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ON WEIGHTED QUASI–ARITHMETIC MEANS WHICH ARE CONVEX

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Abstract. We study convexity in the class of weighted quasi-arithmetic means. It turns out that their convexity depends only on the generator, neither on weights, nor on the number of variables. Connections between the convexity of a mean and the convexity of its increasing generators are considered. We prove that convex means are generated by convex strictly increasing functions. A simple example shows that the converse is not true, so the problem arises when this is the case. Some answers are given under regularity assumptions imposed on the generator.

1. Introduction

Fix any real interval I. Given an integer $n \geqslant 2$ a function $M\colon I^n \to I$ is said to be a *mean on* I if

$$\min\{x_1,\dots,x_n\} \leqslant M(x_1,\dots,x_n) \leqslant \max\{x_1,\dots,x_n\}$$

for all $x_1,\dots,x_n \in I$. We are interested in *convex* means, that is means $M\colon I^n \to I$ satisfying the condition

$$M(t\mathbf{x}+(1-t)\mathbf{y}) \leqslant tM(\mathbf{x})+(1-t)M(\mathbf{y}), \qquad \mathbf{x},\mathbf{y} \in I^n, \tag{1}$$

for every $t \in (0,1)$. If t is fixed then condition (1) defines a t*-convex* mean. $1/2$-convex means are called also *Jensen convex*. Replacing the inequality in (1) by the reverse one we come to the definitions of a *concave*, t*-concave* and *Jensen concave* mean, respectively.

REMARK 1. Let M be a mean on an open interval. Then the following conditions are pairwise equivalent:

(i) M is convex,
(ii) M is t-convex for all $t \in (0,1)$,
(iii) M is t-convex for some $t \in (0,1)$,
(iv) M is Jensen convex.

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Proof. Let $M\colon I^n \to I$. The implications $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are obvious. To prove the implication $(iii) \Rightarrow (iv)$ fix a $t \in (0,1)$. First observe that via a standard argument any real-valued function defined on a convex set is t-convex if and only if its sections along all straight lines are t-convex: given any subset D of $\mathbb{R}^n$ we have

$$f\colon D \to \mathbb{R} \text{ is } t\text{-convex iff } f_{\mathbf{x},\mathbf{e}}\colon I_{\mathbf{x},\mathbf{e}} \to \mathbb{R} \text{ are } t\text{-convex for all } \mathbf{x} \in D \text{ and } \mathbf{e} \in \mathbb{R}^n \setminus \{0\},$$

where $I_{\mathbf{x},\mathbf{e}} := \{\lambda \in \mathbb{R}\colon \mathbf{x}+\lambda\mathbf{e} \in D\}$ and $f_{\mathbf{x},\mathbf{e}}(\lambda) := f(\mathbf{x}+\lambda\mathbf{e})$; of course each function $f_{\mathbf{x},\mathbf{e}}$ is defined on a real interval which is open provided D is open. Now assume that M is t-convex. Then, for any $\mathbf{x} \in I^n$ and $\mathbf{e} \in \mathbb{R}^n \setminus \{0\}$, the section $M_{\mathbf{x},\mathbf{e}}$ is t-convex, and thus, by virtue of a theorem of Kuhn from the paper [5], it is Jensen convex. Consequently, also the mean M is Jensen convex. Another possibility is use here the Daróczy-Páles identity

$$\frac{\mathbf{x}+\mathbf{y}}{2} = t\left(t\frac{\mathbf{x}+\mathbf{y}}{2} + (1-t)\mathbf{x}\right) + (1-t)\left(t\mathbf{y} + (1-t)\frac{\mathbf{x}+\mathbf{y}}{2}\right)$$

probably used in [2, proof of Lemma 1] for the first time:

$$\begin{aligned} M\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) &= M\left(t\left(t\frac{\mathbf{x}+\mathbf{y}}{2} + (1-t)\mathbf{x}\right) + (1-t)\left(t\mathbf{y} + (1-t)\frac{\mathbf{x}+\mathbf{y}}{2}\right)\right) \\ &\leqslant tM\left(t\frac{\mathbf{x}+\mathbf{y}}{2} + (1-t)\mathbf{x}\right) + (1-t)M\left(t\mathbf{y} + (1-t)\frac{\mathbf{x}+\mathbf{y}}{2}\right) \\ &\leqslant \left(t^2 + (1-t)^2\right)M\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) + 2t(1-t)\frac{M(\mathbf{x})+M(\mathbf{y})}{2}, \end{aligned}$$

hence

$$M\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) \leqslant \frac{M(\mathbf{x})+M(\mathbf{y})}{2}$$

for all $\mathbf{x},\mathbf{y} \in I^n$.

Now observe that $M(J^n) \subset J$ for any interval $J \subset I$, and thus M is locally bounded at every point. Making use of the theorem of Bernstein-Doetsch (see [1], also [4, Theorem 6.4.2]) we see that if M is Jensen convex then it is continuous and, consequently, convex (cf., for instance, [4, Theorem 5.3.5 and the comment following it]). This gives the implication $(iv) \Rightarrow (i)$ and completes the proof of the remark. $\square$

Here we study the problem of convexity in the class of weighted quasi-arithmetic means. For any integer $n \geqslant 2$ put $\Delta_n = \{(p_1,\ldots,p_n) \in (0,1)^n\colon\ p_1+\ldots+p_n = 1\}$. Given any continuous strictly monotonic function $\varphi\colon I \to \mathbb{R}$ and a point $\mathbf{p} = (p_1,\ldots,p_n) \in \Delta_n$ the formula

$$A_{\mathbf{p}}^{\varphi}(\mathbf{x}) = \varphi^{-1}\left(p_1\varphi(x_1) + \ldots + p_n\varphi(x_n)\right),$$

where $\mathbf{x} = (x_1,\ldots,x_n)$, defines a mean on I called the *quasi-arithmetic mean generated by φ and weighted by* $\mathbf{p}$. Clearly $A_{\mathbf{p}}^{-\varphi} = A_{\mathbf{p}}^{\varphi}$, so we may always assume that the generator of the mean $A_{\mathbf{p}}^{\varphi}$ is strictly increasing. In fact we know much more:

REMARK 2. Let $\varphi, \psi \colon I \to \mathbb{R}$ be continuous strictly monotonic functions and $\mathbf{p}, \mathbf{q} \in \Delta_n$. Then $A_{\mathbf{p}}^{\varphi} = A_{\mathbf{q}}^{\psi}$ if and only if $\mathbf{p} = \mathbf{q}$ and there exist $a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$ such that $\psi = a\varphi + b$.

This is an immediate consequence of Theorem 1 from [7] which implies that $\mathbf{p} = \mathbf{q}$ and the function $\psi \circ \varphi^{-1}$ is affine. Then, as a continuous function, it is of the form $y \mapsto ay + b$ with some reals a, b (see, for instance, [4, Theorem 13.2.2]). Since $\psi \circ \varphi^{-1}$ is not constant we get $a \neq 0$. Notice also that, in the case when the equality $\mathbf{p} = \mathbf{q}$ is imposed a priori, the assertion of Remark 2 was known already for Hardy, Littlewood and Pólya (see [3]).

As an immediate consequence of Remark 2 we have what follows.

REMARK 3. If a weighted quasi-arithmetic mean is generated by a convex [concave] strictly increasing function, then every its strictly increasing generator is convex [concave].

Many classical means, for instance weighted arithmetic:

$$A_{\mathbf{p}}(\mathbf{x}) = p_1 x_1 + \ldots + p_n x_n, \qquad \mathbf{x} \in \mathbb{R}^n,$$

weighted geometric:

$$G_{\mathbf{p}}(\mathbf{x}) = x_1^{p_1} \cdot \ldots \cdot x_n^{p_n}, \qquad \mathbf{x} \in (0, +\infty)^n,$$

and weighted harmonic:

$$H_{\mathbf{p}}(\mathbf{x}) = \frac{1}{\frac{p_1}{x_1} + \ldots + \frac{p_n}{x_n}}, \qquad \mathbf{x} \in (0, +\infty)^n,$$

are weighted quasi-arithmetic. Their increasing generators are given by

$$\varphi_A(x) = x, \qquad \varphi_G(x) = \log x \qquad \text{and} \qquad \varphi_H(x) = -\frac{1}{x},$$

respectively.

EXAMPLE 1. (i) It is clear that the function $A_{\mathbf{p}} \colon \mathbb{R}^n \to \mathbb{R}$ is affine, so it is simultaneously convex and concave.

(ii) To answer the question on possible convexity of the mean $G_{\mathbf{p}} : (0, +\infty)^2 \to (0, +\infty)$ we study its Hessian matrix. Fixing $\mathbf{x} = (x_1, x_2) \in (0, +\infty)^2$ arbitrarily we easily get

$$\partial_1 G_{\mathbf{p}}(x_1, x_2) = p_1 \left(\frac{x_2}{x_1}\right)^{p_2}, \qquad \partial_2 G_{\mathbf{p}}(x_1, x_2) = p_2 \left(\frac{x_1}{x_2}\right)^{p_1},$$

$$\partial_{11}^2 G_{\mathbf{p}}(x_1, x_2) = -\frac{p_1 p_2}{x_1} \left(\frac{x_2}{x_1}\right)^{p_2}, \qquad \partial_{22}^2 G_{\mathbf{p}}(x_1, x_2) = -\frac{p_1 p_2}{x_2} \left(\frac{x_1}{x_2}\right)^{p_1},$$

$$\partial_{12}^2 G_{\mathbf{p}}(x_1, x_2) = \partial_{21}^2 G_{\mathbf{p}}(x_1, x_2) = \frac{p_1 p_2}{x_1^{p_2} x_2^{p_1}}.$$

Hence $\partial_{11}^2 G_{\mathbf{p}}(x_1,x_2)<0$ and

$$\begin{aligned}&\partial_{11}^2 G_{\mathbf{p}}(x_1,x_2)\,\partial_{22}^2 G_{\mathbf{p}}(x_1,x_2)-\partial_{12}^2 G_{\mathbf{p}}(x_1,x_2)\,\partial_{21}^2 G_{\mathbf{p}}(x_1,x_2)\\&\quad=\frac{p_1^2p_2^2}{x_1x_2}\left(\frac{x_1}{x_2}\right)^{p_1}\left(\frac{x_2}{x_1}\right)^{p_2}-\frac{p_1^2p_2^2}{x_1^{2p_2}x_2^{2p_1}}=\frac{p_1^2p_2^2}{x_1^{2p_2}x_2^{2p_1}}\left(x_1^{p_1+p_2-1}x_2^{p_1+p_2-1}-1\right)=0\end{aligned}$$

for all $(x_1,x_2)\in(0,+\infty)^2$. This means that the Hessian matrix of $G_{\mathbf{p}}$ is negatively semidefinite at every point of $(0,+\infty)^2$ which is equivalent to the concavity of $G_{\mathbf{p}}\colon(0,+\infty)^2\to(0,+\infty)$.

(*iii*) One can easily check that suitable derivatives of the function $H_{\mathbf{p}}\colon(0,+\infty)^2\to(0,+\infty)$ are as follows:

$$\begin{aligned}&\partial_1 H_{\mathbf{p}}(x_1,x_2)=\frac{p_1x_2^2}{(p_2x_1+p_1x_2)^2}, &&\partial_2 H_{\mathbf{p}}(x_1,x_2)=\frac{p_2x_1^2}{(p_2x_1+p_1x_2)^2},\\&\partial_{11}^2 H_{\mathbf{p}}(x_1,x_2)=-\frac{2p_1p_2x_2^2}{(p_2x_1+p_1x_2)^3}, &&\partial_{22}^2 H_{\mathbf{p}}(x_1,x_2)=-\frac{2p_1p_2x_1^2}{(p_2x_1+p_1x_2)^3},\\&\partial_{12}^2 H_{\mathbf{p}}(x_1,x_2)=\partial_{21}^2 H_{\mathbf{p}}(x_1,x_2)=\frac{2p_1p_2x_1x_2}{(p_2x_1+p_1x_2)^3},\end{aligned}$$

and thus $\partial_{11}^2 H_{\mathbf{p}}(x_1,x_2)<0$ and

$$\partial_{11}^2 H_{\mathbf{p}}(x_1,x_2)\,\partial_{22}^2 H_{\mathbf{p}}(x_1,x_2)-\partial_{12}^2 H_{\mathbf{p}}(x_1,x_2)\,\partial_{21}^2 H_{\mathbf{p}}(x_1,x_2)=0$$

for all $x_1,x_2\in(0,+\infty)$. Therefore, also the mean $H_{\mathbf{p}}\colon(0,+\infty)^2\to(0,+\infty)$ is concave.

2. Posing the problem

Our main problem is as follows. *Is there any connection between the convexity of a weighted quasi-arithmetic mean and the convexity of its increasing generator?* Example 1 suggests that maybe this is the case indeed, as we have the following rough information:

increasing generator	mean
affine $\varphi(x)=x$	affine $A_{\mathbf{p}}$
concave $\varphi(x)=\log x$	concave $G_{\mathbf{p}}$
concave $\varphi(x)=-1/x$	concave $H_{\mathbf{p}}$

However, the example below shows that in general the convexity of the increasing generator of weighted quasi-arithmetic mean does not force the convexity of the mean. Nevertheless, in Section 5, in some classes of differentiable functions, we characterize convex strictly increasing functions generating convex weighted quasi-arithmetic means.

EXAMPLE 2. Clearly, the formula

$$\varphi(x)=\begin{cases}x, & x<0,\\ 2x, & x\geqslant 0,\end{cases}$$

defines a convex strictly increasing function with the inverse φ^{-1} given by

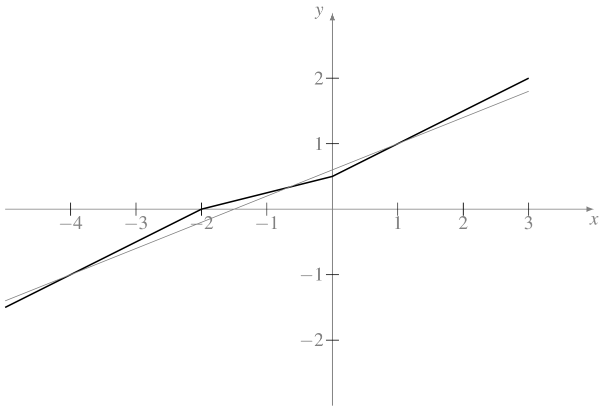
$$\varphi^{-1}(y)=\begin{cases} y, & y<0, \\ y/2, & y\geqslant 0. \end{cases}$$

Taking $p=1/2$ we come to the quasi-arithmetic mean A^{φ} defined by

$$A^{\varphi}(x_1,x_2)=\begin{cases} \frac{x_1+2x_2}{2}, & \text{if } x_1<0\leqslant x_2 \text{ and } x_1+2x_2<0, \\[2ex] \frac{x_1+2x_2}{4}, & \text{if } x_1<0\leqslant x_2 \text{ and } x_1+2x_2\geqslant 0, \\[2ex] \frac{x_1+x_2}{2}, & \text{if either } x_1,x_2<0, \text{ or } x_1,x_2\geqslant 0, \\[2ex] \frac{2x_1+x_2}{2}, & \text{if } x_2<0\leqslant x_1 \text{ and } 2x_1+x_2<0, \\[2ex] \frac{2x_1+x_2}{4}, & \text{if } x_2<0\leqslant x_1 \text{ and } 2x_1+x_2\geqslant 0. \end{cases}$$

Putting here $x_2=1$ we get the section $A^{\varphi}(\cdot,1)$ of A^{φ}:

$$A^{\varphi}(x_1,1)=\begin{cases} \frac{x_1+2}{2}, & \text{if } x_1<-2, \\[2ex] \frac{x_1+2}{4}, & \text{if } -2\leqslant x_1<0, \\[2ex] \frac{x_1+1}{2}, & \text{if } x_1\geqslant 0. \end{cases}$$



Since the section $A^{\varphi}(\cdot,1)$ is not convex, neither is the function A^{φ}. Consequently, the mean A^{φ} is not convex but it is generated by a convex function.

We are interested also in the following converse question. *Do strictly increasing generators of a convex weighted quasi-arithmetic mean have to be convex?* That problem will be positively answered in Section 4.

3. Adjoint classes of means

Given integers $m,n \geqslant 2$ and real intervals I and J let $\{M_{\mathbf{p}}\}_{\mathbf{p}\in\Delta_m}$ and $\{N_{\mathbf{q}}\}_{\mathbf{q}\in\Delta_n}$ be families of means on I and J, respectively. We say that the class $\{M_{\mathbf{p}}\}_{\mathbf{p}\in\Delta_m}$ is *adjoint to* $\{N_{\mathbf{q}}\}_{\mathbf{q}\in\Delta_n}$ if for all $\mathbf{p}\in\Delta_m$ and $\mathbf{q}\in\Delta_n$ the conditions

$$\begin{aligned}M_{\mathbf{p}}&\left(q_1\mathbf{x}^{(1)}+\ldots+q_n\mathbf{x}^{(n)}\right)\\ &\leqslant q_1M_{\mathbf{p}}\left(\mathbf{x}^{(1)}\right)+\ldots+q_nM_{\mathbf{p}}\left(\mathbf{x}^{(n)}\right),\quad \mathbf{x}^{(1)},\ldots,\mathbf{x}^{(n)}\in I^m,\end{aligned}\tag{2}$$

and

$$\begin{aligned}N_{\mathbf{q}}&\left(p_1\mathbf{y}^{(1)}+\ldots+p_m\mathbf{y}^{(m)}\right)\\ &\geqslant p_1N_{\mathbf{q}}\left(\mathbf{y}^{(1)}\right)+\ldots+p_mN_{\mathbf{q}}\left(\mathbf{y}^{(m)}\right),\quad \mathbf{y}^{(1)},\ldots,\mathbf{y}^{(m)}\in J^n,\end{aligned}\tag{3}$$

are equivalent. The following result provides an important example of adjoint classes of means.

PROPOSITION 1. *Let I be a real interval, $\varphi\colon I\to\mathbb{R}$ be a continuous strictly increasing function and let $m,n\geqslant 2$ be integers. Then the class $\left\{A^{\varphi}_{\mathbf{p}}\right\}_{\mathbf{p}\in\Delta_m}$ of φ-generated weighted quasi-arithmetic means in m variables is adjoint to the class $\left\{A^{\varphi^{-1}}_{\mathbf{q}}\right\}_{\mathbf{q}\in\Delta_n}$ of φ^{-1}-generated means in n-variables.*

Proof. Fix $\mathbf{p}\in\Delta_m$ and $\mathbf{q}\in\Delta_n$. Assume condition (2) for $M_{\mathbf{p}}=A^{\varphi}_{\mathbf{p}}$, fix points $\mathbf{y}^{(1)}=\left(y_1^{(1)},\ldots,y_n^{(1)}\right),\ldots,\mathbf{y}^{(m)}=\left(y_1^{(m)},\ldots,y_n^{(m)}\right)\in J^n$, where $J:=\varphi(I)$, and put

$$\mathbf{x}^{(1)}:=\left(\varphi^{-1}\left(y_1^{(1)}\right),\ldots,\varphi^{-1}\left(y_1^{(m)}\right)\right),\ldots,\mathbf{x}^{(n)}:=\left(\varphi^{-1}\left(y_n^{(1)}\right),\ldots,\varphi^{-1}\left(y_n^{(m)}\right)\right).$$

Then $\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(n)}\in I^m$ and

$$A^{\varphi}_{\mathbf{p}}\left(q_1\mathbf{x}^{(1)}+\ldots+q_n\mathbf{x}^{(n)}\right)\leqslant q_1A^{\varphi}_{\mathbf{p}}\left(\mathbf{x}^{(1)}\right)+\ldots+q_nA^{\varphi}_{\mathbf{p}}\left(\mathbf{x}^{(n)}\right),\tag{4}$$

that is

$$\begin{aligned}\varphi^{-1}&\left(p_1\varphi\left(q_1x_1^{(1)}+\ldots+q_nx_1^{(n)}\right)+\ldots+p_m\varphi\left(q_1x_m^{(1)}+\ldots+q_nx_m^{(n)}\right)\right)\\ &\leqslant q_1\varphi^{-1}\left(p_1\varphi\left(x_1^{(1)}\right)+\ldots+p_m\varphi\left(x_m^{(1)}\right)\right)+\ldots\\ &\qquad +q_n\varphi^{-1}\left(p_1\varphi\left(x_1^{(n)}\right)+\ldots+p_m\varphi\left(x_m^{(n)}\right)\right).\end{aligned}$$

Since φ is increasing then, using the variables $\mathbf{y}^{(1)},\ldots,\mathbf{y}^{(m)}$, this inequality can be rewritten in the form

$$\begin{aligned}p_1\varphi&\left(q_1\varphi^{-1}\left(y_1^{(1)}\right)+\ldots+q_n\varphi^{-1}\left(y_n^{(1)}\right)\right)+\ldots\\ &\qquad +p_m\varphi\left(q_1\varphi^{-1}\left(y_1^{(m)}\right)+\ldots+q_n\varphi^{-1}\left(y_n^{(m)}\right)\right)\\ &\leqslant\varphi\left(q_1\varphi^{-1}\left(p_1y_1^{(1)}+\ldots+p_my_1^{(m)}\right)+\ldots+q_n\varphi^{-1}\left(p_1y_n^{(1)}+\ldots+p_my_n^{(m)}\right)\right)\end{aligned}$$

or, equivalently,

$$p_1 A_{\mathbf{q}}^{\varphi^{-1}}\left(\mathbf{y}^{(1)}\right)+\ldots+p_m A_{\mathbf{q}}^{\varphi^{-1}}\left(\mathbf{y}^{(m)}\right) \leqslant A_{\mathbf{q}}^{\varphi^{-1}}\left(p_1\mathbf{y}^{(1)}+\ldots+p_m\mathbf{y}^{(m)}\right). \tag{5}$$

Thus we come to (3) for $N_{\mathbf{q}}=A_{\mathbf{q}}^{\varphi^{-1}}$. The implication (3)$\Rightarrow$(2) for $M_{\mathbf{p}}=A_{\mathbf{p}}^{\varphi}$ and $N_{\mathbf{q}}=A_{\mathbf{q}}^{\varphi^{-1}}$ can be proved similarly. $\square$

Using Proposition 1 we can prove the following main result of this section.

THEOREM 1. *Let I be an open real interval and $\varphi\colon I\to\mathbb{R}$ be a continuous strictly increasing function. Then the following statements are pairwise equivalent:*

(i) *there exist an integer $m\geqslant 2$ and $\mathbf{p}\in\Delta_m$ such that the mean $A_{\mathbf{p}}^{\varphi}$ is convex,*

(ii) *the mean $A_{\mathbf{p}}^{\varphi}$ is convex for every integer $m\geqslant 2$ and all $\mathbf{p}\in\Delta_m$,*

(iii) *there exist an integer $n\geqslant 2$ and $\mathbf{q}\in\Delta_n$ such that the mean $A_{\mathbf{q}}^{\varphi^{-1}}$ is concave,*

(iv) *the mean $A_{\mathbf{q}}^{\varphi^{-1}}$ is concave for every integer $n\geqslant 2$ and all $\mathbf{q}\in\Delta_n$.*

Proof. Assume statement (i) and choose an integer $m\geqslant 2$ and a vector $\mathbf{p}\in\Delta_m$ such that the mean $A_{\mathbf{p}}^{\varphi}$ is convex. Fix any integer $n\geqslant 2$ and $\mathbf{q}\in\Delta_n$. Using Jensen's inequality (see, for instance, [4, Theorem 8.1.1]) we infer that inequality (4) holds for all $\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(n)}\in I^m$, and thus, by Proposition 1, inequality (5) is satisfied for all $\mathbf{y}^{(1)},\ldots,\mathbf{y}^{(m)}\in\varphi(I)^n$. Therefore, putting $\mathbf{y}^{(k)}=\mathbf{y}^{(2)}$ for $k=2,\ldots,m$ in (5), we obtain

$$p_1 A_{\mathbf{q}}^{\varphi^{-1}}\left(\mathbf{y}^{(1)}\right)+(1-p_1)A_{\mathbf{q}}^{\varphi^{-1}}\left(\mathbf{y}^{(2)}\right)\leqslant A_{\mathbf{q}}^{\varphi^{-1}}\left(p_1\mathbf{y}^{(1)}+(1-p_1)\mathbf{y}^{(2)}\right)$$

for all $\mathbf{y}^{(1)},\mathbf{y}^{(2)}\in\varphi(I)^n$, which means that $A_{\mathbf{q}}^{\varphi^{-1}}$ is p_1-concave. Now Remark 1 implies that it is concave. This completes the proof of the implication $(i)\Rightarrow(iv)$. Analogously one can prove that $(iii)\Rightarrow(ii)$. Since the implications $(iv)\Rightarrow(iii)$ and $(ii)\Rightarrow(i)$ trivially hold, the proof is completed. $\square$

Theorem 1 shows that the convexity of a weighted quasi-arithmetic mean depends only on its generator, neither on the numbers of variables, nor the weights. Consequently, studying the convexity of a mean $A_{\mathbf{p}}^{\varphi}$ with some $\mathbf{p}\in\Delta_n$ and $n\geqslant 2$ it is enough to deal with the simplest case of the quasi-arithmetic mean $A^{\varphi}\colon I^2\to I$ defined by

$$A^{\varphi}(x_1,x_2)=\varphi^{-1}\left(\frac{\varphi(x_1)+\varphi(x_2)}{2}\right).$$

Of course calculations for A^{φ} are shorter and much more straightforward. For that reason, in what follows we disregard both the number of the variables and the weights of the mean $A_{\mathbf{p}}^{\varphi}$ and refer only to the quasi-arithmetic A^{φ} in two variables.

4. Increasing generators of convex quasi-arithmetic means are convex

We start with a characterization of the convexity of weighted quasi-arithmetic means. This is an immediate consequence of [9, Theorem 4] by Páles and Theorem 1.

THEOREM 2. *Let I be an open real interval and $\varphi\colon I\to\mathbb{R}$ be a continuous strictly increasing function. Then the mean A^{φ} is convex if and only if there exist functions $d_1,d_2\colon I^2\to\mathbb{R}$ such that*

$$\varphi\left(\frac{x_1+x_2}{2}\right)-\varphi\left(\frac{y_1+y_2}{2}\right)\leqslant d_1(y_1,y_2)(\varphi(x_1)-\varphi(y_1))+d_2(y_1,y_2)(\varphi(x_2)-\varphi(y_2)) \tag{6}$$

for all $x_1,x_2,y_1,y_2\in I$.

Now we can positively answer the question posed at the very end of Section 2.

THEOREM 3. *Let I be an open real interval and $\varphi\colon I\to\mathbb{R}$ be a continuous strictly increasing function. If the mean $A_{\mathbf{p}}^{\varphi}$ is convex for some $\mathbf{p}\in\Delta_m$ and an integer $m\geqslant 2$, then the function φ is convex.*

Proof. Assume that $p\in\Delta_m$ with an $m\geqslant 2$ and $A_{\mathbf{p}}^{\varphi}$ is convex. Theorem 1 implies that also A^{φ} is convex. Fix an $x_0\in I$. According to Remarks 2 and 3 we may assume without loss of generality that $\varphi(x_0)=0$. By Theorem 2 there exist functions $d_1,d_2\colon I^2\to\mathbb{R}$ satisfying inequality (6) for all $x_1,x_2,y_1,y_2\in I$. Setting $y_1=y_2=x_0$ in (6) we see that

$$\varphi\left(\frac{x_1+x_2}{2}\right)\leqslant c_1\varphi(x_1)+c_2\varphi(x_2),\qquad x_1,x_2\in I, \tag{7}$$

where $c_1:=d_1(x_0,x_0)$ and $c_2:=d_2(x_0,x_0)$. In particular,

$$\varphi(x)\leqslant(c_1+c_2)\,\varphi(x),\qquad x\in I,$$

and thus, as φ is changing its sign in a neighbourhood of x_0, we have $c_1+c_2=1$. Now, swapping x_1 and x_2 in (7), and summing the obtained inequality with (7), we get that φ is Jensen convex. Since φ is continuous, it is convex. □

5. Convexity of quasi-arithmetic means generated by regular convex functions

First notice the following result which is a direct consequence of [9, Theorem 6] and again Theorem 1.

THEOREM 4. *Let I be an open real interval and $\varphi\colon I\to\mathbb{R}$ be a differentiable function with positive first derivative. Then the mean A^{φ} is convex if and only if the function $E\colon I^2\to\mathbb{R}$, given by*

$$E(x_1,x_2)=\frac{\varphi(x_1)-\varphi(x_2)}{\varphi'(x_2)},$$

is convex.

Observe that Theorem 4 characterizes the convexity of the mean A^{φ} in terms of the convexity of the two-variable function E. The result below reduces the problem to studying the convexity of another functions also built using only the generator φ. Notice however, that both of them are in a single variable, so easier to study. It should be also observed that the assumptions made in both results are different.

THEOREM 5. *Let I be an open real interval and $\varphi\colon I \to \mathbb{R}$ be a twice continuously differentiable function with positive first derivative. Then the following statements are pairwise equivalent:*

(i) *the mean A^{φ} is convex [concave],*

(ii) *either φ is an affine function, or the second derivative φ'' is positive [negative] and the function $\left[(\varphi')^2/\varphi''\right]\circ\varphi^{-1}$ is concave [convex],*

(iii) *either φ is an affine function, or the second derivative φ'' is positive [negative] and the function φ'/φ'' is concave [convex].*

Proof. To prove that (i) is equivalent to (ii) assume that the quasi-arithmetic mean $A^{\varphi}\colon I^2 \to I$ is convex. To find the Hessian matrix of A^{φ} we need to calculate second partial derivatives of A^{φ}. Standard argument shows that for any $x_1,x_2 \in I$ we have

$$\partial_1 A^{\varphi}(x_1,x_2) = \frac{\varphi'(x_1)}{2\varphi'(A^{\varphi}(x_1,x_2))}, \qquad \partial_2 A^{\varphi}(x_1,x_2) = \frac{\varphi'(x_2)}{2\varphi'(A^{\varphi}(x_1,x_2))},$$

$$\partial^2_{11} A^{\varphi}(x_1,x_2) = \frac{2\varphi'(A^{\varphi}(x_1,x_2))^2\,\varphi''(x_1) - \varphi'(x_1)^2\,\varphi''(A^{\varphi}(x_1,x_2))}{4\varphi'(A^{\varphi}(x_1,x_2))^3},$$

$$\partial^2_{22} A^{\varphi}(x_1,x_2) = \frac{2\varphi'(A^{\varphi}(x_1,x_2))^2\,\varphi''(x_2) - \varphi'(x_2)^2\,\varphi''(A^{\varphi}(x_1,x_2))}{4\varphi'(A^{\varphi}(x_1,x_2))^3},$$

and

$$\partial^2_{12} A^{\varphi}(x_1,x_2) = \partial^2_{21} A^{\varphi}(x_1,x_2) = -\frac{\varphi'(x_1)\,\varphi'(x_2)\,\varphi''(A^{\varphi}(x_1,x_2))}{4\varphi'(A^{\varphi}(x_1,x_2))^3}.$$

The convexity of A^{φ} is equivalent to the condition

$$D_1(x_1,x_2) \geqslant 0 \quad \text{and} \quad D_2(x_1,x_2) \geqslant 0, \qquad x_1,x_2 \in I,$$

where

$$D_1 = \partial^2_{11} A^{\varphi} \quad \text{and} \quad D_2 = \partial^2_{11} A^{\varphi}\,\partial^2_{22} A^{\varphi} - \partial^2_{12} A^{\varphi}\,\partial^2_{21} A^{\varphi}.$$

In particular, since

$$D_1(x,x) = \frac{\varphi''(x)}{4\varphi'(x)} \geqslant 0, \qquad x \in I,$$

we see that $\varphi'' \geqslant 0$. If $\varphi'' = 0$ then φ is affine. So we may assume that the open set $\{x \in I\colon \quad \varphi''(x) > 0\}$ is nonempty. Let J be any its connected component. If $x_0 := \sup J < \sup I$ then $\varphi''(x_0) = 0$ and

$$0 \leqslant D_1(x_0,x) = \partial^2_{11} A^{\varphi}(x_0,x) = -\frac{\varphi'(x_0)^2\,\varphi''(A^{\varphi}(x_0,x))}{4\varphi'(A^{\varphi}(x_0,x))^3}, \qquad x \in I,$$

hence $\varphi''(A^{\varphi}(x_0,x)) \leqslant 0$, that is $\varphi''(A^{\varphi}(x_0,x)) = 0$ for all $x \in I$. This, however, is impossible as $A^{\varphi}(x_0,x) \in J$, for $x \in J$ sufficiently close to x_0. Therefore $\sup J = \sup I$. Similarly, we show that $\inf J = \inf I$, and thus $J = I$, i.e. $\varphi'' > 0$. Put $\phi := (\varphi')^2/\varphi''$. Then

$$\partial_{11}^2 A^{\varphi}(x_1,x_2) = \frac{(2\phi(A^{\varphi}(x_1,x_2)) - \phi(x_1))\,\varphi''(x_1)\,\varphi''(A^{\varphi}(x_1,x_2))}{4\varphi'(A^{\varphi}(x_1,x_2))^3}, \tag{8}$$

and

$$\partial_{22}^2 A^{\varphi}(x_1,x_2) = \frac{(2\phi(A^{\varphi}(x_1,x_2)) - \phi(x_2))\,\varphi''(x_2)\,\varphi''(A^{\varphi}(x_1,x_2))}{4\varphi'(A^{\varphi}(x_1,x_2))^3}$$

for all $x_1,x_2 \in I$. Moreover,

$$\begin{aligned} &\frac{8\varphi'(A^{\varphi}(x_1,x_2))^6}{\varphi''(x_1)\,\varphi''(x_2)\,\varphi''(A^{\varphi}(x_1,x_2))^2} D_2(x_1,x_2) \\ &\quad = \phi(A^{\varphi}(x_1,x_2))\left[2\phi(A^{\varphi}(x_1,x_2)) - \phi(x_1) - \phi(x_2)\right], \qquad x_1,x_2 \in I, \end{aligned} \tag{9}$$

and thus, since $\phi > 0$ and $D_2 \geqslant 0$, we have

$$\phi(A^{\varphi}(x_1,x_2)) \geqslant \frac{\phi(x_1) + \phi(x_2)}{2}, \qquad x_1,x_2 \in I, \tag{10}$$

that is

$$(\phi \circ \varphi^{-1})\left(\frac{y_1+y_2}{2}\right) \geqslant \frac{(\phi \circ \varphi^{-1})(y_1) + (\phi \circ \varphi^{-1})(y_2)}{2}, \qquad y_1,y_2 \in \varphi(I).$$

This means that the function $\phi \circ \varphi^{-1}$ is Jensen concave and according to its continuity, concave. In such a way we have proved statement (ii).

If the function φ is affine, then A^{φ} is the arithmetic mean which is clearly convex. So to prove implication $(ii) \Rightarrow (i)$ we may assume that $\varphi'' > 0$ and the function $\phi \circ \varphi^{-1}$ is concave. This gives (10) and, using (9), we infer that $D_2 \geqslant 0$. Condition (10) and the positivity of ϕ imply also that

$$2\phi(A^{\varphi}(x_1,x_2)) \geqslant \phi(x_1), \qquad x_1,x_2 \in I,$$

which, in view of (8), shows that $D_1 \geqslant 0$ as well. Therefore, the Hessian matrix of A^{φ} is positively semidefinite at every point of I^2 which forces the convexity of A^{φ}.

Finally we prove the equivalence of statements (ii) and (iii). Of course we may consider only the case when $\varphi'' > 0$. Assume that the function $\left[(\varphi')^2/\varphi''\right] \circ \varphi^{-1}$ is concave. Then A^{φ} is convex and, by Theorem 1, the mean A^{ψ}, where $\psi := \varphi^{-1}$, is concave. Thus the function $\left[(\psi')^2/\psi''\right] \circ \psi^{-1}$ is convex. We have $\psi^{-1} = \varphi$,

$$\psi' = \frac{1}{\varphi' \circ \varphi^{-1}} \qquad \text{and} \qquad \psi'' = -\frac{\varphi'' \circ \varphi^{-1}}{(\varphi' \circ \varphi^{-1})^3},$$

and thus

$$\frac{(\psi')^2}{\psi''}\circ\psi^{-1} = \left(-\frac{1}{(\varphi'\circ\varphi^{-1})^2}\,\frac{(\varphi'\circ\varphi^{-1})^3}{\varphi''\circ\varphi^{-1}}\right)\circ\varphi = -\frac{\varphi'}{\varphi''}.$$

Consequently, the function φ'/φ'' is concave and the implication $(ii)\Rightarrow(iii)$ has been proved. Reversing this reasoning we see that also the implication $(iii)\Rightarrow(ii)$ holds. This completes the proof. $\square$

As a consequence we obtain the following simple necessary condition of the convexity of the mean A^{φ}.

COROLLARY 1. *Let I be an open real interval and $\varphi\colon I\to\mathbb{R}$ be a twice continuously differentiable function with positive first derivative. If the mean A^{φ} is convex [concave], then either φ is an affine function, or it is strictly convex [strictly concave].*

Recently, in the paper [10], Páles and Pasteczka have proved that statement (iii) is equivalent to the convexity [concavity] of the quasi-arithmetic mean $A^{\varphi}\colon I^m\to I$ for all integers $m\geqslant 2$.

6. Simple applications

The function $\varphi\colon I\to\mathbb{R}$, described in Example 2, is convex and strictly increasing but the mean A^{φ} is neither convex, nor concave. However, φ is not differentiable at 0. Just recently, Małolepszy [8] has asked about a similar example with a generator which is continuously differentiable. He proposed to consider $\varphi\colon I\to\mathbb{R}$ defined by

$$\varphi(x)=\begin{cases} x, & \text{if } x<0,\\ x^2+x, & \text{if } x\geqslant 0.\end{cases}$$

In fact, it is a convex, strictly increasing and continuously differentiable function, and the section $A^{\varphi}(\cdot,1)$ of the mean A^{φ} is given by

$$A^{\varphi}(x_1,1)=\begin{cases} \frac{x_1+2}{2}, & \text{if } x_1<-2,\\[2ex] \frac{\sqrt{2x_1+5}-1}{2}, & \text{if } -2\leqslant x_1<0,\\[2ex] \frac{\sqrt{2x_1^2+2x_1+5}-1}{2}, & \text{if } x_1\geqslant 0.\end{cases}$$

Simple calculations show that its right-hand side derivative at $-2,0$ and 1 equals $1/2, 1/2\sqrt{5}$ and $1/2$, respectively, so this derivative is not monotonic. Therefore $A^{\varphi}(\cdot,1)$ and, consequently, also A^{φ} are neither convex, nor concave.

Encouraged by Małolepszy's question we were looking for similar examples where the regularity of the generator is of higher order.

EXAMPLE 3. For each $n \in \mathbb{N}$ the function $\varphi^{[n]}\colon \mathbb{R} \to \mathbb{R}$, defined by

$$\varphi^{[n]}(x) = \begin{cases} x, & \text{if } x < 0, \\ x^n + x, & \text{if } x \geqslant 0, \end{cases}$$

is convex, strictly increasing and $(n-1)$-times continuously differentiable. Observe that $\varphi^{[1]}$ is the function discussed in Example 2 and $\varphi^{[2]}$ is that one presented at the very beginning of this section. In general, examining the convexity of the mean $A^{\varphi^{[n]}}$, we cannot follow the arguments used for $n = 1$ or $n = 2$. For $n \geqslant 3$ it is usually hard even to determine the form of the inverse $\left(\varphi^{[n]}\right)^{-1}$. However, making use of Corollary 1 we see that the mean $A^{\varphi^{[n]}}$ is neither convex, nor concave for all $n \in \mathbb{N}$.

It turns out that even the analyticity of a convex generator φ does not guarantee the convexity of the mean A^{φ}.

EXAMPLE 4. The formula $\varphi(x) = \mathrm{e}^x + x$ defines a convex strictly increasing function $\varphi\colon \mathbb{R} \to \mathbb{R}$ which is analytic. Moreover,

$$\varphi'(x) = \mathrm{e}^x + 1 > 0 \quad \text{and} \quad \varphi''(x) = \mathrm{e}^x > 0,$$

hence

$$\frac{\varphi'(x)}{\varphi''(x)} = \frac{\mathrm{e}^x + 1}{\mathrm{e}^x} = 1 + \mathrm{e}^{-x}$$

for all $x \in \mathbb{R}$, and thus the function φ'/φ'' is convex. Therefore, using Theorem 5, we infer that the mean A^{φ} is neither convex, nor concave.

In the last example we apply Theorem 5 to examine the convexity of the power means. The result is known due to Losonczi [6].

EXAMPLE 5. Fix an integer $n \geqslant 2$, a vector $\mathbf{p} \in \Delta_n$ and a number $t \in \mathbb{R}$. The formula

$$\varphi_t(x) = \begin{cases} -x^t, & \text{if } t \in (-\infty, 0), \\ \log x, & \text{if } t = 0, \\ x^t, & \text{if } t \in (0, +\infty), \end{cases}$$

defines a continuous strictly increasing function $\varphi_t\colon (0, +\infty) \to \mathbb{R}$. It generates the weighted Hölder mean $H^t_{\mathbf{p}} := A^{\varphi_t}_{\mathbf{p}}$ on the half-line $(0, +\infty)$. Clearly $H^0_{\mathbf{p}} = G_{\mathbf{p}}$ is the $\mathbf{p}$-weighted geometric mean. Since

$$\varphi_0'(x) = \frac{1}{x} \qquad \text{and} \qquad \varphi_0''(x) = -\frac{1}{x^2}, \qquad x \in (0, +\infty),$$

we have

$$\frac{(\varphi_0')^2}{\varphi_0''}\left(\varphi_0^{-1}(y)\right) = -1, \qquad y \in \mathbb{R}, \qquad \text{and} \qquad \frac{\varphi_0'(x)}{\varphi_0''(x)} = -x, \qquad x \in (0, +\infty),$$

and thus Theorem 5 (cf. statement (ii) and/or (iii)) implies the concavity of the mean $H_{\mathbf{p}}^0$. Taking $t=1$ we come to the $\mathbf{p}$-weighted arithmetic mean $A_{\mathbf{p}}$ which is affine, so simultaneously convex and concave.

Now assume that $t \in \mathbb{R} \setminus \{0,1\}$. Then

$$H_{\mathbf{p}}^t(\mathbf{x}) = \left(p_1 x_1^t + \ldots + p_n x_n^t\right)^{\frac{1}{t}}, \qquad \mathbf{x} \in (0,+\infty)^n.$$

Since

$$\varphi_t'(x) = \begin{cases} -tx^{t-1}, \text{ if } t \in (-\infty,0), \\ tx^{t-1}, \text{ if } t \in (0,1) \cup (1,+\infty), \end{cases}$$

and

$$\varphi_t''(x) = \begin{cases} -t(t-1)x^{t-2}, \text{ if } t \in (-\infty,0), \\ t(t-1)x^{t-2}, \text{ if } t \in (0,1) \cup (1,+\infty), \end{cases}$$

for each $x \in (0,+\infty)$, it follows that φ_t'' is positive for $t \in (1,+\infty)$ and negative for $t \in (-\infty,0) \cup (0,1)$. Moreover,

$$\frac{(\varphi_t')^2}{\varphi_t''}\left(\varphi_t^{-1}(y)\right) = \frac{t}{t-1}y$$

for each $y \in \varphi_t((0,+\infty))$ and

$$\frac{\varphi_t'(x)}{\varphi_t''(x)} = \frac{1}{t-1}x$$

for all $x \in (0,+\infty)$. Therefore, by Theorem 5, the mean $H_{\mathbf{p}}^t$ is convex for $t \in (1,+\infty)$ and concave for $t \in (-\infty,0) \cup (0,1)$. Reasumming, if $t \in (-\infty,1]$ then $H_{\mathbf{p}}^t$ is concave and if $t \in [1,+\infty)$ then the mean $H_{\mathbf{p}}^t$ is convex.

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