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ON THE CAUCHY–SCHWARZ INEQUALITY AND SEVERAL INEQUALITIES IN AN INNER PRODUCT SPACE

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Dedicated to Josip Pečarić
on the occasion of his 70th birthday

(*Communicated by S. Varošanec*)

Abstract. The aim of this article is to prove new results related to several inequalities in an inner product space. Among these inequalities we will mention Cauchy-Schwarz inequality. Moreover, we will we obtain some applications of these inequalities.

1. Introduction

Many classical inequalities have been extended for the inner product spaces. Among these inequalities is the inequality of Cauchy-Schwarz [2, 13]:

$$|\langle x,y\rangle| \leqslant \|x\|\,\|y\|, \tag{1.1}$$

for all $x,y\in X$, where X is a complex inner product space.

The Cauchy-Schwarz inequality in the complex case is studied by Dragomir [7]. Using the Cauchy-Schwarz inequality, Pečarić proved a generalization of Hua's inequality in [16].

Aldaz [1] and Niculescu [15], gave the following identity:

$$\langle x,y\rangle = \|x\|\,\|y\|\left(1-\frac{1}{2}\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|^2\right), \tag{1.2}$$

for all $x,y\in X$, $x,y\neq 0$, which implies the Cauchy-Schwarz inequality in the real case.

Another inequality which plays a central role in an inner product space is the triangle inequality,

$$\|x+y\| \leqslant \|x\|+\|y\|, \tag{1.3}$$

for all $x,y\in X$, where X is a complex normed space. Other different results about the triangle inequality have been proven by Pečarić and Rajić in [19]. In [4] Dadipour et al.

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gave a characterization of a generalized triangle inequality in normed spaces. In [12] we show several estimates of the triangle inequality using integrals.

Equality (1.2) can be written in terms of the *norm-angular distance or Clarkson distance* (see e.g. [3]) between nonzero vectors x and $y, \alpha[x,y] = \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|$, thus:

$$\alpha^2[x,y] = \frac{2(\|x\|\,\|y\| - \langle x,y\rangle)}{\|x\|\,\|y\|}. \tag{1.4}$$

In [10], Maligranda proved an inequality which is a refinement of the triangle inequality in a normed space. This can be written in terms of the norm-angular distance as:

$$\frac{\|x-y\| - |\|x\| - \|y\||}{\min\{\|x\|, \|y\|\}} \leqslant \alpha[x,y] \leqslant \frac{\|x-y\| + |\|x\| - \|y\||}{\max\{\|x\|, \|y\|\}}, \tag{1.5}$$

for all $x,y \in X,\ x,y \neq 0$.

By combining the inequalities (1.4) and (1.5) we infer the inequality

$$\begin{aligned} \frac{\max\{\|x\|, \|y\|\}}{2\min\{\|x\|, \|y\|\}} (\|x-y\| - |\|x\| - \|y\||)^2 &\leqslant \|x\|\,\|y\| - \langle x,y\rangle \\ &\leqslant \frac{\min\{\|x\|, \|y\|\}}{2\max\{\|x\|, \|y\|\}} (\|x-y\| + |\|x\| - \|y\||)^2, \end{aligned} \tag{1.6}$$

for all $x,y \in X$, $x,y \neq 0$. This inequality implies the following inequality:

$$\frac{1}{2}(\|x-y\| - |\|x\| - \|y\||)^2 \leqslant \|x\|\,\|y\| - \langle x,y\rangle \leqslant \frac{1}{2}(\|x-y\| + |\|x\| - \|y\||)^2, \tag{1.7}$$

for all $x,y \in X$.

The norm-angular distance was generalized to the p-angular distance in normed spaces in [10], thus: for p in the interval $[0,\infty)$ and for nonzero x and y in X define $\alpha_p[x,y] = \left\| \|x\|^{p-1}x - \|y\|^{p-1}y \right\|$, with $\alpha_0[x,y] = \alpha[x,y]$.

In [6], Dragomir characterizes this distance obtaining new bounds for it. A survey on the results for bounds for the angular distance, named Dunkl-Williams type theorems (see [8, 11, 18]), is given by Moslehian et al. [14].

2. Main results

We will present some results regarding the Cauchy-Schwarz inequality and the triangle inequality. We will also present some characterizations of the relationship between the two inequalities.

THEOREM 1. *If $X = (X, \langle\cdot,\cdot\rangle)$ is an Euclidean space and the norm $\|\cdot\|$ is generated by an inner product $\langle\cdot,\cdot\rangle$, then we have*

$$\max\{\|x\|, \|y\|\}(\|x\| + \|y\| - \|x+y\|) \leqslant \|x\| \cdot \|y\| - \langle x,y\rangle, \tag{2.1}$$

for all vectors x and y in X.

Proof. Without loss of generality, we may assume that $\|y\| \leqslant \|x\|$. The inequality from the statement becomes $\|x\|^2 - \|x\| \cdot \|x+y\| + \langle x,y \rangle \leqslant 0$, which is equivalent to $(2\|x\| - \|x+y\| - \|x-y\|)(2\|x\| - \|x+y\| + \|x-y\|) \leqslant 0$. This inequality is true, because from the triangle inequality, we obtain $2\|x\| \leqslant \|x+y\| + \|x-y\|$ and $\|x+y\| \leqslant 2\|x\| + \|x-y\|$. $\square$

REMARK 1. In the proof of inequality (2.1), we use the inequality $\|x\|^2 + \langle x,y \rangle \leqslant \|x\| \cdot \|x+y\|$, for all vectors x and y in X, which improves the inequality of Cauchy-Schwarz, thus:

$$\langle x,y \rangle \leqslant \|x\| \cdot (\|x+y\| - \|x\|) \leqslant \|x\| \cdot \|y\|. \tag{2.2}$$

COROLLARY 1. *If $X = (X, \langle \cdot,\cdot \rangle)$ is an Euclidean space and the norm $\|\cdot\|$ is generated by an inner product $\langle \cdot,\cdot \rangle$, then we have*

$$0 \leqslant \frac{\|x\| \cdot \|y\| - \langle x,y \rangle}{\|x\| \cdot \|y\|} \leqslant \alpha[x,y], \tag{2.3}$$

for all nonzero vectors x and y in X.

Proof. For $x \to \frac{x}{\|x\|}$ and $y \to \frac{y}{\|y\|}$ in relation (2.1), we obtain $\square$

$$0 \leqslant 2 - \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leqslant \frac{\|x\| \cdot \|y\| + \langle x,y \rangle}{\|x\| \cdot \|y\|},$$

which by simple calculations implies the statements.

REMARK 2. Combining inequalities (1.5) and (2.3), we find the following inequality:

$$0 \leqslant \|x\| \cdot \|y\| - \langle x,y \rangle \leqslant \min\{\|x\| \cdot \|y\|\}(\|x-y\| + |\|x\| - \|y\||), \tag{2.4}$$

for all vectors x and y in X.

THEOREM 2. *If $X = (X, \langle \cdot,\cdot \rangle)$ is an Euclidean space and the norm $\|\cdot\|$ is generated by an inner product $\langle \cdot,\cdot \rangle$, then we have*

$$\max\{\|x\|, \|y\|\}(\|x-y\| - |\|x\| - \|y\||) \geqslant \|x\| \cdot \|y\| - \langle x,y \rangle, \tag{2.5}$$

for all vectors x and y in X.

Proof. Without loss of generality we may assume that $\|y\| \leqslant \|x\|$. The inequality of the statement becomes $\|x\|(\|x-y\| - \|x\| + \|y\|) \geqslant \|x\| \cdot \|y\| - \langle x,y \rangle$, which is equivalent to

$$\|x\|^2 - \|x\| \cdot \|x-y\| - \langle x,y \rangle \leqslant 0. \tag{2.6}$$

If we take $y \to -y$ in the first part of inequality (2.2), we obtain inequality (2.6). $\square$

A consequence of the above theorems is related to the lower bounds of the p-angular distance.

REMARK 3. Combining inequalities (2.1) and (2.5), we find the following inequality:

$$\max\{\|x\|,\|y\|\}(\|x\|+\|y\|-\|x+y\|) \leqslant \|x\|\cdot\|y\|-\langle x,y\rangle \\ \leqslant \max\{\|x\|,\|y\|\}(\|x-y\|-|\|x\|-\|y\||), \tag{2.7}$$

for all vectors x and y in X.

THEOREM 3. *With the above notations, we have*

$$\alpha_p[x,y] \geqslant \|x\|^p+\|y\|^p-\min\{\|x\|^p,\|y\|^p\}\left(1+\frac{\langle x,y\rangle}{\|x\|\,\|y\|}\right), \tag{2.8}$$

for all nonzero vectors x and y in X and

$$\alpha_p[x,y] \geqslant |\|x\|^p-\|y\|^p|+\frac{1}{2}\min\{\|x\|^p,\|y\|^p\}\alpha^2[x,y], \tag{2.9}$$

for all vectors x and y in X.

Proof. In inequality (2.1) if we replace x by $\|x\|^{p-1}x$ and y by $-\|y\|^{p-1}y$, then we deduce the following inequality:

$$\max\{\|x\|^p,\|y\|^p\}(\|x\|^p+\|y\|^p-\alpha_p[x,y]) \leqslant \|x\|^p\cdot\|y\|^p\left(1+\frac{\langle x,y\rangle}{\|x\|\,\|y\|}\right),$$

which is equivalent to

$$\|x\|^p+\|y\|^p-\alpha_p[x,y] \leqslant \min\{\|x\|^p,\|y\|^p\}\left(1+\frac{\langle x,y\rangle}{\|x\|\,\|y\|}\right).$$

Consequently we obtain inequality (2.8).

In inequality (2.5) if we replace x by and y by $\|x\|^{p-1}x$ and y by $\|y\|^{p-1}y$, then we deduce the following inequality:

$$\max\{\|x\|^p,\|y\|^p\}(\alpha_p[x,y]-|\|x\|^p-\|y\|^p|) \geqslant \|x\|^{p-1}\|y\|^{p-1}(\|x\|\cdot\|y\|-\langle x,y\rangle). \tag{2.11}$$

But, using relation (1.4), we have

$$\max\{\|x\|^p,\|y\|^p\}(\alpha_p[x,y]-|\|x\|^p-\|y\|^p|) \geqslant \frac{1}{2}\|x\|^p\|y\|^p\alpha^2[x,y] \\ =\frac{1}{2}\max\{\|x\|^p,\|y\|^p\}\min\{\|x\|^p,\|y\|^p\}\alpha^2[x,y].$$

Therefore, the statement is proven. □

Next, we develop these inequalities for linear combinations of vectors.

Let $\{e_1,e_2,\ldots,e_n\}$ be an orthonormal system of vectors in a unitary space $X=(X,\langle\cdot,\cdot\rangle)$.

For $x \in X$, we put

$$\hat{x} = x - \sum_{k=1}^{n} \langle x, e_k \rangle e_k \text{ and } S_n(x,y) = \langle x,y \rangle - \sum_{k=1}^{n} \langle x, e_k \rangle \langle e_k, y \rangle ,$$

where $x, y \in X$.

In [5], Dragomir proved the following inequality

$$[S_n(x,y)]^2 \leqslant S_n(x,x) S_n(y,y) \tag{2.12}$$

where $x, y \in X$.

In relation (2.12) the equality holds if and only if $\{x, y, e_1, e_2, \ldots, e_n\}$ is linearly dependent.

For $n = 1$, we apply the inequality (2.12) on $C^0[a,b]$ for $e_1 = \frac{1}{\sqrt{b-a}}$, $x = \frac{1}{\sqrt{b-a}} f$, $y = \frac{1}{\sqrt{b-a}} g$, where $f, g \in C^0[a,b]$, and we obtain an inequality in terms of the Chebyshev functional, as follows:

$$[T(f,g)]^2 \leqslant T(f,f) T(g,g), \tag{2.13}$$

where $f, g \in C^0[a,b]$ and

$$T(f,g) = \frac{1}{b-a} \int_a^b f(x) g(x) \, dx - \frac{1}{b-a} \int_a^b f(x) \, dx \frac{1}{b-a} \int_a^b g(x) \, dx.$$

This inequality implies the Grüss inequality, as follows for $f, g \in C^0[a,b]$ with $\gamma_1 \leqslant f(x) \leqslant \Gamma_1$ and $\gamma_2 \leqslant g(x) \leqslant \Gamma_2$, where $\gamma_1, \gamma_2, \Gamma_1, \Gamma_2$ are four constants, we have $T(f,f) \leqslant \frac{1}{4} (\Gamma_1 - \gamma_1)^2$, so we obtain

$$|T(f,g)| \leqslant \frac{1}{4} (\Gamma_1 - \gamma_1)(\Gamma_2 - \gamma_2).$$

We also see an improvement of a Grüss type discrete inequality in an inner product space [17].

THEOREM 4. *With the above notations, we have*

$$\begin{aligned} 0 &\leqslant \max\left\{\sqrt{S_n(x,x)}, \sqrt{S_n(y,y)}\right\} \left(\sqrt{S_n(x,x)} + \sqrt{S_n(y,y)} - \sqrt{S_n(x+y,x+y)}\right) \\ &\leqslant \sqrt{S_n(x,x) S_n(y,y)} - S_n(x,y) \qquad (2.14) \\ &\leqslant \max\left\{\sqrt{S_n(x,x)}, \sqrt{S_n(y,y)}\right\} \left(\sqrt{S_n(x-y,x-y)} - \left|\sqrt{S_n(x,x)} - \sqrt{S_n(y,y)}\right|\right), \end{aligned}$$

$for all x, y \in X$.

Proof. From [9] we have the following identity:

$$\langle \hat{x}, \hat{y} \rangle = \langle x, y \rangle - \sum_{k=1}^{n} \langle x, e_k \rangle \langle e_k, y \rangle = S_n(x,y).$$

But, $\langle \hat{x}, \hat{y} \rangle = \langle \hat{x}, y \rangle = \langle x, \hat{y} \rangle$, so we deduce

$$\|\hat{x}\|^2 = \langle \hat{x}, \hat{x} \rangle = \langle \hat{x}, x \rangle = \langle x, \hat{x} \rangle = S_n(x,x) = \|\hat{x}\|^2 - \sum_{k=1}^{n} \langle x, e_k \rangle^2 .$$

Using Theorem 1, we have

$$\begin{aligned} &\max\{\|\hat{x}\|, \|\hat{y}\|\}(\|\hat{x}\| + \|\hat{y}\| - \|\hat{x}+\hat{y}\|) \leqslant \|\hat{x}\| \cdot \|\hat{y}\| - \langle \hat{x}, \hat{y} \rangle \\ &\leqslant \max\{\|\hat{x}\|, \|\hat{y}\|\}(\|\hat{x}-\hat{y}\| - |\|\hat{x}\| - \|\hat{y}\||), \end{aligned}$$

for all $\hat{x}, \hat{y} \in X$. Therefore, we deduce the inequality of the statement. □

REMARK 4. Multiplying by $\sqrt{S_n(x,x) S_n(y,y)} + S_n(x,y)$ in the inequality (2.14) we obtain an improvement of inequality (2.12).

3. Applications

a) In the inner product space $\left(C^0[a,b], \langle .,. \rangle\right)$, for $f, g \in C^0[a,b]$, we have $\langle f, g \rangle = \int_a^b f(x) g(x) dx$ and $\|f\| = \sqrt{\int_a^b f^2(x) dx}$. Similarly to the ones mentioned above, for $n = 1$, we apply inequality (2.14) on $C^0[a,b]$ for

$$\left\{ e_1 = \frac{1}{\sqrt{b-a}}, x = \frac{1}{\sqrt{b-a}} f, y = \frac{1}{\sqrt{b-a}} g \right\},$$

where $f, g \in C^0[a,b]$, and we obtain an improvement of inequality (2.13):

$$\begin{aligned} 0 &\leqslant \max\left\{\sqrt{T(f,f)}, \sqrt{T(g,g)}\right\} \left(\sqrt{T(f,f)} + \sqrt{T(g,g)} - \sqrt{T(f+g,f+g)}\right) \\ &\leqslant \sqrt{T(f,f) T(g,g)} - T(f,g) \\ &\leqslant \max\left\{\sqrt{T(f,f)}, \sqrt{T(g,g)}\right\} \left(\sqrt{T(f-g,f-g)} - \left|\sqrt{T(f,f)} - \sqrt{T(g,g)}\right|\right), \end{aligned} \tag{3.1}$$

where $f, g, h \in C^0[a,b]$.

Let $X = (X, \langle .,. \rangle)$ be an Euclidean space. For $n = 1$ in inequality (2.12) and the vector $e \in X$ with $\|e\| = 1$, we have

$$[\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle]^2 \leqslant \left(\|x\|^2 - \langle x, e \rangle^2\right)\left(\|y\|^2 - \langle y, e \rangle^2\right). \tag{3.2}$$

Dragomir [6] used this inequality to get a refinement of the Grüss inequality in an inner product space.

From inequality (2.14) we prove an improvement of inequality (3.2), thus:

$$\begin{aligned} 0 &\leqslant A + \langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle \leqslant \sqrt{\left(\|x\|^2 - \langle x, e \rangle^2\right)\left(\|y\|^2 - \langle y, e \rangle^2\right)} \\ &\leqslant B + \langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle, \end{aligned} \tag{3.3}$$

for all $x, y, e \in X$, with $\|e\| = 1$, where

$$A = \max\left\{\sqrt{\|x\|^2 - \langle x, e\rangle^2}, \sqrt{\|y\|^2 - \langle y, e\rangle^2}\right\}$$
$$\left(\sqrt{\|x\|^2 - \langle x, e\rangle^2} + \sqrt{\|y\|^2 - \langle y, e\rangle^2} - \sqrt{\|x+y\|^2 - \langle x+y, e\rangle^2}\right)$$

and

$$B = \max\left\{\sqrt{\|x\|^2 - \langle x, e\rangle^2}, \sqrt{\|y\|^2 - \langle y, e\rangle^2}\right\}$$
$$\left(\sqrt{\|x-y\|^2 - \langle x-y, e\rangle^2} - \left|\sqrt{\|x\|^2 - \langle x, e\rangle^2} - \sqrt{\|y\|^2 - \langle y, e\rangle^2}\right|\right).$$

b) In the inner product space

$$\left(R^n, \langle ., .\rangle\right), \text{ for } x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n),$$

we have $\langle x, y\rangle = x_1y_1 + x_2y_2 + \ldots + x_ny_n$ and $\|x\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$.

If we apply the inequality (2.14), then we deduce the inequality:

$$\begin{aligned} 0 &\leqslant \max\left\{\sqrt{\sum_{i=1}^n x_i^2}, \sqrt{\sum_{i=1}^n y_i^2}\right\}\left(\sqrt{\sum_{i=1}^n x_i^2} + \sqrt{\sum_{i=1}^n y_i^2} - \sqrt{\sum_{i=1}^n (x_i+y_i)^2}\right) \\ &\leqslant \sqrt{\sum_{i=1}^n x_i^2}\sqrt{\sum_{i=1}^n y_i^2} - \sum_{i=1}^n x_iy_i \qquad (3.4) \\ &\leqslant \max\left\{\sqrt{\sum_{i=1}^n x_i^2}, \sqrt{\sum_{i=1}^n y_i^2}\right\}\left(\sqrt{\sum_{i=1}^n (x_i-y_i)^2} - \left|\sqrt{\sum_{i=1}^n x_i^2} - \sqrt{\sum_{i=1}^n y_i^2}\right|\right). \end{aligned}$$

REMARK 5. Inequality (2.14) can be written as:

$$\begin{aligned} 0 &\leqslant \max\left\{\sqrt{\int_a^b f^2(x)\,dx}, \sqrt{\int_a^b g^2(x)\,dx}\right\} \\ &\qquad \left(\sqrt{\int_a^b f^2(x)\,dx} + \sqrt{\int_a^b g^2(x)\,dx} - \sqrt{\int_a^b (f(x)+g(x))^2\,dx}\right) \\ &\leqslant \sqrt{\int_a^b f^2(x)\,dx}\sqrt{\int_a^b g^2(x)\,dx} - \int_a^b f(x)\,g(x)\,dx \qquad (3.5) \\ &\leqslant \max\left\{\sqrt{\int_a^b f^2(x)\,dx}, \sqrt{\int_a^b g^2(x)\,dx}\right\} \\ &\qquad \left(\sqrt{\int_a^b (f(x)-g(x))^2\,dx} - \left|\sqrt{\int_a^b f^2(x)\,dx} - \sqrt{\int_a^b g^2(x)\,dx}\right|\right) \end{aligned}$$

Therefore, we obtain some improvements of the inequality of Cauchy-Schwarz in the discrete version and in the integral version.

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