FOURIER SERIES METHOD RELATED TO
WILKER – CUSA – HUYGENS INEQUALITIES

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(*Communicated by J. Pečarić*)

Abstract. In this paper we present a new approach of Wilker - Cusa - Huygens inequalities using the Fourier series method for even functions. This approach provides new proofs and refinements of these inequalities.

1. Introduction

The starting point of our work is the following remarkable inequalities.

The Cusa - Huygens inequality asserts that for $x \in \left(0, \frac{\pi}{2}\right)$,

$$\frac{\sin x}{x} < \frac{2 + \cos x}{3}.$$

The famous Huygens inequality for the sine and tangent functions states that for $x \in \left(0, \frac{\pi}{2}\right)$,

$$2\frac{\sin x}{x} + \frac{\tan x}{x} > 3.$$

The Wilker inequality asserts that for $x \in \left(0, \frac{\pi}{2}\right)$,

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2.$$

Another inequality which is of interest to us is the following:

$$\frac{\sin x}{x} + \frac{\tan x}{x} > 2, \text{ for every } x \in \left(0, \frac{\pi}{2}\right).$$

These inequalities were extended in different forms in the recent past. We refer to [1] - [11] and closely related references therein. Some of the recent improvements were

obtained using Taylor's expansion or Padé approximation of some trigonometric functions. These methods provide polynomial or rational approximations for the functions involved.

The aim of our work is to refine these classical inequalities. The main idea is that the functions involved in the above inequalities are even, so they can be expanded in Fourier series, e.g.,

$$\frac{\sin x}{x} - \frac{2 + \cos x}{3} = a_1 + b_1 \cos x + c_1 \cos 2x + ...,$$

$$2\frac{\sin x}{x} + \frac{\tan x}{x} - 3 = a_2 + b_2 \cos x + c_2 \cos 2x + ...,$$

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2 = a_3 + b_3 \cos x + c_3 \cos 2x + ...,$$

$$\frac{\sin x}{x} + \frac{\tan x}{x} - 2 = a_4 + b_4 \cos x + c_4 \cos 2x + ...$$

In the following we will present our algorithm for the first function.
We define the function $F_1(x)$ by

$$F_1(x) = a_1 + b_1 \cos x + c_1 \cos 2x.$$

The power series expansion of $\dfrac{\sin x}{x} - \dfrac{2 + \cos x}{3} - F_1(x)$ near 0 is

$$(-a_1 - b_1 - c_1) + x^2 \left(\frac{b_1}{2} + 2c_1\right) + x^4 \left(-\frac{b_1}{24} - \frac{2c_1}{3} - \frac{1}{180}\right) + O\left(x^6\right).$$

In order to increase the speed of the function $F_1(x)$ approximating $\dfrac{\sin x}{x} - \dfrac{2 + \cos x}{3}$ we vanish the first coefficients as follows:

$$\begin{cases} -a_1 - b_1 - c_1 = 0 \\ \dfrac{b_1}{2} + 2c_1 = 0 \\ -\dfrac{b_1}{24} - \dfrac{2c_1}{3} - \dfrac{1}{180} = 0 \end{cases}$$

and we obtain

$$a_1 = -\frac{1}{30}, \ b_1 = \frac{2}{45}, \ c_1 = -\frac{1}{90}.$$

Therefore we have

$$\frac{\sin x}{x} - \frac{2 + \cos x}{3} - \frac{1}{30} + \frac{2}{45} \cos x + \frac{1}{90} \cos 2x = -\frac{x^6}{1512} + \frac{29x^8}{453600} + O\left(x^{10}\right),$$

or, equivalently,

$$\frac{\sin x}{x} - \frac{2 + \cos x}{3} + \frac{1}{45}(1 - \cos x)^2 = -\frac{x^6}{1512} + \frac{29x^8}{453600} + O\left(x^{10}\right).$$

Using the same algorithm, we find

$$2\frac{\sin x}{x} + \frac{\tan x}{x} - 3 - \frac{3}{5}(1 - \cos x)^2 = \frac{11x^6}{140} + \frac{x^8}{50} + O\left(x^{10}\right),$$

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2 - \frac{32}{45}(1 - \cos x)^2 = \frac{76x^6}{945} + \frac{187x^8}{9450} + O\left(x^{10}\right),$$

$$\frac{\sin x}{x} + \frac{\tan x}{x} - 2 - \frac{1}{45}(1 - \cos x)(43 - 28\cos x) = \frac{599x^6}{7560} + \frac{9043x^8}{453600} + O\left(x^{10}\right).$$

2. The results

In order to attain our aim, we first prove a lemma.

LEMMA 1. *For every* $x \in \left(0, \frac{\pi}{2}\right)$, *one has*

$$2\sin\frac{x}{2} - x\cos\frac{x}{2}\cos x > 0.$$

Proof. The function $t : \left(0, \frac{\pi}{2}\right) \to \mathbb{R}$, $t(x) = 2\sin\frac{x}{2} - x\cos\frac{x}{2}\cos x$ has the derivative

$$t'(x) = \sin\frac{x}{2}\left(\sin x + x + \frac{3}{2}x\cos x\right).$$

Evidently, $t' > 0$ on $\left(0, \frac{\pi}{2}\right)$. Then t is strictly increasing on $\left(0, \frac{\pi}{2}\right)$. As $t(0) = 0$, we get $t > 0$ on $\left(0, \frac{\pi}{2}\right)$.

This completes the proof. $\square$

Using the Fourier series method we can establish our main theorems, which are refined variants of the above inequalities. We also remark that our results have simple forms.

THEOREM 1. *For all* $x \in \left(0, \frac{\pi}{2}\right)$, *one has*

$$\frac{\sin x}{x} < \frac{2 + \cos x}{3} - \frac{1}{45}(1 - \cos x)^2.$$

Proof. The inequality $\dfrac{\sin x}{x} < \dfrac{2 + \cos x}{3} - \dfrac{1}{45}(1 - \cos x)^2$ has the equivalent form

$$45\sin x < 29x + 17x\cos x - x\cos^2 x.$$

The function $f_1 : \left(0, \frac{\pi}{2}\right) \to \mathbb{R}$, $f_1(x) = 45\sin x - 29x - 17x\cos x + x\cos^2 x$ has the derivative

$$f_1'(x) = 28(\cos x - 1) - \sin^2 x + 17x\sin x - x\sin 2x$$
$$= 2\sin\frac{x}{2}\left(-28\sin\frac{x}{2} - 2\sin\frac{x}{2}\cos^2\frac{x}{2} + 17x\cos\frac{x}{2} - 2x\cos x\cos\frac{x}{2}\right).$$

We introduce the function $g_1 : \left(0, \dfrac{\pi}{2}\right) \to \mathbb{R}$, $g_1(x) = -28 \sin \dfrac{x}{2} - 2 \sin \dfrac{x}{2} \cos^2 \dfrac{x}{2}$
$+ 17x \cos \dfrac{x}{2} - 2x \cos x \cos \dfrac{x}{2}$.

Then

$$g_1'(x) = \sin \frac{x}{2} \left(7 \sin \frac{x}{2} \cos \frac{x}{2} - \frac{17x}{2} + 4x \cos^2 \frac{x}{2} + x \cos x \right)$$

$$= \sin \frac{x}{2} \left(\frac{7}{2} \sin x - \frac{17x}{2} + 4x \cdot \frac{1 + \cos x}{2} + x \cos x \right)$$

$$= \frac{1}{2} \sin \frac{x}{2} \left(7 (\sin x - x) + 6x (\cos x - 1) \right).$$

Evidently, $g_1' < 0$ on $\left(0, \dfrac{\pi}{2}\right)$. Then g is strictly decreasing on $\left(0, \dfrac{\pi}{2}\right)$. As $g_1(0) = 0$, we find $g_1 < 0$ on $\left(0, \dfrac{\pi}{2}\right)$. It follows that $f_1' < 0$ on $\left(0, \dfrac{\pi}{2}\right)$. Using the same arguments, we obtain that $f_1 < 0$ on $\left(0, \dfrac{\pi}{2}\right)$. $\quad\square$

THEOREM 2. *For all* $x \in \left(0, \dfrac{\pi}{2}\right)$, *one has*

$$2 \frac{\sin x}{x} + \frac{\tan x}{x} - 3 > \frac{3}{5} (1 - \cos x)^2.$$

Proof. The inequality $2 \dfrac{\sin x}{x} + \dfrac{\tan x}{x} - 3 > \dfrac{3}{5} (1 - \cos x)^2$ can be rearranged as follows:

$$2 (\sin x - x) + (\tan x - x) - \frac{3}{5} x (1 - \cos x)^2 > 0.$$

We define the function $f_2 : \left(0, \dfrac{\pi}{2}\right) \to \mathbb{R}$, $f_2(x) = 2 (\sin x - x) + (\tan x - x) - \dfrac{3}{5} x (1 - \cos x)^2$. Then

$$f_2'(x) = (\cos x - 1) \left(\frac{13}{5} - \frac{3}{5} \cos x + \frac{6}{5} x \sin x \right) + \tan^2 x$$

$$= 2 \sin^2 \frac{x}{2} \left(-\frac{13}{5} + \frac{3}{5} \cos x - \frac{6}{5} x \sin x + \frac{2 \cos^2 \frac{x}{2}}{\cos^2 x} \right)$$

$$= 2 \sin^2 \frac{x}{2} \cdot \frac{-13 \cos^2 x + 3 \cos^3 x - 6x \sin x \cos^2 x + 5 + 5 \cos x}{5 \cos^2 x}.$$

We consider the function $g_2 : \left(0, \dfrac{\pi}{2}\right) \to \mathbb{R}$, $g_2(x) = -13 \cos^2 x + 3 \cos^3 x - 6x \sin x \cos^2 x + 5 + 5 \cos x$.

Then

$$g_2(x) = (1 - \cos x) \left(-3 \cos^2 x + 10 \cos x + 5 \right) - 12x \sin \frac{x}{2} \cos \frac{x}{2} \cos^2 x$$

$$= 2 \sin \frac{x}{2} \left(-3 \sin \frac{x}{2} \cos^2 x + 10 \sin \frac{x}{2} \cos x + 5 \sin \frac{x}{2} - 6x \cos \frac{x}{2} \cos^2 x \right).$$

But $5\sin\dfrac{x}{2} > 5\sin\dfrac{x}{2}\cos x$ on $\left(0,\dfrac{\pi}{2}\right)$, hence we can write

$$g_2(x) > 6\sin\frac{x}{2}\cos x\left(-\sin\frac{x}{2}\cos x + 5\sin\frac{x}{2} - 2x\cos x\cos\frac{x}{2}\right),$$

or equivalently,

$$g_2(x) > 6\sin\frac{x}{2}\cos x\left(\sin\frac{x}{2}(1-\cos x) + 2\left(2\sin\frac{x}{2} - x\cos\frac{x}{2}\cos x\right)\right).$$

Using Lemma 1 we find that $g_2 > 0$ on $\left(0,\dfrac{\pi}{2}\right)$, therefore $f_2' > 0$ on $\left(0,\dfrac{\pi}{2}\right)$. As $f_2(0) = 0$, we get $f_2 > 0$ on $\left(0,\dfrac{\pi}{2}\right)$. $\square$

THEOREM 3. *For all $x \in \left(0,\dfrac{\pi}{2}\right)$, one has*

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2 > \frac{32}{45}(1-\cos x)^2.$$

Proof. Using the result of Theorem 2, we obtain

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > \left(\frac{\sin x}{x}\right)^2 + 3 - 2\frac{\sin x}{x} + \frac{3}{5}(1-\cos x)^2.$$

Then we have to prove that

$$\left(\frac{\sin x}{x}\right)^2 + 3 - \frac{2\sin x}{x} + \frac{3}{5}(1-\cos x)^2 > 2 + \frac{32}{45}(1-\cos x)^2,$$

or equivalently,

$$\left(\frac{\sin x}{x} - 1\right)^2 > \frac{1}{9}(1-\cos x)^2.$$

The above inequality can be rearranged as

$$\left(\frac{\sin x}{x} - 1 - \frac{1}{3}(1-\cos x)\right)\left(\frac{\sin x}{x} - \frac{2+\cos x}{3}\right) > 0,$$

which is true according to the Cusa - Huygens inequality. $\square$

THEOREM 4. *For all $x \in \left(0,\dfrac{\pi}{2}\right)$, one has*

$$\frac{\sin x}{x} + \frac{\tan x}{x} - 2 > \frac{1}{45}(1-\cos x)(43 - 28\cos x).$$

Proof. We introduce the function $f_3 : \left(0, \dfrac{\pi}{2}\right) \to \mathbb{R}$, $f_3(x) = (\sin x - x) + (\tan x - x)$
$-\dfrac{1}{45}x(1 - \cos x)(43 - 28\cos x)$.

Easy computations lead us to

$$f_3'(x) = \frac{116}{45}(\cos x - 1) + \tan^2 x - \frac{71x\sin x - 56x\sin x\cos x - 28\sin^2 x}{45}$$
$$= \frac{90\sin x + 112x\cos^5 x - 71x\cos^4 x - 56x\cos^3 x - 187\sin x\cos^3 x + 112\sin x\cos^4 x}{45\cos^3 x}.$$

The function $g_3 : \left(0, \dfrac{\pi}{2}\right) \to \mathbb{R}$, $g_3(x) = 90\sin x + 112x\cos^5 x - 71x\cos^4 x$
$- 56x\cos^3 x - 187\sin x\cos^3 x + 112\sin x\cos^4 x$ has the derivative

$$g_3'(x) = \cos x(90 + 672\cos^4 x - 819\cos^3 x - 504\cos^2 x + 561\cos x - 560x\cos^3 x\sin x$$
$$+ 284x\cos^2 x\sin x + 168x\cos x\sin x).$$

The function g_3' can be rewritten as

$$g_3'(x) = \cos x\Big[3(1 - \cos x)\left(-224\cos^3 x + 49\cos^2 x + 217\cos x + 30\right)$$
$$-560x\cos^3 x \cdot 2\sin\frac{x}{2}\cos\frac{x}{2} + 284x\cos^2 x \cdot 2\sin\frac{x}{2}\cos\frac{x}{2}$$
$$+168x\cos x \cdot 2\sin\frac{x}{2}\cos\frac{x}{2}\Big]$$
$$= \sin\frac{x}{2}\cos x\Big(180\sin\frac{x}{2} + 1302\sin\frac{x}{2}\cos x + 294\sin\frac{x}{2}\cos^2 x - 1344\sin\frac{x}{2}\cos^3 x$$
$$-1120x\cos^3 x\cos\frac{x}{2} + 568x\cos^2 x\cos\frac{x}{2} + 336x\cos x\cos\frac{x}{2}\Big)$$
$$= \sin\frac{x}{2}\cos x\Big[\left(180\sin\frac{x}{2} - 180\sin\frac{x}{2}\cos^3 x\right)$$
$$+ \left(1164\sin\frac{x}{2}\cos x - 1164\sin\frac{x}{2}\cos^3 x\right)$$
$$+ \left(138\sin\frac{x}{2} + 294\sin\frac{x}{2}\cos x - 216x\cos^2 x\cos\frac{x}{2}\right) \cdot \cos x$$
$$+568x\cos^2 x\cos\frac{x}{2}\left(1 - \cos^2 x\right) + 336x\cos x\cos\frac{x}{2}\left(1 - \cos^2 x\right)\Big].$$

We only have to prove that

$$138\sin\frac{x}{2} + 294\sin\frac{x}{2}\cos x - 216x\cos^2 x\cos\frac{x}{2} > 0 \text{ on } \left(0, \frac{\pi}{2}\right).$$

Using the inequality $138\sin\dfrac{x}{2} > 138\sin\dfrac{x}{2}\cos x$ on $\left(0, \dfrac{\pi}{2}\right)$, we obtain

$$138\sin\frac{x}{2} + 294\sin\frac{x}{2}\cos x - 216x\cos^2 x\cos\frac{x}{2} > 216\cos x\left(2\sin\frac{x}{2} - x\cos x\cos\frac{x}{2}\right) > 0,$$

according to the Lemma 1.

Therefore, $g_3' > 0$ on $\left(0, \dfrac{\pi}{2}\right)$. As $g_3(0) = 0$, we get $g_3 > 0$ on $\left(0, \dfrac{\pi}{2}\right)$. Then $f_3' > 0$ on $\left(0, \dfrac{\pi}{2}\right)$. As $f_3(0) = 0$, we find $f_3 > 0$ on $\left(0, \dfrac{\pi}{2}\right)$.

The proof is completed. $\square$

3. Improvement of Shafer's inequality

In our version of Cusa-Huygens inequality

$$\frac{\sin x}{x} < \frac{2 + \cos x}{3} - \frac{1}{45}(1 - \cos x)^2$$

for all $x \in \left(0, \dfrac{\pi}{2}\right)$, we take $x = \tan t$, $t \in \left(0, \dfrac{\pi}{2}\right)$ and we obtain

$$\frac{\arctan t}{t} > \frac{45\sqrt{t^2 + 1}}{17\sqrt{t^2 + 1} + 29t^2 + 28}.$$

The inequality

$$\frac{45\sqrt{t^2 + 1}}{17\sqrt{t^2 + 1} + 29t^2 + 28} > \frac{3}{1 + 2\sqrt{t^2 + 1}}$$

is equivalent to the following true inequality

$$\left(\sqrt{t^2 + 1} - 1\right)^2 > 0,$$

therefore we also improved on $\left(0, \dfrac{\pi}{2}\right)$ the famous Shafer's inequality for the arctangent function([9]):

$$\frac{\arctan x}{x} > \frac{3}{1 + 2\sqrt{x^2 + 1}}.$$

4. Final remarks

We are convinced that the Fourier series method will be useful to refine many others problems concerning inequalities.

Competing interests. The author declares that he has no competing interests.

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(Received August 10, 2018)

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