Mathematical
Inequalities
& Applications
Volume 22, Number 4 (2019), 1081–1089 doi:10.7153/mia-2019-22-74

CONTINUITY PROPERTIES OF K–MIDCONVEX AND K–MIDCONCAVE SET–VALUED MAPS

ELIZA JABŁOŃSKA AND KAZIMIERZ NIKODEM*

Dedicated to Professor Josip Pečarić on the occasion of his 70th birthday

(*Communicated by C. P. Niculescu*)

Abstract. A recent result on the continuity of midconvex functionals upper bounded on a not null-finite set (see [2]) is extended to K-midconvex and K-midconcave set-valued maps.

1. Introduction and preliminaries

Let X and Y be topological vector spaces (real and Hausdorff in the whole paper). Assume that D is a convex subset of X and K is a convex cone in Y (i.e. $K+K\subset K$ and $tK\subset K$ for all $t\geqslant 0$). Denote by $n(Y)$, $\mathscr{B}(Y)$, $\mathscr{BC}(Y)$ and $\mathscr{CC}(Y)$ the families of all nonempty, nonempty bounded, nonempty bounded convex and nonempty compact convex subsets of Y, respectively.

A set-valued map $F: D\to n(Y)$ is called K-*convex*, if

$$tF(x_1)+(1-t)F(x_2)\subset F(tx_1+(1-t)x_2)+K \tag{1}$$

for all $x_1,x_2\in D$ and $t\in[0,1]$. If F satisfies

$$F(tx_1+(1-t)x_2)\subset tF(x_1)+(1-t)F(x_2)+K \tag{2}$$

for all $x_1,x_2\in D$ and $t\in[0,1]$, then it is called K-*concave*.

A set-valued map $F: D\to n(Y)$ is called K-*midconvex* (K-*midconcave*, resp.), if (1) ((2)) is assumed only for $t=\frac{1}{2}$.

Clearly, if F is K-convex with $K=\{0\}$ then it is convex, which means that its graph is a convex subset of $X\times Y$. If F is single-valued and Y is endowed with the relation $\leqslant_K$ of partial order defined by $x\leqslant_K y \Leftrightarrow y-x\in K$, then conditions (1) and (2) reduce to the following conditions:

$$F(tx_1+(1-t)x_2)\leqslant_K tF(x_1)+(1-t)F(x_2)$$

Mathematics subject classification (2010): 26B25, 39B62, 54C60.
Keywords and phrases: K-midconvex set-valued map, K-midconcave set-valued map, K-continuity, K-hemicontinuity, null-finite set, Haar-null set, Haar-meager set.

* Corresponding author.

Paper MIA-22-74

and

$$tF(x_1)+(1-t)F(x_2) \leqslant_K F(tx_1+(1-t)x_2),$$

respectively. In particular, if $Y=\mathbb{R}$ and $K=[0,\infty)$, we obtain the standard definitions of convex and concave functions.

A set-valued map $F:D\longrightarrow n(Y)$ is said to be *K-continuous* at a point $x_0\in D$ if for every neighbourhood W of zero in Y there exists a neighbourhood U of zero in X such that

$$F(x_0)\subset F(x)+W+K \tag{3}$$

and

$$F(x)\subset F(x_0)+W+K \tag{4}$$

for every $x\in(x_0+U)\cap D$. If only condition (3) (condition (4)) is fulfilled, F is called *K-lower semicontinuous* (*K-upper semicontinuous*) at x_0.

Denote by K^* the set of all continuous linear functionals on Y which are nonnegative on K, i.e.

$$K^*=\{y^*\in Y^*: y^*(y)\geqslant 0 \text{ for every } y\in K\}.$$

We say that a set-valued map $F:D\to\mathscr{B}(Y)$ is *K-hemicontinuous* (*K-lower hemicontinuous, K-upper hemicontinuous*) at a point $x_0\in D$ if for every $y^*\in K^*$ the functional $f_{y^*}:D\to\mathbb{R}$ defined by

$$f_{y^*}(x)=\inf y^*(F(x)),\ \ x\in D \tag{5}$$

is continuous (lower semicontinuous, upper semicontinuous) at x_0.

We say that a set-valued map $F:D\to n(Y)$ is *partially K-upper bounded* on a set $A\subset D$ if there exists a bounded set $B\subset Y$ such that $F(x)\cap\mathrm{cl}(B-K)\neq\emptyset$ for all $x\in A$. F is *K-lower bounded* on a set A if there exists a bounded set $B\subset Y$ such that $F(x)\subset\mathrm{cl}(B+K)$ for all $x\in A$.

In [2] a new concept of a null-finite set has been introduced. Let us recall, a subset A of a metric vector space X is called *null-finite* if there exists a sequence $(x_n)_{n\in\mathbb{N}}$ tending to zero in X such that the set $\{n\in\mathbb{N}: x+x_n\in A\}$ is finite for every $x\in X$.

The following crucial property of null-finite sets has been proved in [2].

THEOREM 1. [2, Theorems 5.1 and 6.1] *In a complete abelian metric group each Borel null-finite set is Haar-meager as well as each universally measurable null-finite set is Haar-null.*

Let us recall that a subset B of an abelian Polish group X is called:

- *Haar-meager* if there exist a Borel set $A\supset B$, a compact metric space K and a continuous function $f:K\to X$ such that $f^{-1}(A+x)$ is meager in K for each $x\in X$ (see [5]);

- *Haar-null* if there exists a universally measurable set $A\supset B$ and a σ-additive probability Borel measure μ on X such that $\mu(A+x)=0$ for each $x\in X$ (see [4]).

It has been proved in [4] and [5] that in each locally compact abelian Polish group the notions of a Haar-meager set and a Haar-null set are equivalent to the notions of a meager set and a set of Haar measure zero, respectively.

In 1983 K. Baron and R. Ger [3, p. 239] asked the following question:

Does the upper boundedness of an additive or midpoint convex function on some universally measurable set which is not Haar-null imply the continuity of the function?

This problem has been resolved in [2] thanks to Theorem 1 and the following important result.

THEOREM 2. [2, Theorem 11.1] *If a midpoint convex function $f: D \to \mathbb{R}$ defined on an open convex subset $D \subset X$ of a metric vector space X with an invariant metric is upper bounded on a set $B \subset D$ which is not null-finite in X and whose closure* $\mathrm{cl}\,B$ *is contained in D, then f is continuous.*

In the paper [6] we generalized Theorem 2 as below.

THEOREM 3. [6, Theorem 11] *Let Y be a metric vector space with an invariant metric. If a K-midconvex set-valued map $F: D \to \mathscr{B}(Y)$ defined on an open convex subset D of a metric vector space X with an invariant metric is partially K-upper bounded on a set $B \subset D$, which is not null-finite in X and satisfies* $\mathrm{cl}\,B \subset D$*, then F is K-continuous on D.*

In this paper we show that the above Theorem 3 also holds in the case where X is a Baire topological vector space, Y is a locally convex topological vector space such that $\bigcup_{n\in\mathbb{N}}(B_n - K) = Y$ for some bounded sets $B_n \subset Y$, $n \in \mathbb{N}$, and $F: D \to \mathscr{C}\mathscr{C}(Y)$. But first we prove that if we weaken assumptions about Y in Theorem 3, then we get just K-hemicontinuity of F.

2. Some connections between K-continuity and K-hemicontinuity

Assume that X and Y are topological vector spaces, D is an open subset of X and K is a convex cone in Y. It is known (and easy to prove) that if a set-valued map $F: D \to B(Y)$ is K-continuous at a point $x_0 \in D$, then it is K-hemicontinuous at this point (see [9, Prop. 1]; cf. also [7, Prop. 2.1]), but the converse is not true in general (cf. [1, p. 62]). However, under some additional regularity assumptions, K-midconvex and K-hemicontinuous set-valued maps are K-continuous. Namely, the following result has been proved in [9].

THEOREM 4. [9, Theorem 1] *Let X be a Baire topological vector space and D be a convex open subset of X. Assume that Y is a locally convex topological vector space and K is a convex cone in Y. Assume also that there exist bounded sets $B_n \subset Y$, $n \in \mathbb{N}$, such that*

$$\bigcup_{n\in\mathbb{N}}(B_n - K) = Y. \tag{6}$$

If a set-valued map $F: D \to \mathscr{C}\mathscr{C}(Y)$ is K-midconvex and K-upper hemicontinuous on D, then F is K-continuous on D.

The next theorem shows that a similar result holds also for K-midconcave set-valued maps.

THEOREM 5. *Let X be a Baire topological vector space and D be a convex open subset of X. Assume that Y is a locally convex topological vector space and K is a convex cone in Y. Assume also that there exist open bounded sets $B_n \subset Y$, $n \in \mathbb{N}$, such that*

$$\bigcup_{n\in\mathbb{N}} (B_n + K) \;=\; Y. \tag{7}$$

If a set-valued map $F : D \to \mathscr{C}\mathscr{C}(Y)$ is K-midconcave and K-lower hemicontinuous on D, then F is K-continuous on D.

In the proof of this theorem we will use the following lemma (which is a slight improvement of the Bernstain-Doetsch-type theorem for K-midconcave set-valued maps given in [8, Theorem 4.4]). Recall that if a set-valued map $F : D \to n(Y)$ is K-midconcave and convex-valued, then

$$F(qx_1 + (1-q)x_2) \subset qF(x_1) + (1-q)F(x_2) + K \tag{8}$$

for all $x_1, x_2 \in D$ and all dyadic $q \in [0,1]$ (see [8, Lemma 4.1]).

LEMMA 6. *Let X and Y be topological vector spaces. Assume that D is an open convex subset of X and K is a convex cone in Y. If a set-valued map $F : D \to \mathscr{B}\mathscr{C}(Y)$ is K-midconcave and K-lower bounded on a subset of D with a nonempty interior, then F is K-continuous on D.*

Proof. Let F be K-lower bounded on a set $x_0 + U \subset D$, where $x_0 \in D$ and U is a neighbourhood of zero in X. Then there is a bounded set $B \subset Y$ such that

$$F(x) \subset \operatorname{cl}(B + K), \;\; x \in x_0 + U. \tag{9}$$

We prove that F is K-upper semicontinuous at x_0. So, take an arbitrary neighbourhood W of zero in Y and next choose a balanced neighbourhood V of zero such that $V + V + V \subset W$. Since the sets B and $F(x_0)$ are bounded, there exists a dyadic number $q \in (0,1)$ such that

$$qB \subset V \quad \text{and} \quad qF(x_0) \subset V.$$

Thus, by (9),

$$qF(x) \subset q\operatorname{cl}(B+K) = \operatorname{cl}(qB + qK) \subset \operatorname{cl}(V+K) \subset V + K + V, \quad x \in x_0 + U. \tag{10}$$

Now, fix $u \in U$. Then, using (8) and (10), we obtain

$$\begin{aligned} F(x_0 + qu) &= F((1-q)x_0 + q(u+x_0)) \subset (1-q)F(x_0) + qF(u+x_0) + K \\ &\subset F(x_0) - qF(x_0) + V + V + K \subset F(x_0) - V + V + V + K \\ &\subset F(x_0) + W + K, \end{aligned}$$

which means that F is K-upper semicontinuous at x_0 and, consequently, K-continuous on D (see [8, Theorem 4.5]). □

Proof of Theorem 5. Let B_n, $n \in \mathbb{N}$, be open bounded sets satisfying (7). Define $\widetilde{B}_n = \operatorname{conv}(B_1 \cup \ldots \cup B_n)$, $n \in \mathbb{N}$. Since the convex hull of an open set is open and, in locally convex spaces, the convex hull of a bounded set is bounded, the sets $\widetilde{B}_n$ are open and convex. Moreover, $\widetilde{B}_n \subset \widetilde{B}_{n+1}$, $n \in \mathbb{N}$. Define

$$A_n = \{x \in D : F(x) \subset \operatorname{cl}(\widetilde{B}_n + K)\}, \ n \in \mathbb{N}. \tag{11}$$

Then $\bigcup_{n\in\mathbb{N}} A_n = D$. Indeed, for every fixed $x \in D$ the sets $\widetilde{B}_n + K$, $n \in \mathbb{N}$, form an open covering of $F(x)$. Since $F(x)$ is compact, there exists a finite subcovering of it:

$$F(x) \subset \left(\widetilde{B}_{n_1} + K\right) \cup \ldots \cup \left(\widetilde{B}_{n_p} + K\right) = \widetilde{B}_{n_p} + K,$$

and hence $x \in A_{n_p}$. By the definition of A_n, the set-valued map F is K-lower bounded on every set A_n. We will show that F is also K-lower bounded on the sets $\operatorname{cl} A_n$. To this aim fix an $n \in \mathbb{N}$ and take an $x_0 \in \operatorname{cl} A_n$. By (11) $F(x) \subset \operatorname{cl}(\widetilde{B}_n + K)$ for every $x \in A_n$. We will show that also $F(x_0) \subset \operatorname{cl}(\widetilde{B}_n + K)$.

For the proof by contradiction suppose that there exists $z \in F(x_0) \setminus \operatorname{cl}(\widetilde{B}_n + K)$. Since the set $\operatorname{cl}(\widetilde{B}_n + K)$ is convex and closed, by the separation theorem (see e.g (see [10, Theorem 3.4])) there exists a continuous linear functional $y^* \in Y^*$ such that

$$y^*(z) < \inf y^*\left(\operatorname{cl}(\widetilde{B}_n + K)\right). \tag{12}$$

Note that $y^* \in K^*$. Indeed, in view of (12) we have

$$y^*(k) \geqslant y^*(z) - y^*(b_0) =: M,$$

for all $k \in K$ and arbitrarily fixed $b_0 \in \widetilde{B}_n$. Hence, by the homogeneity of y^*, we get

$$y^*(k) = \frac{1}{m} y^*(mk) \geqslant \frac{1}{m} M, \ m \in \mathbb{N},$$

which proves that $y^*(k) \geqslant 0$ for all $k \in K$. Now, put

$$\varepsilon := \inf y^*\left(\operatorname{cl}(\widetilde{B}_n + K)\right) - y^*(z).$$

By the K-lower hemicontinuity of F at x_0 there exists a neighbourhood $U_{x_0} \subset D$ such that

$$f_{y^*}(x) < f_{y^*}(x_0) + \varepsilon, \ x \in U_{x_0}. \tag{13}$$

Since $x_0 \in \operatorname{cl} A_n$, there exists an $x_1 \in A_n \cap U_{x_0}$. Then, by (13) and the definition of ε, we obtain

$$\begin{aligned} f_{y^*}(x_1) &< f_{y^*}(x_0) + \varepsilon \leqslant y^*(z) + \varepsilon = \inf y^*\left(\operatorname{cl}(\widetilde{B}_n + K)\right) \\ &\leqslant \inf y^*\left(F(x_1)\right) = f_{y^*}(x_1). \end{aligned}$$

This contradiction proves that $F(x) \subset \operatorname{cl}(\widetilde{B}_n + K)$ for every $x \in \operatorname{cl} A_n$.

Hence, F is K-lower bounded on every $\operatorname{cl} A_n$, $n \in \mathbb{N}$. Since Y is a Baire space and $D \subset \bigcup_{n\in\mathbb{N}} \operatorname{cl} A_n$, there exists $n_0 \in \mathbb{N}$ such that $\operatorname{int} \operatorname{cl} A_{n_0} \neq \emptyset$. Thus F is K-lower bounded on a set with nonempty interior and consequently, by Lemma 6, F is K-continuous on D. This finishes the proof. □

3. K-continuity as a consequence of boundedness on not null-finite sets

In this section first we will show that K-midconvex set-valued maps partially K-upper bounded on a not null-finite set, as well as K-midconcave set-valued maps K-lower bounded on a not null-finite set, are K-hemicontinuous. Next we will use these results and Theorems 4, 5 to prove that under some additional assumptions K-midconvex set-valued maps partially K-upper bounded on a not null-finite set, as well as K-midconcave set-valued maps K-lower bounded on a not null-finite set, are K-continuous.

THEOREM 7. *Let X be a metric vector space with an invariant metric, D be an open convex subset of X and $A \subset D$ be a set which is not null-finite and* $\mathrm{cl}A \subset D$. *Assume that Y is a topological vector space and K is a convex cone in Y. If a set-valued map $F : D \to \mathscr{B}(Y)$ is K-midconvex and partially K-upper bounded on A, then F is K-hemicontinuous on D.*

Proof. Assume that F is K-midconvex and partially K-upper bounded on A. Then there exists a bounded set $B \subset Y$ such that

$$F(x) \cap \mathrm{cl}(B-K) \neq \emptyset, \quad x \in A. \tag{14}$$

Fix any $y^* \in K^*$ and take the functional f_{y^*} defined by (12). Since F is K-midconvex and $y^* \in K^*$, we have for all $x_1, x_2 \in D$

$$\begin{aligned}\frac{y^*(F(x_1))+y^*(F(x_2))}{2} &= y^*\left(\frac{F(x_1)+F(x_2)}{2}\right)\\ &\subset y^*\left(F\left(\frac{x_1+x_2}{2}\right)+K\right)\\ &\subset y^*\left(F\left(\frac{x_1+x_2}{2}\right)\right)+[0,\infty).\end{aligned}$$

Hence

$$\begin{aligned}\frac{f_{y^*}(x_1)+f_{y^*}(x_2)}{2} &= \frac{\inf y^*(F(x_1))+\inf y^*(F(x_2))}{2}\\ &= \inf\left(\frac{y^*(F(x_1))+y^*(F(x_2))}{2}\right)\\ &\geqslant \inf y^*\left(F\left(\frac{x_1+x_2}{2}\right)\right)\\ &= f_{y^*}\left(\frac{x_1+x_2}{2}\right),\end{aligned}$$

which means that f_{y^*} is midconvex. By (14), for every $x \in A$ we have

$$y^*(F(x)) \cap y^*\big(\mathrm{cl}(B-K)\big) \neq \emptyset.$$

Since y^* is continuous, $y^*\big(\mathrm{cl}(B-K)\big) \subset \mathrm{cl}\big(y^*(B-K)\big)$. Therefore

$$y^*(F(x)) \cap \mathrm{cl}\big(y^*(B-K)\big) \neq \emptyset$$

and hence

$$y^*(F(x)) \cap \mathrm{cl}\left(y^*(B) + (-\infty, 0]\right) \neq \emptyset. \tag{15}$$

Since continuous linear functionals map bounded sets into bounded sets (see [10, Theorem 1.32]), the set $y^*(B)$ is bounded. Assume that $y^*(B) \subset [m, M]$. Then, by (15),

$$y^*(F(x)) \cap (-\infty, M] \neq \emptyset,$$

and hence

$$f_{y^*}(x) \leqslant M, \quad x \in A.$$

Consequently, in view of Theorem 2, f_{y^*} is continuous on D. This means that F is K-hemicontinuous on D. $\square$

THEOREM 8. *Let X be a metric vector space with an invariant metric, D be an open convex subset of X and $A \subset D$ be a set which is not null-finite and $\mathrm{cl} A \subset D$. Assume that Y is a topological vector space and K is a convex cone in Y. If a set-valued map $F : D \to \mathscr{B}(Y)$ is K-midconcave and K-lower bounded on A, then F is K-hemicontinuous on D.*

Proof. Since F is K-lower bounded on A, there exists a bounded set $B \subset Y$ such that

$$F(x) \subset \mathrm{cl}(B + K), \quad x \in A. \tag{16}$$

Fix any $y^* \in K^*$ and take the functional f_{y^*} defined by (12). Since F is K-midconcave and $y^* \in K^*$,

$$\begin{aligned} y^*\left(F\left(\frac{x_1+x_2}{2}\right)\right) &\subset y^*\left(\frac{F(x_1)+F(x_2)}{2} + K\right) \\ &\subset y^*\left(\frac{F(x_1)+F(x_2)}{2}\right) + [0, \infty) \\ &= \frac{y^*(F(x_1)) + y^*(F(x_2))}{2} + [0, \infty) \end{aligned}$$

for $x_1, x_2 \in D$. Hence

$$\begin{aligned} f_{y^*}\left(\frac{x_1+x_2}{2}\right) &= \inf y^*\left(F\left(\frac{x_1+x_2}{2}\right)\right) \\ &\geqslant \inf \frac{y^*(F(x_1)) + y^*(F(x_2))}{2} \\ &= \frac{f_{y^*}(x_1) + f_{y^*}(x_2)}{2} \end{aligned}$$

which means that f_{y^*} is midconcave. By (16) and the continuity of y^*,

$$y^*(F(x)) \subset y^*\left(\mathrm{cl}(B+K)\right) \subset \mathrm{cl}\left(y^*(B) + [0, \infty)\right) \text{ for each } x \in A.$$

Clearly the set $y^*(B)$ is bounded, so $y^*(B) \subset [m, M]$ and then

$$f_{y^*}(x) \geqslant m, \quad x \in A.$$

Hence, since $-f_{y^*}$ is midconvex, according to Theorem 2, f_{y^*} is continuous on D. It means that F is K-hemicontinuous on D. $\square$

Now, using Theorems 4 and 7 we obtain the following result (cf. [6, Theorem 11]).

COROLLARY 9. *Let X be a Baire metric vector space with an invariant metric, D be an open convex subset of X and $A \subset D$ be a set which is not null-finite and* $\operatorname{cl} A \subset D$. *Assume that Y is a locally convex topological vector space and K is a convex cone in Y. Assume also that there exist bounded sets $B_n \subset Y$, $n \in \mathbb{N}$, satisfying* (6). *If a set-valued map $F : D \to \mathscr{C}\mathscr{C}(Y)$ is K-midconvex and partially K-upper bounded on A, then F is K-continuous on D.*

Analogously, by Theorems 5 and 8 we get the following result.

COROLLARY 10. *Let X be a Baire metric vector space with an invariant metric, D be an open convex subset of X and $A \subset D$ be a set which is not null-finite and* $\operatorname{cl} A \subset D$. *Assume that Y is a locally convex topological vector space and K is a convex cone in Y. Assume also that there exist open bounded sets $B_n \subset Y$, $n \in \mathbb{N}$, satisfying* (7). *If a set-valued map $F : D \to \mathscr{C}\mathscr{C}(Y)$ is K-midconcave and K-lower bounded on A, then F is K-continuous on D.*

Finally, we use Theorem 1 and the above Corollaries 9 and 10 to answer Baron's and Ger's question in the case of set-valued functions.

COROLLARY 11. *Let X be a complete metric vector space with an invariant metric, D be an open convex subset of X and $A \subset D$ be a universally measurable set which is not Haar-null or a Borel set which is not Haar-meager. Let Y be a locally convex topological vector space and K be a convex cone in Y. Assume also that there exist bounded sets $B_n \subset Y$, $n \in \mathbb{N}$, satisfying* (6). *If a set-valued map $F : D \to \mathscr{C}\mathscr{C}(Y)$ is K-midconvex and partially K-upper bounded on A, then F is K-continuous on D.*

COROLLARY 12. *Let X be a complete metric vector space with an invariant metric, D be an open convex subset of X and $A \subset D$ be a universally measurable set which is not Haar-null or a Borel set which is not Haar-meager. Let Y be a locally convex topological vector space and K be a convex cone in Y. Assume also that there exist open bounded sets $B_n \subset Y$, $n \in \mathbb{N}$, satisfying* (7). *If a set-valued map $F : D \to \mathscr{C}\mathscr{C}(Y)$ is K-midconcave and K-lower bounded on A, then F is K-continuous on D.*

REFERENCES

[1] J. P. AUBIN, A. CELLINA, *Differential Inclusions*, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1984.

[2] T. BANAKH, E. JABŁOŃSKA, *Null-finite sets in metric groups and their applications*, Israel J. Math. (in press); `arXiv:1706.08155v2 [math.GN]` 27 Jun 2017.

[3] K. BARON, R. GER, *Problem (P239)*, in: The 21st International Symposium on Functional Equations, August 6–13, 1983, Konolfingen, Switzerland, Aequationes Math. **26** (1984), 225–294.

[4] J. P. R. CHRISTENSEN, *On sets of Haar measure zero in abelian Polish groups*, Israel J. Math. **13** (1972), 255–260.

[5] U. B. DARJI, *On Haar meager sets*, Topology Appl. **160** (2013), 2396–2400.
[6] E. JABŁOŃSKA, K. NIKODEM, *K-midconvex and K-midconcave set-valued maps bounded on "large" sets*, J. Convex Anal. (in press).
[7] D. T. LUC, *Continuity properties of cone-convex functions*, Acta Math. Hung. **55** (1990) 57–61.
[8] K. NIKODEM, *K-convex and K-concave set-valued functions*, Zeszyty Nauk. Politechniki Łódzkiej Mat. **559**; Rozprawy Mat. 114, Łódź 1989.
[9] K. NIKODEM, *Continuity properties of midconvex set-valued maps*, Aequationes Math. **62** (2001), 175–183.
[10] W. RUDIN, *Functional Analysis*, McGraw-Hill, Inc., 1973.

(Received July 9, 2018)

Eliza Jabłońska
Institute of Mathematics
Pedagogical University of Cracow
Podchorążych 2, 30–084 Kraków, Poland
e-mail: eliza.jablonska@up.krakow.pl

Kazimierz Nikodem
Department of Mathematics
University of Bielsko-Biala
ul. Willowa 2, 43–309 Bielsko-Biała, Poland
e-mail: knikodem@ath.bielsko.pl