesssim A$.

2. Boundedness and essential norm of $D^n_{\varphi,u} : A^2_\alpha \to A^2_\beta$

In this section, we give another characterization for the boundedness, compactness and essential norm of the operator $D^n_{\varphi,u} : A^2_\alpha \to A^2_\beta$. Hence, we first state some lemmas which will be used in the proofs of the main results in this section.

LEMMA 2.1. [14] *Let* μ *be a positive measure on* $\mathbb{D}$, $m \in \mathbb{Z}$ *and* $-1 < \alpha < \infty$. *Then* μ *is a bounded* $2+\alpha+2m$*-Carleson measure if and only if there is a positive constant* C, *depending only on* α *and* m *such that*

$$\int_{\mathbb{D}} |f^{(m)}(z)|^2 d\mu(z) \leqslant C\|f\|^2_{A^2_\alpha}$$

for all $f \in A^2_\alpha$. *Moreover, if* μ *is a bounded* $2+\alpha+2m$*-Carleson measure, then* $C = C_1C_2$, *where* $C_1 > 0$ *depends only on* α *and* m *and*

$$C_2 = \sup_I \frac{\mu(S(I))}{|I|^{2+\alpha+2m}}.$$

Let $0 < s < \infty$. The bounded s-Carleson measure can be characterized by a global integral condition (see [1]), namely,

$$\sup_I \frac{\mu(S(I))}{|I|^s} \approx \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |\sigma_a'(z)|^s d\mu(z). \tag{1}$$

LEMMA 2.2. [15] *Let* $0 < \rho < 1, 1 \leqslant s < \infty$ *and let* μ *be a positive Borel measure on* $\mathbb{D}$. *Then*

$$\sup_I \frac{\mu(S(I)\backslash\Delta(0,\rho))}{|I|^s} \lesssim \sup_{|b|\geqslant\rho} \int_{\mathbb{D}} |\sigma_b'(z)|^s d\mu(z),$$

where $\Delta(0,\rho) := \{z : |z| < \rho\}$.

LEMMA 2.3. [2] *Let* g *and* u *be positive measurable functions on* $\mathbb{D}$, *and let* φ *be an analytic self-map of* $\mathbb{D}$. *Then*

$$\int_{\mathbb{D}} (g\circ\varphi)(z)|\varphi'(z)|^2 u(z)dA(z) = \int_{\mathbb{D}} g(w)U(\varphi,w)dA(z),$$

where $U(\varphi,w) = \sum_{z\in\varphi^{-1}(w)} u(z)$ *for* $w \in \mathbb{D}\backslash\{\varphi(0)\}$.

For an $f(z) = \sum_{k=0}^\infty a_k z^k \in H(\mathbb{D})$, *define*

$$T_j f(z) = \sum_{k=0}^{j} a_k z^k, \qquad R_j f(z) = \sum_{k=j+1}^{\infty} a_k z^k.$$

LEMMA 2.4. *Let* $m \in \mathbb{Z}$ *and* $-1 < \alpha < \infty$. *For each* $w \in \mathbb{D}$, *positive integer* j *and* $f \in A^2_\alpha$,

$$\left|(R_j f(w))^{(m)}\right| \lesssim \|f\|_{A^2_\alpha} \sum_{k=j+1}^{\infty} \frac{\Gamma(k+\alpha+2+m)}{k!\Gamma(\alpha+2+m)}|w|^k,$$

where Γ *denotes the Gamma function.*

Proof. Since $f \in A^2_\alpha$, it is clear that $R_j f \in A^2_\alpha$. Hence

$$(R_j f)(w) = \int_{\mathbb{D}} (R_j f)(z) K_\alpha(w,z) dA_\alpha(z),$$

where $K_\alpha(w,z)=\frac{1}{(1-w\bar{z})^{a+2}}$ is the Bergman Kernel function. Thus, by the orthogonality of monomials z^γ with respect to dA_α,

$$\begin{aligned}(R_jf)^{(m)}(w) &= \int_{\mathbb{D}} R_jf(z)\frac{\Gamma(\alpha+2+m)}{\Gamma(\alpha+2)}\frac{\bar{z}^m}{(1-\bar{z}w)^{\alpha+2+m}}dA_\alpha(z)\\ &= \int_{\mathbb{D}} f(z)\frac{\Gamma(\alpha+2+m)}{\Gamma(\alpha+2)}R_j\left(\frac{\bar{z}^m}{(1-\bar{z}w)^{\alpha+2+m}}\right)dA_\alpha(z).\end{aligned}$$

Using Hölder's inequality, we get

$$\begin{aligned}\left|(R_jf)^{(m)}(w)\right| &\leqslant \frac{\Gamma(\alpha+2+m)}{\Gamma(\alpha+2)}\int_{\mathbb{D}}|f(z)|\left|R_j\left(\frac{\bar{z}^m}{(1-\bar{z}w)^{\alpha+2+m}}\right)\right|dA_\alpha(z)\\ &\approx \int_{\mathbb{D}}|f(z)|\left|\sum_{k=j+1}^{\infty}\frac{\Gamma(k+m+\alpha+2)}{k!\Gamma(m+\alpha+2)}w^k\bar{z}^{k+m}\right|dA_\alpha(z)\\ &\leqslant \|f\|_{A^2_\alpha}\left(\int_{\mathbb{D}}\left(\sum_{k=j+1}^{\infty}\frac{\Gamma(k+m+\alpha+2)}{k!\Gamma(m+\alpha+2)}|w|^k|z|^{k+m}\right)^2dA_\alpha(z)\right)^{\frac{1}{2}}\\ &\leqslant \|f\|_{A^2_\alpha}\sum_{k=j+1}^{\infty}\frac{\Gamma(k+m+\alpha+2)}{k!\Gamma(m+\alpha+2)}|w|^k. \quad \square\end{aligned}$$

THEOREM 2.1. *Let φ be an analytic self-map of $\mathbb{D}$, $-1<\alpha,\beta<\infty, n\in\mathbb{Z}$ and $u\in H(\mathbb{D})$. Then $D^n_{\varphi,u}:A^2_\alpha\to A^2_\beta$ is bounded if and only if*

$$\sup_{b\in\mathbb{D}}\int_{\mathbb{D}}|\sigma_b'(\varphi(z))|^{2+\alpha+2n}|u(z)|^2dA_\beta(z)<\infty. \tag{2}$$

Proof. First we assume that (2) holds. Let $d\mu(z)=|u(z)|^2dA_\beta(z)$. By (1) and Lemma 2.3 we have

$$\begin{aligned}\sup_I\frac{\mu\circ\varphi^{-1}(S(I))}{|I|^{2+\alpha+2n}} &\approx \sup_{b\in\mathbb{D}}\int_{\mathbb{D}}|\sigma_b'(w)|^{2+\alpha+2n}d\mu\circ\varphi^{-1}(w)\\ &= \sup_{b\in\mathbb{D}}\int_{\mathbb{D}}|\sigma_b'(\varphi(z))|^{2+\alpha+2n}|u(z)|^2dA_\beta(z)<\infty. \end{aligned}\tag{3}$$

For any $f\in A^2_\alpha$, by Lemmas 2.1 and 2.3, and (3) we have

$$\begin{aligned}\|D^n_{\varphi,u}f\|^2_{A^2_\beta} &\approx \int_{\mathbb{D}}|D^n_{\varphi,u}f(z)|^2dA_\beta(z)=\int_{\mathbb{D}}|f^{(n)}(w)|^2d\mu\circ\varphi^{-1}\\ &\lesssim \sup_I\frac{\mu\circ\varphi^{-1}(S(I))}{|I|^{2+\alpha+2n}}\|f\|^2_{A^2_\alpha}<\infty.\end{aligned}$$

Thus $D^n_{\varphi,u}:A^2_\alpha\to A^2_\beta$ is bounded.

Conversely, assume that $D^n_{\varphi,u}:A^2_\alpha\to A^2_\beta$ is bounded. For any $a\in\mathbb{D}$, set

$$f_a(z)=\left(\frac{1-|a|^2}{(1-\bar{a}z)^2}\right)^{\frac{\alpha+2}{2}},\quad z\in\mathbb{D}.$$

Then $\|f_a\|_{A^2_\alpha} \approx 1$. Let $I \subset \partial\mathbb{D}$, and let $\zeta \in \partial\mathbb{D}$ be the center of arc I and $b = (1-|I|)\zeta \in \mathbb{D}$. Then

$$f_b^{(n)}(z) = \frac{\Gamma(2+\alpha+n)}{\Gamma(2+\alpha)} \frac{(1-|b|^2)^{\frac{\alpha+2}{2}}\bar{b}^n}{(1-\bar{b}z)^{2+\alpha+n}}$$

and

$$|f_b^{(n)}(z)|^2 \gtrsim \frac{1}{(1-|b|)^{2+\alpha+2n}}, \qquad z \in S(I).$$

Thus, by the boundedness of $D^n_{\varphi,u} : A^2_\alpha \to A^2_\beta$, we get

$$\begin{aligned}\infty > \|D^n_{\varphi,u}\|^2\|f_b\|^2_{A^2_\alpha} \geqslant \|D^n_{\varphi,u}f_b\|^2_{A^2_\beta} &= \int_{\mathbb{D}} |f_b^{(n)}(\varphi(z))|^2|u(z)|^2 dA_\beta(z)\\ = \int_{\mathbb{D}} |f_b^{(n)}(w)|^2 d\mu\circ\varphi^{-1}(w) &\gtrsim \int_{S(I)} \frac{1}{(1-|b|)^{2+\alpha+2n}} d\mu\circ\varphi^{-1}(w) \approx \frac{\mu\circ\varphi^{-1}(S(I))}{|I|^{2+\alpha+2n}},\end{aligned}$$

for all $I \subset \partial\mathbb{D}$. By (1) and Lemma 2.3 we have

$$\begin{aligned}\sup_{b\in\mathbb{D}} \int_{\mathbb{D}} |\sigma_b'(\varphi(z))|^{2+\alpha+2n}|u(z)|^2 dA_\beta(z) &= \sup_{b\in\mathbb{D}} \int_{\mathbb{D}} |\sigma_b'(w)|^{2+\alpha+2n} d\mu\circ\varphi^{-1}\\ &\approx \sup_I \frac{\mu\circ\varphi^{-1}(S(I))}{|I|^{2+\alpha+2n}} < \infty.\end{aligned}$$

This completes the proof of this theorem. □

THEOREM 2.2. *Let φ be an analytic self-map of $\mathbb{D}$, $-1 < \alpha, \beta < \infty, n \in \mathbb{Z}$ and $u \in H(\mathbb{D})$. Suppose that $D^n_{\varphi,u} : A^2_\alpha \to A^2_\beta$ is bounded. Then*

$$\|D^n_{\varphi,u}\|^2_{e,A^2_\alpha\to A^2_\beta} \approx T,$$

where

$$T := \limsup_{|b|\to 1} \int_{\mathbb{D}} |\sigma_b'(\varphi(z))|^{2+\alpha+2n}|u(z)|^2 dA_\beta(z).$$

Proof. First we prove that $\|D^n_{\varphi,u}\|^2_{e,A^2_\alpha\to A^2_\beta} \gtrsim T$. Let $b \in \mathbb{D}$. Set

$$f_b(z) = \left(\frac{1-|b|^2}{(1-\bar{b}z)^2}\right)^{\frac{\alpha+2}{2}}, \ z \in \mathbb{D}.$$

We have $\|f_b\|_{A^2_\alpha} \approx 1$ and $f_b \to 0$ weakly in A^2_α as $|b| \to 1$. Thus $\|K(f_b)\|_{A^2_\beta} \to 0$ as $|b| \to 1$ for every compact operator $K : A^2_\alpha \to A^2_\beta$. Thus,

$$\begin{aligned}\|D^n_{\varphi,u} - K\|^2_{A^2_\alpha\to A^2_\beta} &\geqslant \limsup_{|b|\to 1} \|D^n_{\varphi,u}(f_b) - K(f_b)\|^2_{A^2_\beta}\\ &\geqslant \limsup_{|b|\to 1} \|D^n_{\varphi,u}(f_b)\|^2_{A^2_\beta} - \limsup_{|b|\to 1} \|K(f_b)\|^2_{A^2_\beta} = \limsup_{|b|\to 1} \|D^n_{\varphi,u}(f_b)\|^2_{A^2_\beta}\end{aligned}$$

for every compact operator $K : A^2_\alpha \to A^2_\beta$. By Lemma 2.3 we have

$$\begin{aligned}
\limsup_{|b|\to 1} \|D^n_{\varphi,u}(f_b)\|^2_{A^2_\beta} &\approx \limsup_{|b|\to 1} \int_{\mathbb{D}} |f_b^{(n)}(w)|^2 d\mu\circ\varphi^{-1} \\
&= \limsup_{|b|\to 1} \int_{\mathbb{D}} \Big| \frac{\Gamma(2+\alpha+n)}{\Gamma(2+\alpha)} \frac{(1-|b|^2)^{\frac{\alpha+2}{2}} \bar{b}^n}{(1-\bar{b}w)^{2+\alpha+n}} \Big|^2 d\mu\circ\varphi^{-1} \\
&\gtrsim \limsup_{|b|\to 1} \int_{\mathbb{D}} |\sigma_b'(w)|^{2+\alpha+2n} d\mu\circ\varphi^{-1} = T.
\end{aligned}$$

Therefore, from the definition of the essential norm, we obtain

$$\|D^n_{\varphi,u}\|^2_{e,A^2_\alpha\to A^2_\beta} = \inf_J \|D^n_{\varphi,u} - J\|^2_{A^2_\alpha\to A^2_\beta} \gtrsim T.$$

Next, we prove that $\|D^n_{\varphi,u}\|^2_{e,A^2_\alpha\to A^2_\beta} \lesssim T$. It is clear that

$$\begin{aligned}
\|D^n_{\varphi,u}\|_{e,A^2_\alpha\to A^2_\beta} &= \|D^n_{\varphi,u}(T_j+R_j)\|_{e,A^2_\alpha\to A^2_\beta} \leqslant \|D^n_{\varphi,u}T_j\|_{e,A^2_\alpha\to A^2_\beta} + \|D^n_{\varphi,u}R_j\|_{e,A^2_\alpha\to A^2_\beta} \\
&= \|D^n_{\varphi,u}R_j\|_{e,A^2_\alpha\to A^2_\beta} \leqslant \|D^n_{\varphi,u}R_j\|_{A^2_\alpha\to A^2_\beta}.
\end{aligned}$$

Here we used the fact that T_j is compact on A^2_α. Hence

$$\|D^n_{\varphi,u}\|_{e,A^2_\alpha\to A^2_\beta} \leqslant \liminf_{j\to\infty} \|D^n_{\varphi,u}R_j\|_{A^2_\alpha\to A^2_\beta}.$$

For an $f(z) = \sum_{k=0}^\infty a_k z^k \in H(\mathbb{D})$, by Lemma 2.3 we have

$$\begin{aligned}
\|D^n_{\varphi,u}\|^2_{e,A^2_\alpha\to A^2_\beta} &\leqslant \liminf_{j\to\infty} \|D^n_{\varphi,u}R_j\|^2_{A^2_\alpha\to A^2_\beta} \leqslant \liminf_{j\to\infty} \sup_{\|f\|_{A^2_\alpha}\leqslant 1} \|D^n_{\varphi,u}(R_j f)\|^2_{A^2_\beta} \\
&\approx \liminf_{j\to\infty} \sup_{\|f\|_{A^2_\alpha}\leqslant 1} \int_{\mathbb{D}} |(R_j f)^{(n)}(\varphi(z))|^2 |u(z)|^2 dA_\beta(z) \\
&= \liminf_{j\to\infty} \sup_{\|f\|_{A^2_\alpha}\leqslant 1} \int_{\mathbb{D}} |(R_j f)^{(n)}(w)|^2 d\mu\circ\varphi^{-1}. \qquad (4)
\end{aligned}$$

Let $r \in (0,1)$. For each $f \in A^2_\alpha$, by Lemma 2.4 we have

$$\begin{aligned}
&\int_{|w|\leqslant r} |(R_j f)^{(n)}(w)|^2 d\mu\circ\varphi^{-1} \\
&\lesssim \|f\|^2_{A^2_\alpha} \int_{|w|\leqslant r} \Big(\sum_{k=j+1}^\infty \frac{\Gamma(k+\alpha+2+n)}{k!\Gamma(\alpha+2+n)} |w|^k \Big)^2 d\mu\circ\varphi^{-1}(w) \\
&\leqslant \|f\|^2_{A^2_\alpha} \Big(\sum_{k=j+1}^\infty \frac{\Gamma(k+\alpha+2+n)}{k!\Gamma(\alpha+2+n)} r^k \Big)^2 \int_{|w|\leqslant r} d\mu\circ\varphi^{-1}.
\end{aligned}$$

By the boundedness of $D^n_{\varphi,u} : A^2_\alpha \to A^2_\beta$ is bounded, we have $u \in A^2_\beta$. Hence by Lemma 2.3 we have

$$\int_{|w|\leqslant r} d\mu\circ\varphi^{-1} = \int_{|\varphi(z)|\leqslant r} |u(z)|^2 dA_\beta(z) < \infty.$$

Hence

$$\liminf_{j\to\infty} \sup_{\|f\|_{A^2_\alpha}\leqslant 1} \int_{|w|\leqslant r} |(R_j f)^{(n)}(w)|^2 d\mu\circ\varphi^{-1} = 0. \tag{5}$$

We now estimate $\int_{|w|>r} |(R_j f)^{(n)}(w)|^2 d\mu\circ\varphi^{-1}$. By Lemmas 2.1, 2.2 and 2.3 we obtain

$$\begin{aligned}\int_{|w|>r} |(R_j f)^{(n)}(w)|^2 d\mu\circ\varphi^{-1} &\lesssim \|R_j f\|^2_{A^2_\alpha} \sup_I \frac{\int_{S(I)\setminus\Delta(0,r)} d\mu\circ\varphi^{-1}}{|I|^{2+\alpha+2n}} \\ &\lesssim \|R_j f\|^2_{A^2_\alpha} \sup_{|b|\geqslant r} \int_{\mathbb{D}} |\sigma_b'(w)|^{2+\alpha+2n} d\mu\circ\varphi^{-1} \\ &= \|R_j f\|^2_{A^2_\alpha} \sup_{|b|\geqslant r} \int_{\mathbb{D}} |\sigma_b'(\varphi(z))|^{2+\alpha+2n} |u(z)|^2 dA_\beta(z). \end{aligned} \tag{6}$$

Using (4), (5) and (6), for any $r\in(0,1)$ we get

$$\begin{aligned}\|D^n_{\varphi,u}\|^2_{e,A^2_\alpha\to A^2_\beta} &\leqslant \liminf_{j\to\infty} \sup_{\|f\|_{A^2_\alpha}\leqslant 1} \int_{|w|>r} |(R_j f)^{(n)}(w)|^2 d\mu\circ\varphi^{-1} \\ &\lesssim \liminf_{j\to\infty} \sup_{\|f\|_{A^2_\alpha}\leqslant 1} \|R_j f\|^2_{A^2_\alpha} \sup_{|b|\geqslant r} \int_{\mathbb{D}} |\sigma_b'(\varphi(z))|^{2+\alpha+2n} |u(z)|^2 dA_\beta(z) \\ &\leqslant \sup_{|b|\geqslant r} \int_{\mathbb{D}} |\sigma_b'(\varphi(z))|^{2+\alpha+2n} |u(z)|^2 dA_\beta(z). \end{aligned}$$

Taking the limit as $r\to 1$, we get the desired result. The proof is complete. □

From Theorem 2.2, we immediately get the following result.

THEOREM 2.3. *Let φ be an analytic self-map of $\mathbb{D}$, $-1<\alpha,\beta<\infty, n\in\mathbb{Z}$ and $u\in H(\mathbb{D})$. Suppose that $D^n_{\varphi,u}: A^2_\alpha\to A^2_\beta$ is bounded. Then $D^n_{\varphi,u}: A^2_\alpha\to A^2_\beta$ is compact if and only if*

$$\limsup_{|b|\to 1} \int_{\mathbb{D}} |\sigma_b'(\varphi(z))|^{2+\alpha+2n} |u(z)|^2 dA_\beta(z) = 0.$$

3. Hilbert-Schmidt operator $D^n_{\varphi,u}: A^2_\alpha\to A^2_\beta$

When $\alpha=\beta=0$, Čučković and Zhao [3] proved that $uC_\varphi: A^2\to A^2$ is a Hilbert-Schmidt operator if and only if

$$\int_{\mathbb{D}} \frac{|u(z)|^2}{(1-|\varphi(z)|^2)^2} dA(z) < \infty.$$

In this section, we generalize the above result and study the Hilbert-Schmidt operator $D^n_{\varphi,u}: A^2_\alpha\to A^2_\beta$. For the case $u=1$ see also [2]. The following result was essentially proved in [24], but since there are some minor differences and for the completeness we present a proof of it.

THEOREM 3.1. *Let φ be an analytic self-map of $\mathbb{D}$, $-1<\alpha,\beta<\infty, n\in\mathbb{Z}$ and $u\in H(\mathbb{D})$. Assume that $D^n_{\varphi,u}:A^2_\alpha\to A^2_\beta$ is bounded. Then $D^n_{\varphi,u}:A^2_\alpha\to A^2_\beta$ is a Hilbert-Schmidt operator if and only if*

$$\int_{\mathbb{D}}\frac{|u(z)|^2}{(1-|\varphi(z)|^2)^{\alpha+2+2n}}dA_\beta(z)<\infty.$$

Proof. Let $e^\alpha_m(z)=\sqrt{\frac{\Gamma(m+\alpha+2)}{m!\Gamma(\alpha+2)}}z^m$. Then $\{e^\alpha_m\}_{m=0}^\infty$ is an orthonormal basis for A^2_α. We have

$$\begin{aligned}
&D^n_{\varphi,u}:A^2_\alpha\to A^2_\beta \quad \text{is Hilbert-Schmidt}\\
\Leftrightarrow&\sum_{m=0}^\infty\|D^n_{\varphi,u}(e^\alpha_m)\|^2_{A^2_\beta}<\infty\\
\Leftrightarrow&\sum_{m=0}^\infty\int_{\mathbb{D}}|u(z)|^2|(e^\alpha_m)^{(n)}(\varphi(z))|^2dA_\beta(z)<\infty\\
\Leftrightarrow&\int_{\mathbb{D}}|u(z)|^2\sum_{m=n}^\infty\frac{\Gamma(m+\alpha+2)}{m!\Gamma(\alpha+2)}\Big(\prod_{j=o}^{n-1}(m-j)\Big)^2|\varphi(z)|^{2m-2n}dA_\beta(z)<\infty\\
\Leftrightarrow&\int_{\mathbb{D}}\frac{|u(z)|^2}{(1-|\varphi(z)|^2)^{\alpha+2+2n}}dA_\beta(z)<\infty. \quad \square
\end{aligned}$$

From the last theorem, we easily get the following result.

COROLLARY 3.1. *Let φ be an analytic self-map of $\mathbb{D}$ such that $\|\varphi\|_\infty<1$, $-1<\alpha,\beta<\infty$ and $n\in\mathbb{Z}$. Then for any $u\in A^2_\beta$, $D^n_{\varphi,u}:A^2_\alpha\to A^2_\beta$ is a Hilbert-Schmidt operator.*

THEOREM 3.2. *Let φ be an analytic self-map of $\mathbb{D}$ and $u\in H(\mathbb{D})$. Let $-1<\alpha,\beta<\infty$, $n\in\mathbb{Z}$ such that $\beta>2n-1+\alpha$. If*

$$\int_{\mathbb{D}}|u(z)|^2(1-|z|^2)^{\beta-\alpha-2-2n}dA(z)<\infty,$$

then $D^n_{\varphi,u}:A^2_\alpha\to A^2_\beta$ is a Hilbert-Schmidt operator.

Proof. From page 41 of [2], we have

$$\frac{1-|z|^2}{1-|\varphi(z)|^2}\leqslant 2\frac{1+|\varphi(0)|}{1-|\varphi(0)|},$$

which implies that

$$\begin{aligned}
&\int_{\mathbb{D}}\frac{|u(z)|^2(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+2+2n}}dA(z)\\
\leqslant\ &2^{\alpha+2+2n}\int_{\mathbb{D}}\frac{|u(z)|^2(1-|z|^2)^\beta}{(1-|z|^2)^{\alpha+2+2n}}\Big(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\Big)^{\alpha+2+2n}dA(z)\\
\lesssim\ &\int_{\mathbb{D}}|u(z)|^2(1-|z|^2)^{\beta-\alpha-2-2n}dA(z)<\infty.
\end{aligned}$$

Here we use the fact that $\frac{1+|\varphi(0)|}{1-|\varphi(0)|}$ is a constant. By Theorem 3.1, we see that $D^n_{\varphi,u}: A^2_\alpha \to A^2_\beta$ is a Hilbert-Schmidt operator. $\square$

The above theorem gives a sufficient condition for $D^n_{\varphi,u}: A^2_\alpha \to A^2_\beta$ to be a Hilbert-Schmidt operator for any φ. However, when φ is an automorphism of $\mathbb{D}$, we prove that this condition is also a necessary condition.

THEOREM 3.3. *Let $u \in H(\mathbb{D})$, $-1<\alpha,\beta<\infty$ and n be a nonnegative integer such that $\beta > 2n-1+\alpha$. Assume that φ is an automorphism of $\mathbb{D}$. Then $D^n_{\varphi,u}: A^2_\alpha \to A^2_\beta$ is a Hilbert-Schmidt operator if and only if*

$$\int_{\mathbb{D}} |u(z)|^2(1-|z|^2)^{\beta-\alpha-2-2n}dA(z) < \infty.$$

Proof. We only need to prove the necessary part. Suppose that $D^n_{\varphi,u}: A^2_\alpha \to A^2_\beta$ is a Hilbert-Schmidt operator. For $a \in \mathbb{D}$, let $\varphi(z) = \lambda \frac{a-z}{1-\overline{a}z}$ where $|\lambda| = 1$. After some calculation, we have

$$(1-|z|^2) \geqslant \frac{1-|a|}{1+|a|}(1-|\varphi(z)|^2).$$

Hence by Theorem 3.1 and the fact that $\frac{1+|a|}{1-|a|}$ is a constant, we get

$$\int_{\mathbb{D}} |u(z)|^2(1-|z|^2)^{\beta-\alpha-2-2n}dA(z) \lesssim \left(\frac{1+|a|}{1-|a|}\right)^{\alpha+2+2n} \int_{\mathbb{D}} \frac{|u(z)|^2(1-|z|^2)^{\beta}}{(1-|\varphi(z)|^2)^{\alpha+2+2n}}dA(z)$$
$$< \infty. \quad \square$$

4. Order boundedness of $D^n_{\varphi,u}: A^2_\alpha \to A^2_\beta$

In this section, we investigate the order boundedness of $D^n_{\varphi,u}: A^2_\alpha \to A^2_\beta$.

THEOREM 4.1. *Let φ be an analytic self-map of $\mathbb{D}$, $-1<\alpha,\beta<\infty$, $n \in \mathbb{Z}$ and $u \in H(\mathbb{D})$. The operator $D^n_{\varphi,u}: A^2_\alpha \to A^2_\beta$ is order bounded if and only if*

$$\int_{\mathbb{D}} \frac{|u(z)|^2}{(1-|\varphi(z)|^2)^{2+\alpha+2n}}dA_\beta(z) < \infty. \tag{7}$$

Proof. First we assume that $D^n_{\varphi,u}: A^2_\alpha \to A^2_\beta$ is order bounded. Then, for any $f \in A^2_\alpha$ with $\|f\|_{A^2_\alpha} \leqslant 1$, there exists a nonnegative function $g \in L^2(\mathbb{D}, dA_\beta)$ such that

$$|D^n_{\varphi,u}f(z)| \leqslant g(z)$$

for almost every $z \in \mathbb{D}$. For any $z \in \mathbb{D}$, set

$$h_z(a) = \left(\frac{1-|\varphi(z)|^2}{(1-a\overline{\varphi(z)})^2}\right)^{\frac{\alpha+2}{2}}, \qquad a \in \mathbb{D}.$$

A simple computation shows that $h_z \in A^2_\alpha$ with $\|h_z\|_{A^2_\alpha} \leqslant 1$. So

$$\frac{|\varphi(z)|^n |u(z)|}{(1-|\varphi(z)|^2)^{\frac{(\alpha+2)}{2}+n}} \lesssim |D^n_{\varphi,u} h_z(z)| \leqslant g(z).$$

Since $g \in L^2(\mathbb{D}, dA_\beta)$, the above inequality implies

$$\int_{|\varphi(z)|>1/2} \frac{|u(z)|^2}{(1-|\varphi(z)|^2)^{\alpha+2+2n}} dA_\beta(z) \lesssim \int_{\mathbb{D}} |g(z)|^2 dA_\beta(z) < \infty. \tag{8}$$

On the other hand, set

$$k(z) = \frac{z^n}{\|z^n\|_{A^2_\alpha}}, \qquad z \in \mathbb{D}.$$

Here

$$\|z^n\|_{A^2_\alpha} = \sqrt{\frac{(\alpha+1)\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma(n+2+\alpha)}}.$$

It is clear that $k \in A^2_\alpha$ with $\|k\|^2_{A^2_\alpha} = 1$. So,

$$|u(z)| \lesssim |D^n_{\varphi,u} k(z)| \leqslant g(z), \;\; z \in \mathbb{D}.$$

Since $g \in L^2(\mathbb{D}, dA_\beta)$, the above inequality implies $u \in A^2_\beta$. Hence

$$\int_{|\varphi(z)|\leqslant 1/2} \frac{|u(z)|^2}{(1-|\varphi(z)|^2)^{\alpha+2+2n}} dA_\beta(z) \lesssim \int_{|\varphi(z)|\leqslant 1/2} |u(z)|^2 dA_\beta(z) < \infty. \tag{9}$$

From (8) and (9), we get

$$\int_{\mathbb{D}} \frac{|u(z)|^2}{(1-|\varphi(z)|^2)^{\alpha+2+2n}} dA_\beta(z) < \infty.$$

Conversely, assume that (7) holds. By a classical estimate (see, e.g., a general point-value estimation in Lemma 5 of [17]), for any $f \in A^2_\alpha$, we have

$$|D^n_{\varphi,u} f(z)| = |u(z)| \cdot |f^{(n)}(\varphi(z))| \leqslant c_{n,\alpha} \frac{|u(z)|}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{2}+n}} \|f\|_{A^2_\alpha}, \; z \in \mathbb{D}, \tag{10}$$

and so

$$\|D^n_{\varphi,u} f\|^2_{A^2_\beta} \leqslant c_{n,\alpha} \int_{\mathbb{D}} \frac{|u(z)|^2}{(1-|\varphi(z)|^2)^{2+\alpha+2n}} dA_\beta(z) \cdot \|f\|^2_{A^2_\alpha} < \infty.$$

Here $c_{n,\alpha}$ is a constant depending only on n and α. Therefore $D^n_{\varphi,u} : A^2_\alpha \to A^2_\beta$ is bounded.

Now take a function $f \in A^2_\alpha$ with $\|f\|_{A^2_\alpha} \leqslant 1$. From (10),

$$|D^n_{\varphi,u} f(z)| \leqslant \frac{c_{n,\alpha}|u(z)|}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{2}+n}},$$

for any $z \in \mathbb{D}$. Set

$$g = c_{n,\alpha}|u|(1-|\varphi|^2)^{-\frac{\alpha+2}{2}-n}.$$

Then the assumed condition implies $g \in L^2(\mathbb{D}, dA_\beta)$ and $g \geqslant 0$. Moreover, $|D^n_{\varphi,u} f| \leqslant g$. That is, $D^n_{\varphi,u} : A^2_\alpha \to A^2_\beta$ is order bounded. This completes the proof. □

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