Mathematical
Inequalities
& Applications
Volume 22, Number 4 (2019), 1145–1160 doi:10.7153/mia-2019-22-79

A GENERALIZATION OF G–METRIC SPACES AND RELATED FIXED POINT THEOREMS

KAPIL JAIN AND JATINDERDEEP KAUR*

(*Communicated by M. Praljak*)

Abstract. The idea of b-metric was proposed from the works of Bourbaki and Bakhtin. Czerwik gave an axiom which was weaker than the triangular inequality and formally defined b-metric spaces with a view of generalizing the Banach contraction mapping theorem. Further, in 2006, Mustafa and Sims have introduced an alternative more robust generalization of metric spaces to overcome fundamental flaws in B.C. Dhage$'s$ theory of generalized metric spaces and named it as G-metric spaces. In this paper, inspired by the concept of b-metric spaces and G-metric spaces, a new generalization of G-metric spaces (named as G_b-metric spaces) are introduced that recovers a large class of topological spaces including standard metric spaces, b-metric spaces, G-metric spaces *etc*. In such spaces, a new version of known fixed point theorems in b-metric spaces as well as in G-metric spaces have been proved. As an application of our result, we establish an existence and uniqueness result for system of linear equations in G_b-complete metric spaces.

1. Introduction

In 1989, Bakhtin [1] established b-metric spaces as an extension of metric spaces by defining a b-metric constant($s \geqslant 1$) in triangle inequality of metric axiom. This idea gave researchers to think in a magnificent way for their fixed point results.

DEFINITION 1. [1] Let X be a non empty set and let $s \geqslant 1$ be a given real number. A function $d : X \times X \to [0,\infty)$ is called a b-metric if for all $x,y,z \in X$,

(i) $d(x,y) = 0 \quad if\ and\ only\ if \quad x = y$,

(ii) $d(x,y) = d(y,x)$,

(iii) $d(x,z) \leqslant s[d(x,y)+d(y,z)]$.

A pair (X,d) is called a b-metric space.

Mathematics subject classification (2010): 47H10, 54H25.
Keywords and phrases: Fixed point, complete metric space, system of linear equations, contraction mappings.
* Corresponding author.

Paper MIA-22-79

EXAMPLE 1. Let $X = L_p[0,1]$ be the space of all real functions $x(t)$, $t \in [0,1]$ such that $\int_0^1 |x(t)|^p dt < \infty$ with $0 < p < 1$. Define $d : X \times X \to [0,\infty)$ as

$$d(x,y) = \left(\int_0^1 |x(t) - y(t)|^p dt\right)^{1/p}.$$

Then d is a b-metric space with coefficient $s = 2^{1/p}$.

EXAMPLE 2. Let $X = \mathbb{R}$ be the set of real numbers. Define $d : X \times X \to [0,\infty)$ as

$$d(x,y) = (x-y)^2.$$

Then d is a b-metric space with coefficient $s = 2$.

The above examples show that the class of b-metric spaces is larger than the class of metric spaces. When $s = 1$, the concept of b-metric spaces coincide with the concept of metric spaces.

DEFINITION 2. Let (X,d) be a b-metric space. A sequence $\{x_n\}$ in X is said to be:

(I) Cauchy if and only if $d(x_n,x_m) \to 0$ as $n,m \to \infty$.

(II) Convergent if and only if there exists $x \in X$ such that $d(x_n,x) \to 0$ as $n \to \infty$ and we write $\lim_{n\to\infty} x_n = x$.

(III) The b-metric space $(X,\ d)$ is complete if every Cauchy sequence is convergent.

The extension of Banach contraction principle in case of b-metric spaces proved in [11] reads as.

THEOREM 1. *Let (X,d) be a complete b-metric space with constant $s \geqslant 1$, such that b-metric is a continuous functional. Let $T : X \to X$ be a contraction having contraction constant $k \in [0,1)$ such that $ks < 1$. Then T has a unique fixed point.*

Generalizations of metric spaces were also proposed by Gahler ([7], [8]) (called 2-metric spaces) and Dhage ([4], [5], [6]) (called D-metric spaces) to extend the known fixed point theorems from metric spaces to these spaces. But later, different authors proved that these attempts are invalid (for detail see [9], [12], [17]). In 2005, Mustafa and Sims ([14]) introduced G-metric spaces as a generalization of metric spaces as follows:

DEFINITION 3. ([14]) Let X be a non empty set and $G : X \times X \times X \to [0,\infty)$ be a function satisfying:

(i) $G(x,y,z) = 0 \ if \ \ x = y = z,$

(ii) $0 < G(x,x,y)$; *for all* $x,y \in X$, *with* $x \neq y$,

(iii) $G(x,x,y) \leqslant G(x,y,z)$; *for all* $x,y,z \in X$, *with* $z \neq y$,

(iv) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \ldots($ *symmetric in all three variables*$)$,

(v) $G(x,y,z) \leqslant G(x,a,a) + G(a,y,z)$ *for all* $x,y,z,a \in X$, $($ *rectangle inequality*$)$,

then the function G is called a generalized metric or a G-metric on X, and the pair (X,G) is a G-metric space.

In this paper, a new class of metric spaces have been introduced which is generalization of standard metric spaces, b-metric spaces, G-metric spaces etc as below:

DEFINITION 4. Let X be a non empty set and $s \geqslant 1$ be a given real number. Let $G_b : X \times X \times X \to [0,\infty)$ be a function satisfying:

(i) $G_b(x,y,z) = 0$ *if* $x = y = z$,

(ii) $0 < G_b(x,x,y)$; *for all* $x,y \in X$, *with* $x \neq y$,

(iii) $G_b(x,x,y) \leqslant s\, G_b(x,y,z)$; *for all* $x,y,z \in X$, *with* $z \neq y$,

(iv) $G_b(x,y,z) = G_b(x,z,y) = G_b(y,z,x) = \ldots($ *symmetric in all three variables*$)$,

(v) $G_b(x,y,z) \leqslant s[G_b(x,a,a) + G_b(a,y,z)]$ *for all* $x,y,z,a \in X$,

then the function G_b is called generalized-G metric or a G_b-metric on X, and the pair (X,G_b) is a G_b-metric space.

REMARK 1. Every G-metric is a G_b-metric with $s = 1$. But converse need not be true.

The following example shows the existence of a G_b-metric which is not a G-metric.

EXAMPLE 3. Define a mapping $G_b : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ as :

$$G_b(x,y,z) = |x-y|^2 + |y-z|^2 + |z-x|^2, \quad for\ all\ x,y,z \in \mathbb{R},$$

then this mapping is not a G-metric, however, it is a G_b-metric with $s = 2$.

PROPOSITION 1. *Let* (X,G_b) *be a* G_b*-metric space. Then mapping* $d_{G_b} : X \times X \to [0,\infty)$ *defined by*

$$d_{G_b}(x,y) = G_b(x,x,y) + G_b(x,y,y), \quad for\ all\ x,y \in X,$$

is a b*-metric on* X.

DEFINITION 5. Let (X,G_b) be a G_b-metric space. Then for $x_0 \in X,\ \ r>0$, the open ball in (X,G_b) with center x_0 and radius r is

$$B_{G_b}(x_0,r)=\{y\in X : G(x_0,y,y)<r\}.$$

PROPOSITION 2. *Let (X,G_b) be a G_b-metric space with constant $s\geqslant 1$. Then for $x_0\in X,\ \ r>0$,*

$$B_{G_b}(x_0,r)\subseteq B_{d_{G_b}}(x_0,(1+2s)r)\subseteq B_{G_b}(x_0,(1+2s)r).$$

PROPOSITION 3. *Let (X,G_b) be a G_b-metric space with constant $s\geqslant 1$. Then for all $x,y\in X$,*

(i) $G_b(x,y,y)\leqslant 2sG_b(x,x,y)$,

(ii) $\frac{2s+1}{2s}G_b(x,y,y)\leqslant d_{G_b}(x,y)\leqslant (1+2s)G_b(x,y,y)$.

DEFINITION 6. Let (X,G_b) be a G_b-metric space. Then a subset A of X is said to be G_b-open if for each $a\in A$, there exists $r>0$ such that $B_{G_b}(a,r)\subseteq A$.

REMARK 2. A G_b-open ball in G_b-metric space need not be a G_b-open set.

DEFINITION 7. Let (X,G_b) be a G_b-metric space. Then a sequence $\{x_n\}$, $x_n\in X$, is said to be G_b-convergent to $x\in X$ if for each $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ (here $\mathbb{N}$ be the set of natural numbers) such that

$$G_b(x_n,x_n,x)<\varepsilon \qquad for\ \ all\ \ n\geqslant n_0.$$

PROPOSITION 4. *Let (X,G_b) be a G_b-metric space. Then following statements are equivalent.*

(i) *Sequence $\{x_n\}$ is G_b-convergent to x.*

(ii) $d_{G_b}(x_n,x)\to 0$ *as* $n\to\infty$.

(iii) $G_b(x_n,x_n,x)\to 0$ *as* $n\to\infty$.

(iv) $G_b(x_n,x,x)\to 0$ *as* $n\to\infty$.

(v) $G_b(x_n,x_m,x)\to 0$ *as* $n,m\to\infty$.

DEFINITION 8. Let (X,G_b) and (X',G'_b) be two G_b-metric spaces. Then a mapping $f:X\to X'$ is said to be G_b-continuous at $x_0\in X$ if for each $\varepsilon>0$, there exists $\delta>0$ such that

$$G'_b(f(x_0),f(x),f(x))<\varepsilon \quad whenever\ \ G_b(x_0,x,x)<\delta \quad for\ \ all\ \ x\in X.$$

REMARK 3. Continuity defined above is not equivalent to continuity of a function in (X,G_b) as a topological space.

PROPOSITION 5. *Let* (X,G_b) *and* (X',G'_b) *be two* G_b*-metric spaces. Then a mapping* $f:X\to X'$ *is* G_b*-continuous at* $x\in X$ *if and only if whenever sequence* $\{x_n\}$ *is* G_b*-convergent to* x, $\{f(x_n)\}$ *is* G'_b*-convergent to* $f(x)$.

DEFINITION 9. Let (X,G_b) be a G_b-metric space. Then a sequence $\{x_n\}$, $x_n\in X$, is said to be G_b-Cauchy sequence if for each $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that

$$G_b(x_n,x_m,x_l)<\varepsilon \quad for\ all\ \ n,m,l\geqslant n_0.$$

PROPOSITION 6. *Let* (X,G_b) *be a* G_b*-metric space. Then following statements are equivalent.*

(i) Sequence $\{x_n\}$ *is a* G_b*-Cauchy sequence.*

(ii) For each $\varepsilon>0$, *there exists* $n_0\in\mathbb{N}$ *such that* $G_b(x_n,x_m,x_m)<\varepsilon$ *for all* $n,m\geqslant n_0$.

(iii) Sequence $\{x_n\}$ *is a Cauchy sequence in* b*-metric space* $(X,\ d_{G_b})$.

PROPOSITION 7. *Every* G_b*-convergent sequence is a* G_b*-Cauchy sequence.*

DEFINITION 10. A G_b-metric space (X,G_b) is said to be G_b-complete if every G_b-Cauchy sequence is G_b-convergent sequence.

EXAMPLE 4. Let $X=\mathbb{R}$, and $G_b:X\times X\times X\to\mathbb{R}_+$ be defined as :

$$G_b(x,y,z)=|x-y|^2+|y-z|^2+|z-x|^2,\ \ for\ all\ \ x,y,z\in\mathbb{R}.$$

Then $(\mathbb{R},G_b)$ is a G_b-metric space with $s=2$.
We can easily prove that $(\mathbb{R},G_b)$ is G_b-complete with the help of Proposition 4 and Proposition 6.

2. Lemmas

We need the following lemmas for the proofs of our main results:

LEMMA 1. *Let* (X,G_b) *be a* G_b*-metric space with constant* $s\geqslant 1$ *and* $\{x_n\}$ *be any sequence in* (X,G_b). *Then*

$$G_b(x_0,x_k,x_k)\leqslant s^n\sum_{i=0}^{k-1}G_b(x_i,x_{i+1},x_{i+1})$$

for every $n\in\mathbb{N}$ *and* $k\in\{1,2,3,...,2^n\}$.

Proof. We denote $P(n)$ by the statement: For every $n\in\mathbb{N}$ and $k\in\{1,2,3,...,2^n\}$,

$$G_b(x_0,x_k,x_k)\leqslant s^n\sum_{i=0}^{k-1}G_b(x_i,x_{i+1},x_{i+1}).$$

It is easy to prove that $P(0)$ and $P(1)$ are true. Now we prove that $P(n)$ implies $P(n+1)$.
Case 1: If $k\in\{1,2,3,...,2^n\}$, then using $P(n)$ and $s\geqslant 1$, we have

$$G_b(x_0,x_k,x_k)\leqslant s^n\sum_{i=0}^{k-1}G_b(x_i,x_{i+1},x_{i+1})\leqslant s^{n+1}\sum_{i=0}^{k-1}G_b(x_i,x_{i+1},x_{i+1}).$$

Case 2: If $k\in\{2^n+1,2^n+2,2^n+3,...,2^{n+1}\}$, then using $P(n)$, we have

$$\begin{aligned}G_b(x_0,x_k,x_k)&\leqslant s(G_b(x_0,x_{2^n},x_{2^n})+G_b(x_{2^n},x_k,x_k))\\&\leqslant s\left(s^n\sum_{i=0}^{2^n-1}G_b(x_i,x_{i+1},x_{i+1})+s^n\sum_{i=2^n}^{k-1}G_b(x_i,x_{i+1},x_{i+1})\right)\\&=s^{n+1}\sum_{i=0}^{k-1}G_b(x_i,x_{i+1},x_{i+1}).\quad\square\end{aligned}$$

LEMMA 2. *Let (X,G_b) be a G_b-metric space with constant $s\geqslant 1$ and $\{x_n\}$ be any sequence in (X,G_b) such that*

$$G_b(x_n,x_{n+1},x_{n+1})\leqslant kG_b(x_{n-1},x_n,x_n)$$

for every $n\in\mathbb{N}$ and for some $k\in[0,1)$, then $\{x_n\}$ is a G_b-Cauchy sequence.

Proof. Applying induction on given condition we have,

$$G_b(x_n,x_{n+1},x_{n+1})\leqslant k^nG_b(x_0,x_1,x_1). \tag{1}$$

Let $m,l\in\mathbb{N}$ and $p=[\log_2 l]$. Consider,

$$\begin{aligned}&G_b(x_{m+1},x_{m+l},x_{m+l})\\&\leqslant sG_b(x_{m+1},x_{m+2},x_{m+2})+sG_b(x_{m+2},x_{m+l},x_{m+l})\\&\leqslant sG_b(x_{m+1},x_{m+2},x_{m+2})+s^2G_b(x_{m+2},x_{m+2^2},x_{m+2^2})+s^2G_b(x_{m+2^2},x_{m+l},x_{m+l})\\&\leqslant sG_b(x_{m+1},x_{m+2},x_{m+2})+s^2G_b(x_{m+2},x_{m+2^2},x_{m+2^2})+s^3G_b(x_{m+2^2},x_{m+2^3},x_{m+2^3})\\&\quad+s^3G_b(x_{m+2^3},x_{m+l},x_{m+l})\\&\leqslant\ldots\leqslant\sum_{n=1}^{p}s^nG_b(x_{m+2^{n-1}},x_{m+2^n},x_{m+2^n})+s^{p+1}G_b(x_{m+2^p},x_{m+l},x_{m+l})\\&\leqslant\sum_{n=1}^{p}s^{2n}\left(\sum_{i=m}^{m+2^{n-1}-1}G_b(x_{2^{n-1}+i},x_{2^{n-1}+i+1},x_{2^{n-1}+i+1})\right)\end{aligned}$$

$$+s^{2(p+1)}\left(\sum_{i=m}^{m+l-2^p-1} G_b(x_{2^p+i},x_{2^p+i+1},x_{2^p+i+1})\right)$$
$$\leqslant \sum_{n=1}^{p+1} s^{2n}\left(\sum_{i=m}^{m+2^{n-1}-1} G_b(x_{2^{n-1}+i},x_{2^{n-1}+i+1},x_{2^{n-1}+i+1})\right),$$

$$\begin{aligned}
G_b(x_{m+1},x_{m+l},x_{m+l}) &\leqslant G_b(x_0,x_1,x_1)\sum_{n=1}^{p+1} s^{2n}\left(\sum_{i=0}^{2^{n-1}-1} k^{m+2^{n-1}+i}\right)\\
&\leqslant \frac{G_b(x_0,x_1,x_1)k^m}{1-k}\sum_{n=1}^{p+1} s^{2n}k^{2^{n-1}}\\
&\leqslant \frac{G_b(x_0,x_1,x_1)k^m}{1-k}\sum_{n=1}^{\infty} k^{2n\log_k s+2^{n-1}}.
\end{aligned}$$

Since $\lim\limits_{n\to\infty}(2n\log_k s+2^{n-1}-n)=\infty$, so for $\lambda>0$, there exists $n_0\in\mathbb{N}$ such that

$$2n\log_k s+2^{n-1}-n>\lambda \qquad for\ \ all\ \ n\geqslant n_0,$$

which implies that

$$k^{2n\log_k s+2^{n-1}}\leqslant k^{\lambda+n} \qquad for\ \ all\ \ n\geqslant n_0.$$

Hence, the series $\sum\limits_{n=1}^{\infty} k^{2n\log_k s+2^{n-1}}$ is convergent and denote the sum by μ, then

$$G_b(x_{m+1},x_{m+k},x_{m+k})\leqslant \frac{G_b(x_0,x_1,x_1)k^m\mu}{1-k}$$

for all $m,k\in\mathbb{N}$. But $k\in[0,1)$, so by using Proposition 6, $\{x_n\}$ is a G_b-Cauchy sequence. □

3. Fixed point theorems in the context of G_b - metric spaces

Our first result is the following theorem.

THEOREM 2. *Let (X,G_b) be a G_b-complete metric space with constant $s\geqslant 1$ and $T:X\to X$ be a mapping such that*

$$G_b(Tx,Ty,Tz)\leqslant kG_b(x,y,z) \quad for\ \ all\ \ x,y,z\in X,$$

where $k\in[0,\frac{1}{s})$. Then T has a unique fixed point.

Proof. Let $x_0\in X$ be arbitrary. Define a sequence $\{x_n\}$ in X by

$$x_n=T^n(x_0) \quad for\ \ all \quad n\in\mathbb{N}.$$

Then for each $n \in \mathbb{N}$, we have

$$G_b(x_n, x_{n+1}, x_{n+1}) \leqslant kG_b(x_{n-1}, x_n, x_n).$$

Now an easy induction gives that

$$G_b(x_n, x_{n+1}, x_{n+1}) \leqslant k^n G_b(x_0, x_1, x_1).$$

Let $n, m \in \mathbb{N}$ with $n < m$, then we have,

$$\begin{aligned}
&G_b(x_n, x_m, x_m) \\
&\leqslant s[G_b(x_n, x_{n+1}, x_{n+1}) + G_b(x_{n+1}, x_m, x_m)] \\
&\leqslant sG_b(x_n, x_{n+1}, x_{n+1}) + s^2[G_b(x_{n+1}, x_{n+2}, x_{n+2}) + G_b(x_{n+2}, x_m, x_m)] \\
&\leqslant \ldots \leqslant sG_b(x_n, x_{n+1}, x_{n+1}) + s^2 G_b(x_{n+1}, x_{n+2}, x_{n+2}) + \ldots + s^{m-n} G_b(x_{m-1}, x_m, x_m) \\
&\leqslant (sk^n + s^2 k^{n+1} + \ldots + s^{m-n} k^{m-1}) G_b(x_0, x_1, x_1) \leqslant \frac{sk^n}{1 - sk} G_b(x_0, x_1, x_1),
\end{aligned}$$

so $G_b(x_n, x_m, x_m) \longrightarrow 0$ as $n, m \to \infty$. Thus $\{x_n\}$ is a G_b-Cauchy sequence. But (X, G_b) is a G_b-complete, therefore, there exists $x' \in X$ such that $x_n \to x'$. Now $T(x') = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = x'$. Thus x' is a fixed point of T. Let y be another fixed point of T. Then

$$G_b(x', y, y) = G_b(Tx', Ty, Ty) \leqslant kG_b(x', y, y)$$

which implies that $G_b(x', y, y) = 0$ as $k \in [0, 1)$, therefore, $x' = y$, that is, x' is a unique fixed point of T. $\square$

EXAMPLE 5. Let $X = \mathbb{R}$, and $G_b : X \times X \times X \to \mathbb{R}_+$ be defined as:

$$G_b(x, y, z) = |x - y|^2 + |y - z|^2 + |z - x|^2, \ \ for \ \ all \ \ x, y, z \in \mathbb{R}.$$

Then (X, G_b) is a G_b-complete metric space with $s = 2$.
Define $T : X \to X$ by

$$T(x) = \frac{x}{2}, \quad x \in X.$$

Then

$$G_b(Tx, Ty, Tz) = G_b(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}) = \left|\frac{x}{2} - \frac{y}{2}\right|^2 + \left|\frac{y}{2} - \frac{z}{2}\right|^2 + \left|\frac{z}{2} - \frac{x}{2}\right|^2 \leqslant kG_b(x, y, z),$$

where $k = \frac{1}{4} \in [0, \frac{1}{s})$. Also T has a unique fixed point namely 0.

EXAMPLE 6. Let $X = \{\alpha, \beta, \gamma\}$ and define a mapping $G_b : X \times X \times X \to \mathbb{R}_+$ as follows:

$G_b(x, x, x) = 0$ for all $x \in X$,

$$G_b(\alpha,\beta,\beta)=G_b(\beta,\alpha,\beta)=G_b(\beta,\beta,\alpha)=G_b(\alpha,\alpha,\beta)=G_b(\alpha,\beta,\alpha)=G_b(\beta,\alpha,\alpha)=1,$$
$$G_b(\alpha,\gamma,\gamma)=G_b(\gamma,\alpha,\gamma)=G_b(\gamma,\gamma,\alpha)=G_b(\alpha,\alpha,\gamma)=G_b(\alpha,\gamma,\alpha)=G_b(\gamma,\alpha,\alpha)=1.2,$$
$$G_b(\beta,\gamma,\gamma)=G_b(\gamma,\beta,\gamma)=G_b(\gamma,\gamma,\beta)=G_b(\beta,\beta,\gamma)=G_b(\beta,\gamma,\beta)=G_b(\gamma,\beta,\beta)=1.3,$$
$$G_b(x,y,z)=3.3 \text{ for all } x,y,z\in X \text{ with } x\neq y\neq z\neq x.$$

Then it is easy to prove that (X,G_b) is a G_b-metric space with constant $s=1.5$ (here 1.5 is the smallest possible value of s). Also we notice that, with $x=\alpha,\ y=\beta,\ z=\gamma$,

$$G_b(x,y,z)\nleqslant G_b(x,\beta,\beta)+G_b(\beta,y,z),$$

however,

$$G_b(x,y,z)\leqslant s[G_b(x,\beta,\beta)+G_b(\beta,y,z)].$$

Also it is very easy to show that (X,G_b) is a G_b-complete. Define a mapping $T:X\to X$ by $T\alpha=\alpha,\ T\beta=\alpha,\ T\gamma=\beta$. Now for $k=\frac{5}{6}\in[0,1)$, it is not difficult to prove that

$$G_b(Tx,Ty,Tz)\leqslant kG_b(x,y,z) \quad for\ \ all\ \ x,y,z\in X,$$

and $\frac{5}{6}$ is the smallest value of such k. Here T has a unique fixed point, namely α, however $k\notin[0,\frac{1}{s})$. So it is need to extend the interval $[0,\frac{1}{s})$ in Theorem 2.

In this theorem, we consider $k\in[0,1)$ rather $k\in[0,\frac{1}{s})$.

THEOREM 3. *Let (X,G_b) be a G_b-complete metric space with constant $s\geqslant 1$ and $T:X\to X$ be a mapping such that*

$$G_b(Tx,Ty,Tz)\leqslant kG_b(x,y,z) \quad for\ \ all\ \ x,y,z\in X, \tag{2}$$

where $k\in[0,1)$. Then T has a unique fixed point.

Proof. Let $x_0\in X$ be arbitrary. Define a sequence $\{x_n\}$ in X by

$$x_n=T^n(x_0) \quad for\ \ all\ \ n\in\mathbb{N}.$$

Then for each $n\in\mathbb{N}$, we have

$$G_b(x_n,x_{n+1},x_{n+1})\leqslant kG_b(x_{n-1},x_n,x_n),$$

so by Lemma 2, $\{x_n\}$ is a G_b-Cauchy sequence. But (X,G_b) is a G_b-complete, therefore, there exists $x'\in X$ such that $x_n\to x'$. Now $T(x')=T(\lim_{n\to\infty}x_n)=\lim_{n\to\infty}T(x_n)=\lim_{n\to\infty}x_{n+1}=x'$. Thus x' is a fixed point of T. Let y be another fixed point of T. Then

$$G_b(x',y,y)=G_b(Tx',Ty,Ty)\leqslant kG_b(x',y,y)$$

which implies that $G_b(x',y,y)=0$ as $k\in[0,1)$, therefore, $x'=y$, that is, x' is a unique fixed point of T. $\square$

The next theorems are generalization of Theorem 2 and Theorem 3.

THEOREM 4. *Let (X,G_b) be a G_b-complete metric space with constant $s \geqslant 1$ and $T : X \to X$ be a mapping such that*

$$G_b(Tx,Ty,Tz) \leqslant k[G_b(x,y,z)+G_b(x,Tx,Tx)+G_b(y,Ty,Ty)+G_b(z,Tz,Tz)]$$

for all $x,y,z \in X$, where $k \in [0,\lambda)$ and $\lambda = min\{\frac{1}{4},\frac{1}{2s}\}$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}$ in X by

$$x_n = T^n(x_0) \quad for \;\; all \;\; n \in \mathbb{N}.$$

Then for each $n \in \mathbb{N}$, we have

$$G_b(x_n,x_{n+1},x_{n+1}) \leqslant k[2G_b(x_{n-1},x_n,x_n)+2G_b(x_n,x_{n+1},x_{n+1})]$$

which implies that

$$G_b(x_n,x_{n+1},x_{n+1}) \leqslant \frac{2k}{1-2k}G_b(x_{n-1},x_n,x_n).$$

As $k < \frac{1}{4}$, therefore, $\frac{2k}{1-2k} < 1$ and hence by Lemma 2, $\{x_n\}$ is a G_b-Cauchy sequence. But (X,G_b) is a G_b-complete, therefore, there exists $x' \in X$ such that $x_n \to x'$. Now consider,

$$\begin{aligned} G_b(x',Tx',Tx') &\leqslant s[G_b(x',x_n,x_n)+G_b(x_n,Tx',Tx')] \\ &\leqslant sG_b(x',x_n,x_n)+sk[G_b(x_{n-1},x',x')+G_b(x_{n-1},x_n,x_n)+2G_b(x',Tx',Tx')] \end{aligned}$$

which gives that

$$(1-2ks)G_b(x',Tx',Tx') \leqslant sG_b(x',x_n,x_n)+sk[G_b(x_{n-1},x',x')+G_b(x_{n-1},x_n,x_n)].$$

Letting $n \to \infty$, we have

$$G_b(x',Tx',Tx') = 0$$

which immediately tells that $Tx' = x'$, that is, x' is a fixed point of T. Let y be another fixed point of T. Then

$$G_b(x',y,y) = G_b(Tx',Ty,Ty) \leqslant kG_b(x',y,y)$$

which implies that $G_b(x',y,y) = 0$ as $k < \frac{1}{4}$, therefore, $x' = y$, that is, x' is a unique fixed point of T. □

THEOREM 5. *Let (X,G_b) be a G_b-complete metric space with constant $s \geqslant 1$ and $T : X \to X$ be a mapping such that*

$$G_b(Tx,Ty,Tz) \leqslant AG_b(x,y,z)+BG_b(x,Tx,Tx)+CG_b(y,Ty,Ty)+DG_b(z,Tz,Tz) \tag{3}$$

for all $x,y,z \in X$, where $A+B+C+D < 1$, $s(C+D) < 1$ and $A+B \geqslant 0$. Then either T has a unique fixed point or all elements of X are fixed points of T.

Proof. Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}$ in X by

$$x_n = T^n(x_0) \quad for \ all \ n \in \mathbb{N}.$$

Then for each $n \in \mathbb{N}$, we have

$$\begin{aligned} &G_b(x_n, x_{n+1}, x_{n+1}) \\ \leqslant &AG_b(x_{n-1}, x_n, x_n) + BG_b(x_{n-1}, x_n, x_n) + CG_b(x_n, x_{n+1}, x_{n+1}) + DG_b(x_n, x_{n+1}, x_{n+1}), \end{aligned}$$

which implies that

$$G_b(x_n, x_{n+1}, x_{n+1}) \leqslant \frac{A+B}{1-(C+D)} G_b(x_{n-1}, x_n, x_n).$$

As $A+B+C+D < 1$, $A+B \geqslant 0$, therefore, by Lemma 2, $\{x_n\}$ is a G_b-Cauchy sequence. But (X, G_b) is a G_b-complete, therefore, there exists $x' \in X$ such that $x_n \to x'$. Now consider,

$$\begin{aligned} &G_b(x', Tx', Tx') \\ \leqslant\ &s[G_b(x', x_n, x_n) + G_b(x_n, Tx', Tx')] \\ \leqslant\ &sG_b(x', x_n, x_n) + sAG_b(x_{n-1}, x', x') + sBG_b(x_{n-1}, x_n, x_n) + s(C+D)G_b(x', Tx', Tx'), \end{aligned}$$

which gives that

$$(1-s(C+D))G_b(x', Tx', Tx') \leqslant sG_b(x', x_n, x_n) + s[AG_b(x_{n-1}, x', x') + BG_b(x_{n-1}, x_n, x_n)].$$

Letting $n \to \infty$, we have

$$G_b(x', Tx', Tx') = 0,$$

which gives that $Tx' = x'$, that is, x' is a fixed point of T. Now if all elements of X are not fixed points of T, then there exists some $x^* \in X$ such that $Tx^* \neq x^*$, so on putting $x = y = z = x^*$ in (3), we have that $0 \leqslant (B+C+D)\ G_b(x^*, Tx^*, Tx^*)$ which implies that $B+C+D \geqslant 0$. Thus $A < 1$ as $A+B+C+D < 1$. Let y be another fixed point of T. Then

$$\begin{aligned} G_b(x', y, y) &= G_b(Tx', Ty, Ty) \leqslant AG_b(x', y, y) + BG_b(x', Tx', Tx') + (C+D)G_b(y, Ty, Ty) \\ &= AG_b(x', y, y), \end{aligned}$$

which implies that $G_b(x', y, y) = 0$ as $A < 1$, therefore, $x' = y$, that is, x' is a unique fixed point of T. $\square$

Now, we furnish some examples related to Theorem 5.

EXAMPLE 7. Let $X = \{\alpha, \beta, \gamma, \delta\}$ and define a mapping $G_b : X \times X \times X \to \mathbb{R}_+$ as follows:

$$G_b(x, x, x) = 0 \text{ for all } x \in X,$$
$$G_b(\alpha, \beta, \beta) = G_b(\beta, \alpha, \beta) = G_b(\beta, \beta, \alpha) = G_b(\alpha, \alpha, \beta) = G_b(\alpha, \beta, \alpha)$$

$= G_b(\beta,\alpha,\alpha) = 2,$
$G_b(\alpha,\gamma,\gamma) = G_b(\gamma,\alpha,\gamma) = G_b(\gamma,\gamma,\alpha) = G_b(\alpha,\alpha,\gamma) = G_b(\alpha,\gamma,\alpha) = G_b(\gamma,\alpha,\alpha)$
$= 1,$
$G_b(\alpha,\delta,\delta) = G_b(\delta,\alpha,\delta) = G_b(\delta,\delta,\alpha) = G_b(\alpha,\alpha,\delta) = G_b(\alpha,\delta,\alpha)$
$= G_b(\delta,\alpha,\alpha) = 1,$
$G_b(\beta,\gamma,\gamma) = G_b(\gamma,\beta,\gamma) = G_b(\gamma,\gamma,\beta) = G_b(\beta,\beta,\gamma) = G_b(\beta,\gamma,\beta) = G_b(\gamma,\beta,\beta)$
$= 2.1,$
$G_b(\beta,\delta,\delta) = G_b(\delta,\beta,\delta) = G_b(\delta,\delta,\beta) = G_b(\beta,\beta,\delta) = G_b(\beta,\delta,\beta)$
$= G_b(\delta,\beta,\beta) = 1.3,$
$G_b(\gamma,\delta,\delta) = G_b(\delta,\gamma,\delta) = G_b(\delta,\delta,\gamma) = G_b(\gamma,\gamma,\delta) = G_b(\gamma,\delta,\gamma) = G_b(\delta,\gamma,\gamma)$
$= 4.3,$
$G_b(x,y,z) = 5$ for all $x,y,z \in X$ with $x \neq y \neq z \neq x$.

Then it is easy to prove that (X,G_b) is a G_b-metric space with constant $s = 2.6$. Also we notice that, with $x = \delta,\ y = \beta,\ z = \gamma$,

$$G_b(x,y,z) \nleqslant G_b(x,\beta,\beta) + G_b(\beta,y,z),$$

however,

$$G_b(x,y,z) \leqslant s[G_b(x,\beta,\beta) + G_b(\beta,y,z)].$$

Also it is quite obvious that (X,G_b) is a G_b-complete. Define a mapping $T : X \to X$ by $T\alpha = \alpha,\ T\beta = \delta,\ T\gamma = \delta,\ T\delta = \alpha$. Now for $A = 0.7,\ B = 0.07,\ C = 0.08,\ D = 0.09$, we have $A+B+C+D < 1,\ s(C+D) < 1$ and $A+B \geqslant 0$. Also it is not hard to prove that (3) holds true. But for $x = y = \alpha,\ z = \gamma$, we notice that (2) does not hold true for any $k \in [0,1)$, however T has only one fixed point, namely α.

EXAMPLE 8. Let (X,G_b) be a G_b-complete metric space as in previous Example 7. Define a mapping $T : X \to X$ by $T\alpha = \alpha,\ T\beta = \beta,\ T\gamma = \gamma,\ T\delta = \delta$. Then for $A = 1.4,\ B = -0.2,\ C = -3,\ D = -1$, we have $A+B+C+D < 1,\ s(C+D) < 1$ and $A+B \geqslant 0$. Also it is easy to see that (3) holds true, however all elements of X are fixed points of T.

THEOREM 6. *Let (X,G_b) be a G_b-complete metric space with constant $s \geqslant 1$ and $T : X \to X$ be a mapping such that:*

(i) $G_b(Tx,Ty,Tz) \leqslant k\max\{G_b(x,y,z),\ G_b(x,Ty,Ty),\ G_b(y,Tx,Tx),\ G_b(z,Tz,Tz)\}$ *for all $x,\ y,\ z \in X$ and for some $k \in \left[0, \frac{1}{2s}\right)$;*

(ii) *T is G_b-continuous or $G_b(x_n,y,z) \to G_b(x,y,z)$ whenever $x_n \to x$ for all $y,z \in X$*

then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}$ in X by

$$x_n = T^n(x_0) \quad for\ \ all\ \ n \in \mathbb{N}.$$

Then for each $n \in \mathbb{N}$, we have

$$\begin{aligned}
&G_b(x_n,x_{n+1},x_{n+1}) \\
&\leqslant k\max\{G_b(x_{n-1},x_n,x_n),\ G_b(x_{n-1},x_{n+1},x_{n+1}),\ G_b(x_n,x_n,x_n),\ G_b(x_n,x_{n+1},x_{n+1})\} \\
&\leqslant k\max\{G_b(x_{n-1},x_n,x_n),\ G_b(x_{n-1},x_{n+1},x_{n+1})\}.
\end{aligned}$$

Case I: If $\max\{G_b(x_{n-1},x_n,x_n),\ G_b(x_{n-1},x_{n+1},x_{n+1})\} = G_b(x_{n-1},x_n,x_n)$, then

$$G_b(x_n,x_{n+1},x_{n+1}) \leqslant kG_b(x_{n-1},x_n,x_n).$$

Case II: If $\max\{G_b(x_{n-1},x_n,x_n),\ G_b(x_{n-1},x_{n+1},x_{n+1})\} = G_b(x_{n-1},x_{n+1},x_{n+1})$, then

$$G_b(x_n,x_{n+1},x_{n+1}) \leqslant kG_b(x_{n-1},x_{n+1},x_{n+1}) \leqslant ks[G_b(x_{n-1},x_n,x_n) + G_b(x_n,x_{n+1},x_{n+1})],$$

which implies that

$$G_b(x_n,x_{n+1},x_{n+1}) \leqslant \frac{ks}{1-ks}G_b(x_{n-1},x_n,x_n).$$

As $k \in [0,\frac{1}{2s})$, so in both cases by Lemma 2, $\{x_n\}$ is a G_b-Cauchy sequence. But (X,G_b) is a G_b-complete, therefore, there exists $x' \in X$ such that $x_n \to x'$.
Case I: If T is G_b-continuous, then $T(x') = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = x'$.
Case II: If G_b is continuous with respect to first variable, then

$$\begin{aligned}
G_b(x',Tx',Tx') &\leqslant s[G_b(x',x_n,x_n) + G_b(x_n,Tx',Tx')] \\
&\leqslant sG_b(x',x_n,x_n) + sk\max\big\{G_b(x_{n-1},x',x'),\ G_b(x_{n-1},Tx',Tx'), \\
&\qquad G_b(x',x_n,x_n),\ G_b(x',Tx',Tx')\big\}.
\end{aligned}$$

Letting $n \to \infty$ and using the fact that G_b is continuous with respect to first variable, we have

$$G_b(x',Tx',Tx') \leqslant skG_b(x',Tx',Tx').$$

But $sk < 1$, therefore,

$$G_b(x',Tx',Tx') = 0$$

which implies that $Tx' = x'$, that is, x' is a fixed point of T. Let y be another fixed point of T. Then, in view of Proposition 3,

$$\begin{aligned}
G_b(x',y,y) &= G_b(Tx',Ty,Ty) \\
&\leqslant k\max\{G_b(x',y,y),\ G_b(x',Ty,Ty),\ G_b(y,Tx',Tx'),\ G_b(y,Ty,Ty)\} \\
&= k\max\{G_b(x',y,y),\ G_b(x',y,y),\ G_b(y,x',x'),\ G_b(y,y,y)\} \leqslant kG_b(y,x',x') \\
&\leqslant k[2sG_b(x',y,y)],
\end{aligned}$$

which gives that $G_b(x',y,y) = 0$ as $2ks < 1$, therefore, $x' = y$. Thus T has a unique fixed point. $\square$

4. Application

As an application of the fixed point theorem for contractions on a G_b-complete metric space, we provide an existence and uniqueness result for system of linear equations.

THEOREM 7. *In a system of linear equations*

$$Ax = b \tag{4}$$

where $A = [a_{ij}]$ *is a* $n \times n$ *matrix and* $b = [b_i]$ *is a column vector of constants and* $x = [x_i]$ *is a column matrix of* n *unknowns, if*

$$|a_{ii} + 1| + \sum_{j=1, j \neq i}^{n} |a_{ij}| < 1 \qquad for\ \ all\ \ i = 1, 2, \dots, n. \tag{5}$$

then system has a unique solution.

Proof. Let $X = \{[x_i] \mid\ x_i\ is\ real\ for\ all\ i = 1\ to\ n,\ n\ being\ fixed\}$ and $G_b : X \times\ X \times\ X \to \mathbb{R}_+$ be defined as:

$$G_b(x, y, z) = \max_{i=1}^{n} |x_i - y_i|^2 + \max_{i=1}^{n} |y_i - z_i|^2 + \max_{i=1}^{n} |z_i - x_i|^2,$$

for all $x = [x_i],\ y = [y_i],\ z = [z_i] \in X$. Then clearly (X, G_b) is a G_b-complete metric space with constant $s = 2$. Now define a $n \times n$ matrix $C = [c_{ij}]$ by

$$c_{ij} = \begin{cases} a_{ij} + 1, & if\ \ i = j \\ a_{ij}, & if\ \ i \neq j. \end{cases}$$

Then given system (4) reduces to

$$x = Cx - b. \tag{6}$$

Also given condition (5) becomes

$$\sum_{j=1}^{n} |c_{ij}| < 1 \qquad for\ \ all\ \ i = 1, 2, \dots, n. \tag{7}$$

Now define a mapping $T : X \to\ X$ by

$$Tx = Cx - b, \qquad where\ \ x \in X.$$

Now for $x = [x_i],\ \ y = [y_i],\ \ z = [z_i] \in\ X$, set $p = Tx,\ \ q = Ty,\ \ r = Tz$ and suppose that $p = [p_i],\ \ q = [q_i],\ \ r = [r_i]$, then

$$p_i = \sum_{j=1}^{n} a_{ij} x_i - b_i \quad (i = 1, 2, \dots, n) \quad etc.$$

Consider,

$$
\begin{aligned}
& G_b(Tx,Ty,Tz) \\
&= \max_{i=1}^{n}|p_i-q_i|^2+\max_{i=1}^{n}|q_i-r_i|^2+\max_{i=1}^{n}|r_i-p_i|^2 \\
&= \max_{i=1}^{n}\left|\sum_{j=1}^{n} c_{ij}(x_i-y_i)\right|^2+\max_{i=1}^{n}\left|\sum_{j=1}^{n} c_{ij}(y_i-z_i)\right|^2+\max_{i=1}^{n}\left|\sum_{j=1}^{n} c_{ij}(z_i-x_i)\right|^2 \\
&\leqslant \max_{i=1}^{n}\left(\sum_{j=1}^{n}|c_{ij}||x_i-y_i|\right)^2+\max_{i=1}^{n}\left(\sum_{j=1}^{n}|c_{ij}||y_i-z_i|\right)^2+\max_{i=1}^{n}\left(\sum_{j=1}^{n}|c_{ij}||z_i-x_i|\right)^2 \\
&\leqslant \left(\max_{k=1}^{n}|x_k-y_k|^2+\max_{k=1}^{n}|y_k-z_k|^2+\max_{k=1}^{n}|z_k-x_k|^2\right)\max_{i=1}^{n}\left(\sum_{j=1}^{n}|c_{ij}|\right)^2 \\
&= G_b(x,y,z)\max_{i=1}^{n}\left(\sum_{j=1}^{n}|c_{ij}|\right)^2 = \alpha G_b(x,y,z),
\end{aligned}
$$

where

$$
\alpha = \max_{i=1}^{n}\left(\sum_{j=1}^{n}|c_{ij}|\right)^2 .
$$

By the condition (7), $\alpha \in [0,1)$, therefore, using Theorem 3, T has a unique fixed point and hence system (4) has a unique solution. $\square$

REFERENCES

[1] I. A. BAKHTIN, *The Contraction Mapping Principle in Almost Metric Spaces*, Functional Analysis **30**, (1989), 26–37.

[2] N. BOURBAKI, *Topologie Generale* Herman: Paris, France, (1974).

[3] S. CZERWIK, *Contraction Mappings in b-Metric Spaces*, Acta Mathematica et Informatica Universitatis Ostraviensis **1**, (1993), 5–11.

[4] B. C. DHAGE, *A Study of Some Fixed Point Theorem* Ph.D. Thesis, Marathwada University of Aurangabad, India, (1984).

[5] B. C. DHAGE, *On Generalized Metric Spaces and Topological Structure-II*, Pure and Applied Mathematika Sciences, **40**, (1994) 37–41.

[6] B. C. DHAGE, *Generalized Metric Spaces and Topological Structure I*, Analele stintifice Ale Universitatii Alexandru Ioan Cuza Iasi Tomul Xlvi,s.I a Matematica, **46**, (2000), 3–24.

[7] S. GAHLER, *2-Metrische Raume Und Ihre Topologische Struktur*, Mathematische Nachrichten, **26**, (1963), 115–148.

[8] S. GAHLER, *Zur Geometric 2-Metriche Raume*, Revue Roumaine de Mathematiques Pures et Appliquees, **40** (1966), 664–669.

[9] K. S. HA, Y. J. CHO, AND A. WHITE, *Strictly Convex and Strictly 2-Convex 2-Normed Spaces*, Mathematica Japonica, **33**, (1988), 375–384.

[10] M. JLELI AND B. SAMET, *Remarks on G-Metric Spaces and Fixed Point Theorems*, Fixed Point Theory and Applications **2012**, (2012):210.

[11] N. KIR AND H. KIZILTUN, *On Some Well Known Fixed Point Theorems In b-Metric Spaces*, Turkish Journal of Analysis and Number Theory,**1**, (2013), 13–16.

[12] Z. MUSTAFA AND B. SIMS, *Some Remarks Concerning D-Metric Spaces*, Proceedings of the International Conferences on Fixed Point Theory and Applications, Valencia (Spain), July(2003), 189–198.

[13] Z. MUSTAFA, *A New Structure for Generalized Metric Spaces with Applications to Fixed Point Theory*, Ph.D. Thesis, The University of Newcastle, Callaghan, Australia, (2005).

[14] Z. MUSTAFA AND B. SIMS, *A New Approach to Generalized Metric Spaces*, Journal of Nonlinear and Convex Analysis, **7**, (2006), 289–297.

[15] Z. MUSTAFA, H. OBIEDAT, AND F. AWAWDEH, *Some Fixed Point Theorem for Mapping on Complete G-Metric Spaces*, Fixed Point Theory and Applications, (2008) Article ID 189870 12 pages.

[16] Z. MUSTAFA, W. SHATANAWI, AND M. BATAINEH, *Existence of Fixed Point Results in G-Metric Spaces*, International Journal of Mathematics and Mathematical Sciences, (2009) Article ID 283028 10 pages.

[17] S. V. R. NAIDU, K. P. R. RAO, AND N. S. RAO, *On the Concepts of Balls in a D-Metric Space*, International Journal of Mathematics and Mathematical Sciences, **2005** (1) (2005), 133–141.

(Received September 13, 2018)

Kapil Jain
School of Mathematics
Thapar Institute of Engineering and Technology
Patiala
e-mail: kjain_phdp16@thapar.edu

Jatinderdeep Kaur
School of Mathematics
Thapar Institute of Engineering and Technology
Patiala
e-mail: jkaur@thapar.edu