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FURTHER IMPROVEMENT OF AN EXTENSION OF HÖLDER–TYPE INEQUALITY

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Abstract. In 1995 Pearce and Pečarić proved an extension of Hölder's inequality. In this paper we extend their result in a measure theoretic sense and further improve it using log-convexity of related linear functionals. Moreover, we study the action of related linear functionals on families of exponentially convex functions.

1. Introduction

In [10] Pearce and Pečarić proved an extension of Hölder's inequality using the following generalization of Steffensen's inequality.

THEOREM 1.

(i) Suppose that f and g are integrable functions on $[a,b]$, f is nonincreasing and $\lambda > 0$. If a positive function g satisfies the condition

$$\lambda \int_a^x g(t)dt \leqslant (x-a) \int_a^b g(t)dt \tag{1}$$

for every $x \in [a,b]$, then

$$\frac{\int_a^b f(t)g(t)dt}{\int_a^b g(t)dt} \leqslant \lambda^{-1} \int_a^{a+\lambda} f(t)dt, \tag{2}$$

while if a positive function g satisfies

$$\lambda \int_x^b g(t)dt \leqslant (b-x) \int_a^b g(t)dt \tag{3}$$

for every $x \in [a,b]$, then

$$\lambda^{-1} \int_{b-\lambda}^b f(t)dt \leqslant \frac{\int_a^b f(t)g(t)dt}{\int_a^b g(t)dt}. \tag{4}$$

In either case equality holds if f is constant.

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(ii) If f is nondecreasing, the reverse inequalities hold in (2) and (4).

In the following theorem we recall the aforementioned extension of Hölder's inequality from [10].

THEOREM 2. *Let f and g be two integrable and positive functions defined on $[a,b]$ and let M, K be real numbers satisfying $a \leqslant K < M \leqslant b$.*

(i) Suppose that for every $x \in [K,b]$ we have

$$\frac{1}{x-K}\int_K^x g(t)dt \leqslant \frac{1}{M-K}\int_K^b g(t)dt, \tag{5}$$

that $p > 1$, $p^{-1}+q^{-1}=1$ and that f is nonincreasing. Then

$$\int_a^b f(t)g(t)dt \leqslant \left(\int_a^M f^p(t)dt\right)^{1/p}\left(\int_a^M \hat{g}^q(t)dt\right)^{1/q}, \tag{6}$$

where

$$\hat{g}(t) = \begin{cases} g(t), & a \leqslant t < K \\ \frac{1}{M-K}\int_K^b g(t)dt, & K \leqslant t \leqslant M. \end{cases} \tag{7}$$

The inequality in (6) is reversed if $p < 1$ and f is a nondecreasing function. In both cases, equality holds in (6) if

$$f^p(t) = c\hat{g}^q(t), \quad a \leqslant t \leqslant M$$

(where c is constant) and

$$f(t) = f(K), \quad t \in [K,b].$$

(ii) Suppose that for every $x \in [a,M]$ we have

$$\frac{1}{M-x}\int_x^M g(t)dt \leqslant \frac{1}{M-K}\int_a^M g(t)dt, \tag{8}$$

that $p > 1$, $p^{-1}+q^{-1}=1$ and that f is nondecreasing. Then

$$\int_a^b f(t)g(t)dt \leqslant \left(\int_K^b f^p(t)dt\right)^{1/p}\left(\int_K^b \hat{g}^q(t)dt\right)^{1/q}, \tag{9}$$

where

$$\hat{g}(t) = \begin{cases} \frac{1}{M-K}\int_a^M g(t)dt, & K \leqslant t \leqslant M \\ g(t), & M < t \leqslant b. \end{cases} \tag{10}$$

The inequality in (9) is reversed if $p < 1$ and f is a nonincreasing function. In both cases, equality holds in (9) if

$$f^p(t) = c\hat{g}^q(t), \quad K \leqslant t \leqslant b$$

(where c is constant) and

$$f(t) = f(M), \quad t \in [a,M].$$

In [13] Pečarić and Smoljak improved the above extension of Hölder's inequality using log-convexity. The aim of this paper is to obtain further improvement of an extension of Hölder's inequality in measure theory settings. For this purpose we use generalization of Steffensen's inequality for positive measures. First, let us recall that over the years Steffensen's inequality has been generalized in measure theory settings for various motivations, for example see [3], [4], [5], [7], [8], etc. For a complete survey see [14]. Throughout the paper by $\mathscr{B}([a,b])$ we denote Borel σ-algebra on $[a,b]$.

Let us conclude the introduction by generalization of Steffensen's inequality for positive measures proved in [9].

THEOREM 3. *Let μ be a finite, positive measure on $\mathscr{B}([a,b])$, $f:[a,b]\to\mathbb{R}$ nonincreasing, right-continuous function . Then*

$$\frac{\int_{[a,b]} f(t)G(t)d\mu(t)}{\int_{[a,b]} G(t)d\mu(t)} \leqslant \frac{\int_{[a,a+\lambda]} f(t)d\mu(t)}{\mu([a,a+\lambda])} \tag{11}$$

if and only if $G:[a,b]\to\mathbb{R}$ is $\mu-$integrable function and λ is a positive constant such that

$$\frac{\int_{[a,x)} G(t)d\mu(t)}{\int_{[a,b]} G(t)d\mu(t)} \leqslant \frac{\mu([a,x))}{\mu([a,a+\lambda])} \quad \text{and} \quad \int_{[x,b]} G(t)d\mu(t) \geqslant 0, \tag{12}$$

for every $x\in[a,b]$, assuming $\int_{[a,b]} G(t)d\mu(t)>0$.
For an increasing, right-continuous function $f:[a,b]\to\mathbb{R}$ inequality (11) is reversed.

2. Extension of Hölder's inequality for positive measures

By simple modification of Theorem 3 for an interval $[a,a+\lambda]$ and using similar reasoning for an interval $(b-\lambda,b]$ we have the following generalization of Steffensen's inequality for positive measures.

THEOREM 4.

(i) Let μ be a finite, positive measure on $\mathscr{B}([a,b])$. Suppose that g is $\mu-$integrable function on $[a,b]$, f is nonincreasing, right-continuous function on $[a,b]$ and λ is a positive constant. If a positive function g satisfies the condition

$$\mu([a,a+\lambda])\int_{[a,x)} g(t)d\mu(t) \leqslant \mu([a,x))\int_{[a,b]} g(t)d\mu(t) \tag{13}$$

for every $x\in[a,b]$, then

$$\frac{\int_{[a,b]} f(t)g(t)d\mu(t)}{\int_{[a,b]} g(t)d\mu(t)} \leqslant \frac{\int_{[a,a+\lambda]} f(t)d\mu(t)}{\mu([a,a+\lambda])} \tag{14}$$

while if a positive function g satisfies the condition

$$\mu((b-\lambda,b])\int_{[x,b]} g(t)d\mu(t) \leqslant \mu([x,b])\int_{[a,b]} g(t)d\mu(t) \tag{15}$$

for every $x \in [a,b]$*, then*

$$\frac{\int_{(b-\lambda,b]} f(t)d\mu(t)}{\mu((b-\lambda,b])} \leqslant \frac{\int_{[a,b]} f(t)g(t)d\mu(t)}{\int_{[a,b]} g(t)d\mu(t)}. \tag{16}$$

In either case equality holds if f *is constant.*

(ii) If f *is nondecreasing, right-continuous function, the reverse inequalities hold in (14) and (16).*

Using the above generalization of Steffensen's inequality we obtain the following extension of Hölder's inequality for positive measures.

THEOREM 5. *Let* μ *be a finite, positive measure on* $\mathscr{B}([a,b])$*. Let* f *and* g *be two* $\mu-$*integrable and positive functions defined on* $[a,b]$ *and let* M*,* K *be real numbers satisfying* $a \leqslant K < M \leqslant b$*.*

(i) Suppose that for every $x \in [K,b]$ *we have*

$$\frac{1}{\mu([K,x))} \int_{[K,x)} g(t)d\mu(t) \leqslant \frac{1}{\mu([K,M])} \int_{[K,b]} g(t)d\mu(t), \tag{17}$$

that $p > 1$*,* $p^{-1} + q^{-1} = 1$ *and that* f *is a nonincreasing, right-continuous function. Then*

$$\int_{[a,b]} f(t)g(t)d\mu(t) \leqslant \left(\int_{[a,M]} f^p(t)d\mu(t)\right)^{1/p} \left(\int_{[a,M]} \hat{g}^q(t)d\mu(t)\right)^{1/q}, \tag{18}$$

where

$$\hat{g}(t) = \begin{cases} g(t), & a \leqslant t < K \\ \frac{1}{\mu([K,M])} \int_{[K,b]} g(t)d\mu(t), & K \leqslant t \leqslant M. \end{cases} \tag{19}$$

The inequality in (18) is reversed if $p < 1$ *and* f *is a nondecreasing, right-continuous function.*

(ii) Suppose that for every $x \in [a,M]$ *we have*

$$\frac{1}{\mu([x,M])} \int_{[x,M]} g(t)d\mu(t) \leqslant \frac{1}{\mu((K,M])} \int_{[a,M]} g(t)d\mu(t), \tag{20}$$

that $p > 1$*,* $p^{-1} + q^{-1} = 1$ *and* f *is a nondecreasing, right-continuous function. Then*

$$\int_{[a,b]} f(t)g(t)d\mu(t) \leqslant \left(\int_{(K,b]} f^p(t)d\mu(t)\right)^{1/p} \left(\int_{(K,b]} \hat{g}^q(t)d\mu(t)\right)^{1/q}, \tag{21}$$

where

$$\hat{g}(t) = \begin{cases} \frac{1}{\mu((K,M])} \int_{[a,M]} g(t)d\mu(t), & K \leqslant t \leqslant M \\ g(t), & M < t \leqslant b. \end{cases} \tag{22}$$

The inequality in (21) is reversed if $p < 1$ *and* f *is a nonincreasing, right-continuous function.*

Proof.

(i) Let $\lambda = M-K$ and replace a by K in Theorem 4 (i). Now, by (17) we have that condition (13) is satisfied. Hence, (14) holds, that is

$$\begin{aligned}\int_{[K,b]} f(t)g(t)d\mu(t) &\leqslant \frac{1}{\mu([K,M])}\int_{[K,b]} g(t)d\mu(t)\int_{[K,M]} f(t)d\mu(t)\\ &= \int_{[K,M]} f(t)\hat{g}(t)d\mu(t).\end{aligned}$$

So,

$$\begin{aligned}\int_{[a,b]} f(t)g(t)d\mu(t) &= \int_{[a,K)} f(t)g(t)d\mu(t) + \int_{[K,b]} f(t)g(t)d\mu(t)\\ &\leqslant \int_{[a,K)} f(t)\hat{g}(t)d\mu(t) + \int_{[K,M]} f(t)\hat{g}(t)d\mu(t)\\ &= \int_{[a,M]} f(t)\hat{g}(t)d\mu(t).\end{aligned}$$

Now using Hölder's inequality, inequality (18) follows.

(ii) Let $\lambda = M-K$ and replace b by M in Theorem 4 (ii). Now, by (20) we have that condition (15) is satisfied. Since f is nondecreasing reversed inequality in (16) holds, that is

$$\begin{aligned}\int_{[a,M]} f(t)g(t)d\mu(t) &\leqslant \frac{1}{\mu((K,M])}\int_{[a,M]} g(t)d\mu(t)\int_{(K,M]} f(t)d\mu(t)\\ &= \int_{(K,M]} f(t)\hat{g}(t)d\mu(t).\end{aligned}$$

So

$$\begin{aligned}\int_{[a,b]} f(t)g(t)d\mu(t) &= \int_{[a,M]} f(t)g(t)d\mu(t) + \int_{(M,b]} f(t)g(t)d\mu(t)\\ &\leqslant \int_{(K,M]} f(t)\hat{g}(t)d\mu(t) + \int_{(M,b]} f(t)\hat{g}(t)d\mu(t)\\ &= \int_{(K,b]} f(t)\hat{g}(t)d\mu(t).\end{aligned}$$

Now using Hölder's inequality, inequality (21) follows.

The other cases follow similarly. □

COROLLARY 1.

(i) Suppose the assumptions of Theorem 5(i) are satisfied and further g is nonincreasing. Then Theorem 5(i) is also valid if condition (17) is replaced by

$$g(K) \leqslant \frac{1}{\mu([K,M])}\int_{[K,b]} g(t)d\mu(t).$$

(ii) Suppose the assumptions of Theorem 5(ii) are satisfied and further g is nondecreasing. Then Theorem 5(ii) is also valid if condition (20) is replaced by

$$g(M) \leqslant \frac{1}{\mu((K,M])} \int_{[a,M]} g(t)d\mu(t).$$

Proof. If g is nonincreasing, then

$$\frac{1}{\mu([K,x])} \int_{[K,x)} g(t)d\mu(t) \leqslant g(K) \leqslant \frac{1}{\mu([K,M])} \int_{[K,b]} g(t)d\mu(t),$$

that is, (17) holds. Similarly, if g is nondecreasing, then

$$\frac{1}{\mu([x,M])} \int_{[x,M]} g(t)d\mu(t) \leqslant g(M) \leqslant \frac{1}{\mu((K,M])} \int_{[a,M]} g(t)dt,$$

that is, (20) holds. □

3. Applications

We begin with definitions and properties of classes of exponentially convex and n-exponentially convex functions. For more details see [1], [2], [6] and [11].

DEFINITION 1. A function $f: I \to \mathbb{R}$ is *exponentially convex on* I if it is continuous on I and

$$\sum_{i,j=1}^{n} \xi_i \xi_j f(x_i + x_j) \geqslant 0$$

for all $n \in \mathbb{N}$ and all choices $\xi_i \in \mathbb{R}, i = 1, \ldots, n$ such that $x_i + x_j \in I$, $1 \leqslant i, j \leqslant n$.

DEFINITION 2. A function $f: I \to \mathbb{R}$ is $n-$*exponentially convex in the Jensen sense on* I if

$$\sum_{i,j=1}^{n} \xi_i \xi_j f\left(\frac{x_i + x_j}{2}\right) \geqslant 0$$

for all choices $\xi_i \in \mathbb{R}$, $x_i \in I$. f is $n-$*exponentially convex on* I if it is $n-$exponentially convex in the Jensen sense and continuous on I.

REMARK 1.

(i) $f(x) = e^{\alpha x}$ is exponentially convex on $\mathbb{R}$, for any $\alpha \in \mathbb{R}$.

(ii) $g(x) = x^{-\alpha}$ is exponentially convex on $(0, \infty)$, for any $\alpha > 0$.
$\sum_{i,j=1}^{n} \xi_i \xi_j f\left(\frac{x_i + x_j}{2}\right) = \left(\sum_{i=1}^{n} \xi_i e^{\alpha \frac{x_i}{2}}\right)^2 \geqslant 0$ or from Theorem **??**

In the following lemma we give a family of functions which is useful in constructing exponentially convex functions. This family was also used in [9] so we omit the proof.

LEMMA 1. *For* $p \in \mathbb{R}$ *let* $\varphi_p : (0,\infty) \to \mathbb{R}$ *be defined with*

$$\varphi_p(x) = \begin{cases} \frac{x^p}{p}, & p \neq 0; \\ \log x, & p = 0. \end{cases} \tag{23}$$

Then $x \mapsto \varphi_p(x)$ *is increasing on* $(0,\infty)$ *for each* $p \in \mathbb{R}$ *and* $p \mapsto \varphi_p(x)$ *is exponentially convex on* $(0,\infty)$ *for each* $x \in (0,\infty)$.

Using characterization of convexity by monotonicity of first order divided differences it follows (see [12, p. 4]):

THEOREM 6. *Let* $I \subseteq \mathbb{R}$ *be an open interval. Let* $f : I \to (0,\infty)$ *be log-convex, differentiable function on* I *and* $M : I \times I \to (0,\infty)$ *be defined with*

$$M(x,y) = \begin{cases} \left(\frac{f(x)}{f(y)}\right)^{\frac{1}{x-y}}, & x \neq y; \\ \exp\left(\frac{f'(x)}{f(x)}\right), & x = y. \end{cases}$$

If $x_1, x_2, y_1, y_2 \in I$ *such that* $x_1 \leqslant x_2$, $y_1 \leqslant y_2$ *then*

$$M(x_1,y_1) \leqslant M(x_2,y_2).$$

Let φ_p be defined by (23). Under assumptions of Theorem 5 (i), let us define the following linear functional

$$\mathfrak{L}(\varphi_p \circ f) = \int_{[a,b]} \varphi_p(f(t))g(t)d\mu(t) - \int_{[a,M]} \varphi_p(f(t))\hat{g}(t)d\mu(t) \tag{24}$$

which is positive on a class of nondecreasing, right-continuous functions f.

Also, we have that $-\mathfrak{L}(\varphi_p \circ f)$ is positive on a class of nonincreasing, right-continuous functions f.

THEOREM 7. *Let* $\Phi : \mathbb{R} \to \mathbb{R}$ *be defined with*

$$\Phi(p) = \mathfrak{L}(\varphi_p \circ f)$$

where $\mathfrak{L}$ *is defined with (24),* φ_p *is defined by (23) and* f *is a nondecreasing, right-continuous function. Then the following statements hold:*

(i) The function Φ *is continuous on* $\mathbb{R}$.

(ii) If $n \in \mathbb{N}$ *and* $p_1, \dots, p_n \in \mathbb{R}$ *are arbitrary, then the matrix*

$$\left[\Phi\left(\frac{p_j + p_k}{2}\right)\right]_{j,k=1}^{n}$$

is positive semidefinite. Particularly,

$$det\left[\Phi\left(\frac{p_j + p_k}{2}\right)\right]_{j,k=1}^{n} \geqslant 0.$$

(iii) The function Φ is exponentially convex on $\mathbb{R}$.

(iv) The function Φ is log-convex on $\mathbb{R}$.

(v) If $r,s,t \in \mathbb{R}$ are such that $r<s<t$, then

$$\Phi(s)^{t-r} \leqslant \Phi(r)^{t-s}\Phi(t)^{s-r}.$$

Proof. (i) Continuity of the function $p \mapsto \Phi(p)$ is obvious for $p \in \mathbb{R}\setminus\{0\}$. For $p=0$ it is directly checked using Heine characterization.

(ii) Let $n \in \mathbb{N}$, $\xi_i, p_i \in \mathbb{R}$, $(i=1,\dots,n)$ be arbitrary and define auxiliary function $\Psi : (0,\infty) \to \mathbb{R}$ by

$$\Psi(x) = \sum_{j,k=1}^{n} \xi_j \xi_k \varphi_{\frac{p_j+p_k}{2}}(x).$$

Since

$$\Psi'(x) = \left(\sum_{j=1}^{n} \xi_j x^{\frac{p_j-1}{2}} \right)^2 \geqslant 0$$

we have that Ψ is increasing on $(0,\infty)$.

By (17), condition (13) is satisfied with $\lambda = M-K$ and a replaced by K. Hence, by Theorem 4, the reverse inequality in (14) holds, so for a nondecreasing function $\Psi \circ f$ we obtain

$$\int_{[K,b]} \Psi(f(t))g(t)d\mu(t) \geqslant \int_{[K,M]} \Psi(f(t))\hat{g}(t)d\mu(t)t.$$

By definition

$$\int_{[a,K)} \Psi(f(t))g(t)d\mu(t) = \int_{[a,K)} \Psi(f(t))\hat{g}(t)d\mu(t),$$

so we obtain

$$\int_{[a,b]} \Psi(f(t))g(t)d\mu(t) \geqslant \int_{[a,M]} \Psi(f(t))\hat{g}(t)d\mu(t),$$

that is, $\mathfrak{L}(\Psi \circ f) \geqslant 0$. This is means that

$$\left[\Phi\left(\frac{p_j+p_k}{2} \right) \right]_{j,k=1}^{n}$$

is a positive semi-definite matrix.
(iii), (iv), (v) are simple consequences of (i) and (ii). □

REMARK 2. Similarly as in Theorem 7 we obtain that for a nonincreasing, right-continuous function f statements of Theorem 7 hold for $-\Phi(p)$.

Hence, the following inequality holds true

$$|\Phi(s)|^{t-r} \leqslant |\Phi(r)|^{t-s}|\Phi(t)|^{s-r} \tag{25}$$

for every choice $r,s,t \in \mathbb{R}$ such that $r<s<t$.

In the following theorem we obtain an improvement of Hölder-type inequality in measure theory settings.

THEOREM 8. *Let μ be a finite, positive measure on $\mathscr{B}([a,b])$. Let f and g be two $\mu-$integrable and positive functions defined on $[a,b]$, let $\hat{g}$ be defined by (19) and let M, K be real numbers satisfying $a \leqslant K < M \leqslant b$. Suppose that for every $x \in [K,b]$ we have (17).*

(i) Suppose that $p > 1$, $p^{-1} + q^{-1} = 1$, $1 < s < t$ and that f is a nonincreasing, right-continuous function. Then

$$\left(\int_{[a,M]} f^p(t)d\mu(t)\right)^{1/p}\left(\int_{[a,M]} \hat{g}^q(t)d\mu(t)\right)^{1/q} - \int_{[a,b]} f(t)g(t)d\mu(t) \geqslant [-\Phi(s)]^{\frac{t-1}{t-s}}\,[-\Phi(t)]^{\frac{1-s}{t-s}}. \tag{26}$$

If $p < 1$ and f is a nondecreasing, right-continuous function, then

$$\int_{[a,b]} f(t)g(t)d\mu(t) - \left(\int_{[a,M]} f^p(t)d\mu(t)\right)^{1/p}\left(\int_{[a,M]} \hat{g}^q(t)d\mu(t)\right)^{1/q} \geqslant [\Phi(s)]^{\frac{t-1}{t-s}}\,[\Phi(t)]^{\frac{1-s}{t-s}}. \tag{27}$$

(ii) Suppose that $p > 1$, $p^{-1} + q^{-1} = 1$, $r < s < 1$ and that f is a nonincreasing, right-continuous function. Then

$$\left(\int_{[a,M]} f^p(t)d\mu(t)\right)^{1/p}\left(\int_{[a,M]} \hat{g}^q(t)d\mu(t)\right)^{1/q} - \int_{[a,b]} f(t)g(t)d\mu(t) \geqslant [-\Phi(s)]^{\frac{1-r}{s-r}}\,[-\Phi(r)]^{\frac{s-1}{s-r}}.$$

If $p < 1$ and f is a nondecreasing, right-continuous function, then

$$\int_{[a,b]} f(t)g(t)d\mu(t) - \left(\int_{[a,M]} f^p(t)d\mu(t)\right)^{1/p}\left(\int_{[a,M]} \hat{g}^q(t)d\mu(t)\right)^{1/q} \geqslant [\Phi(s)]^{\frac{1-r}{s-r}}\,[\Phi(r)]^{\frac{s-1}{s-r}}.$$

Proof.

(i) Taking substitution $r \to 1$ in (25) and then raising both sides of inequality (25) to the power $\frac{1}{t-s}$ we obtain

$$|\Phi(1)| \geqslant |\Phi(s)|^{\frac{t-1}{t-s}}|\Phi(t)|^{\frac{1-s}{t-s}}.$$

For a nonincreasing function f, we have

$$|\Phi(1)| = -\Phi(1) = \int_{[a,M]} f(t)\hat{g}(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) \geqslant 0.$$

Now by Hölder's inequality we have

$$\begin{aligned}\left(\int_{[a,M]} f^p(t)d\mu(t)\right)^{1/p}\left(\int_{[a,M]} \hat{g}^q(t)d\mu(t)\right)^{1/q} - \int_{[a,b]} f(t)g(t)d\mu(t)\\ \geqslant \int_{[a,M]} f(t)\hat{g}(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t)\\ = -\Phi(1) \geqslant [-\Phi(s)]^{\frac{t-1}{t-s}}[-\Phi(t)]^{\frac{1-s}{t-s}}.\end{aligned}$$

Hence, we obtain (26).

For a nondecreasing function f, we have

$$|\Phi(1)| = \Phi(1) = \int_{[a,b]} f(t)g(t)d\mu(t) - \int_{[a,M]} f(t)\hat{g}(t)d\mu(t) \geqslant 0.$$

Now by Hölder's inequality for $p<1$ we have

$$\begin{aligned}\int_{[a,b]} f(t)g(t)d\mu(t) - \left(\int_{[a,M]} f^p(t)d\mu(t)\right)^{1/p}\left(\int_{[a,M]} \hat{g}^q(t)d\mu(t)\right)^{1/q}\\ \geqslant \int_{[a,b]} f(t)g(t)d\mu(t) - \int_{[a,M]} f(t)\hat{g}(t)d\mu(t)\\ = \Phi(1) \geqslant [\Phi(s)]^{\frac{t-1}{t-s}}[\Phi(t)]^{\frac{1-s}{t-s}}.\end{aligned}$$

Hence, we obtain (27).

(ii) Similar to the proof of (i), taking substitution $t \to 1$. □

Let φ_p be defined by (23). Under assumptions of Theorem 5 (ii), let us define the following linear functional

$$\mathfrak{N}(\varphi_p \circ f) = \int_{(K,b]} \varphi_p(f(t))\hat{g}(t)d\mu(t) - \int_{[a,b]} \varphi_p(f(t))g(t)d\mu(t) \tag{28}$$

which is positive on a class of nondecreasing, right-continuous functions f.

Also, we have that $-\mathfrak{N}(\varphi_p \circ f)$ is positive on a class of nonincreasing, right-continuous functions f.

THEOREM 9. *Let $\Upsilon : \mathbb{R} \to \mathbb{R}$ be defined with*

$$\Upsilon(p) = \mathfrak{N}(\varphi_p \circ f)$$

where $\mathfrak{N}$ is defined with (28), φ_p is defined by (23) and f is a nondecreasing, right-continuous function. Then the following statements hold:

(i) The function Υ is continuous on $\mathbb{R}$.

(ii) If $n \in \mathbb{N}$ and $p_1,\dots,p_n \in \mathbb{R}$ are arbitrary, then the matrix

$$\left[\Upsilon\left(\frac{p_j+p_k}{2}\right)\right]_{j,k=1}^n$$

is positive semidefinite. Particularly,

$$det\left[\Upsilon\left(\frac{p_j+p_k}{2}\right)\right]_{j,k=1}^n \geqslant 0.$$

(iii) *The function* Υ *is exponentially convex on* $\mathbb{R}$.

(iv) *The function* Υ *is log-convex on* $\mathbb{R}$.

(v) *If* $r,s,t \in \mathbb{R}$ *are such that* $r < s < t$, *then*

$$\Upsilon(s)^{t-r} \leqslant \Upsilon(r)^{t-s}\Upsilon(t)^{s-r}.$$

Proof. Similar to the proof of Theorem 7. □

REMARK 3. Similarly as in Theorem 9 we obtain that for a nonincreasing, right-continuous function f statements of Theorem 9 hold for $-\Upsilon(p)$.

Hence, the following inequality holds true

$$|\Upsilon(s)|^{t-r} \leqslant |\Upsilon(r)|^{t-s}|\Upsilon(t)|^{s-r} \tag{29}$$

for every choice $r,s,t \in \mathbb{R}$ such that $r < s < t$.

In the following theorem we obtain another improvement of Hölder-type inequality in measure theory settings.

THEOREM 10. *Let* μ *be a finite, positive measure on* $\mathscr{B}([a,b])$. *Let* f *and* g *be two* $\mu-$*integrable and positive functions defined on* $[a,b]$, *let* $\hat{g}$ *be defined by (22) and let* M, K *be real numbers satisfying* $a \leqslant K < M \leqslant b$. *Suppose that for every* $x \in [a,M]$ *we have (20).*

(i) *Suppose that* $p > 1$, $p^{-1}+q^{-1} = 1$, $1 < s < t$ *and that* f *is a nondecreasing, right-continuous function. Then*

$$\left(\int_{(K,b]} f^p(t)d\mu(t)\right)^{1/p}\left(\int_{(K,b]} \hat{g}^q(t)d\mu(t)\right)^{1/q} - \int_{[a,b]} f(t)g(t)d\mu(t) \geqslant [\Upsilon(s)]^{\frac{t-1}{t-s}}[\Upsilon(t)]^{\frac{1-s}{t-s}}.$$

If $p < 1$ *and* f *is a nonincreasing, right-continuous function, then*

$$\int_{[a,b]} f(t)g(t)d\mu(t) - \left(\int_{(K,b]} f^p(t)d\mu(t)\right)^{1/p}\left(\int_{(K,b]} \hat{g}^q(t)d\mu(t)\right)^{1/q} \geqslant [-\Upsilon(s)]^{\frac{t-1}{t-s}}[-\Upsilon(t)]^{\frac{1-s}{t-s}}.$$

(ii) *Suppose that* $p > 1$, $p^{-1}+q^{-1} = 1$, $r < s < 1$ *and that* f *is a nondecreasing, right-continuous function. Then*

$$\left(\int_{(K,b]} f^p(t)d\mu(t)\right)^{1/p}\left(\int_{(K,b]} \hat{g}^q(t)d\mu(t)\right)^{1/q} - \int_{[a,b]} f(t)g(t)d\mu(t) \geqslant [\Upsilon(s)]^{\frac{1-r}{s-r}}[\Upsilon(r)]^{\frac{s-1}{s-r}}.$$

If $p < 1$ *and* f *is a nonincreasing, right-continuous function, then*

$$\int_{[a,b]} f(t)g(t)d\mu(t) - \left(\int_{(K,b]} f^p(t)d\mu(t)\right)^{1/p}\left(\int_{(K,b]} \hat{g}^q(t)d\mu(t)\right)^{1/q} \geqslant [-\Upsilon(s)]^{\frac{1-r}{s-r}}[-\Upsilon(t)]^{\frac{s-1}{s-r}}.$$

Proof. Similar to the proof of Theorem 8. □

We continue with Lagrange-type mean value theorems.

THEOREM 11. *Let f be a nondecreasing, right-continuous function and let $\psi \in C^1[f(a), f(b)]$. Let $\mathfrak{L}$ be a linear functional defined with (24). Then there exists $\xi \in [f(a), f(b)]$ such that*

$$\mathfrak{L}(\psi \circ f) = \psi'(\xi)\mathfrak{L}(\mathrm{id} \circ f),$$

where $\mathrm{id}(x) = x$.

Proof. Since $\psi \in C^1[f(a), f(b)]$ there exist

$$m = \min_{x \in [f(a), f(b)]} \psi'(x) \quad \text{and} \quad M = \max_{x \in [f(a), f(b)]} \psi'(x).$$

Denote $h_1(x) = Mx - \psi(x)$ and $h_2(x) = \psi(x) - mx$. Then

$$h_1'(x) = M - \psi'(x) \geqslant 0$$
$$h_2'(x) = \psi'(x) - m \geqslant 0$$

so h_1 and h_2 are nondecreasing on $[f(a), f(b)]$, which means that $\mathfrak{L}(h_1 \circ f) \geqslant 0$ and $\mathfrak{L}(h_2 \circ f) \geqslant 0$ i.e.

$$m\mathfrak{L}(\mathrm{id} \circ f) \leqslant \mathfrak{L}(\psi \circ f) \leqslant M\mathfrak{L}(\mathrm{id} \circ f).$$

If $\mathfrak{L}(\mathrm{id} \circ f) = 0$, the proof is complete. If $\mathfrak{L}(\mathrm{id} \circ f) > 0$, then

$$m \leqslant \frac{\mathfrak{L}(\psi \circ f)}{\mathfrak{L}(\mathrm{id} \circ f)} \leqslant M$$

and the existence of $\xi \in [f(a), f(b)]$ follows. □

THEOREM 12. *Let f be a nondecreasing, right-continuous function and let $\psi \in C^1[f(a), f(b)]$. Let $\mathfrak{N}$ be a linear functional defined with (28). Then there exists $\eta \in [f(a), f(b)]$ such that*

$$\mathfrak{N}(\psi \circ f) = \psi'(\eta)\mathfrak{N}(\mathrm{id} \circ f),$$

where $\mathrm{id}(x) = x$.

Proof. Similar to the proof of Theorem 11. □

Using, standard, Cauchy-type mean value theorem we obtain the following corollary.

COROLLARY 2. *Let f be a nondecreasing, right-continuous function and let ψ_1, $\psi_2 \in C^1[f(a), f(b)]$. Then there exist $\xi, \eta \in [f(a), f(b)]$, such that*

$$\frac{\psi_1'(\xi)}{\psi_2'(\xi)} = \frac{\mathfrak{L}(\psi_1 \circ f)}{\mathfrak{L}(\psi_2 \circ f)} \quad \text{and} \quad \frac{\psi_1'(\eta)}{\psi_2'(\eta)} = \frac{\mathfrak{N}(\psi_1 \circ f)}{\mathfrak{N}(\psi_2 \circ f)}, \tag{30}$$

provided that the denominator on right sides is non-zero, where $\mathfrak{L}$ and $\mathfrak{N}$ are linear functionals defined with (24) and (28).

REMARK 4. If the inverse of ψ_1'/ψ_2' exists then various kinds of means can be defined by (30). That is

$$\xi = \left(\frac{\psi_1'}{\psi_2'}\right)^{-1}\left(\frac{\mathfrak{L}(\psi_1\circ f)}{\mathfrak{L}(\psi_2\circ f)}\right) \quad \text{and} \quad \eta = \left(\frac{\psi_1'}{\psi_2'}\right)^{-1}\left(\frac{\mathfrak{N}(\psi_1\circ f)}{\mathfrak{N}(\psi_2\circ f)}\right). \tag{31}$$

Particularly, if we substitute $\psi_1(x) = \varphi_p(x)$, $\psi_2(x) = \varphi_q(x)$, where φ_p is defined by (23), in (31) and use continuous extension, the following expressions are obtained:

$$M(p,q) = \begin{cases} \left(\frac{\mathfrak{L}(\varphi_p\circ f)}{\mathfrak{L}(\varphi_q\circ f)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{\mathfrak{L}(\varphi_0\cdot(\varphi_p\circ f))}{\mathfrak{L}(\varphi_p\circ f)} - \frac{1}{p}\right), & p = q, \end{cases}$$

and

$$\widehat{M}(p,q) = \begin{cases} \left(\frac{\mathfrak{N}(\varphi_p\circ f)}{\mathfrak{N}(\varphi_q\circ f)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{\mathfrak{N}(\varphi_0\cdot(\varphi_p\circ f))}{\mathfrak{N}(\varphi_p\circ f)} - \frac{1}{p}\right), & p = q. \end{cases}$$

By Theorem 6, if $p,q,u,v \in (0,\infty)$ such that $p \leqslant u$, $q \leqslant v$ then,

$$M(p,q) \leqslant M(u,v) \quad \text{and} \quad \widehat{M}(p,q) \leqslant \widehat{M}(u,v).$$

Similar as in [7] we see that we can further refine obtained results by dropping some of analytical properties of family of functions from Lemma 1. Proofs are similar to the ones in [7] so we omit the details.

By

$$\mathscr{C} = \{\psi_p : \ \psi_p : [a,b] \to \mathbb{R},\ p \in J\}$$

let us define a family of functions from $C([a,b])$ such that $p \mapsto [x_0,x_1;\psi_p]$ is log-convex in the Jensen sense on J for every choice of two distinct points $x_0,x_1 \in [a,b]$.

THEOREM 13. *Let $G_i : J \to \mathbb{R}$, be defined with*

$$G_1(p) = \mathfrak{L}(\psi_p\circ f) \text{ and } G_2(p) = \mathfrak{N}(\psi_p\circ f) \tag{32}$$

where functionals $\mathfrak{L}$ and $\mathfrak{N}$ are defined with (24) and (28), $\psi_p \in \mathscr{C}$ and f is a non-decreasing right-continuous function. Then the following statements hold, for every $i = 1,2$:

(i) G_i is log-convex in the Jensen sense on J.

(ii) If G_i is continuous on J, then it is log-convex on J and for $p,q,r \in J$ such that $p < q < r$, we have

$$G_i(q)^{r-p} \leqslant G_i(p)^{r-q} G_i(r)^{q-p}. \tag{33}$$

(iii) If G_i is positive and differentiable on J, then for every $p,q,r \in J$ such that $p \leqslant u$, $q \leqslant v$, we have

$$\widetilde{M}_i(p,q) \leqslant \widetilde{M}_i(u,v) \tag{34}$$

where $\widetilde{M}_i(p,q)$ *is defined with*

$$\widetilde{M}_i(p,q)=\begin{cases}\left(\frac{G_i(p)}{G_i(q)}\right)^{\frac{1}{p-q}}, & p\neq q;\\ \exp\left(\frac{\frac{d}{dp}(G_i(p))}{G_i(p)}\right), & p=q.\end{cases} \tag{35}$$

REMARK 5. Using inequality (33) we could also obtain improvements of Hölder-type inequality as in Theorems 8 and 10.

By

$$\mathscr{D}=\{\psi_p:\ \psi_p:[a,b]\to\mathbb{R},\ p\in J\},$$

let us define a family of functions from $C([a,b])$ such that $p\mapsto[x_0,x_1;\psi_p]$ is exponentially convex on J for every choice of two distinct points $x_0,x_1\in[a,b]$.

THEOREM 14. *Let* $H_i:J\to\mathbb{R}$, *be defined with*

$$H_i(p)=\mathfrak{L}(\psi_p\circ f) \text{ and } H_2(p)=\mathfrak{N}(\psi_p\circ f)$$

where functionals $\mathfrak{L}$ *and* $\mathfrak{N}$ *are defined with (24) and (28),* $\psi_p\in\mathscr{D}$ *and* f *is a non-decreasing right-continuous function. Then the following statements hold for every* $i=1,2$*:*

(i) If $n\in\mathbb{N}$ *and* $p_1,\dots,p_n\in\mathbb{R}$ *are arbitrary, then the matrix*

$$\left[H_i\left(\frac{p_k+p_m}{2}\right)\right]_{k,m=1}^{n}$$

is positive semidefinite. Particularly,

$$det\left[H_i\left(\frac{p_k+p_m}{2}\right)\right]_{k,m=1}^{n}\geqslant 0.$$

(ii) If the function H_i *is continuous on* J, *then* H_i *is exponentially convex on* J.

(iii) If H_i *is positive and differentiable on* J, *then for every* $p,q,r\in J$ *such that* $p\leqslant u$, $q\leqslant v$, *we have*

$$\widetilde{N}_i(p,q)\leqslant\widetilde{N}_i(u,v)$$

where $\widetilde{N}_i(p,q)$ *is defined with*

$$\widetilde{N}_i(p,q)=\begin{cases}\left(\frac{H_i(p)}{H_i(q)}\right)^{\frac{1}{p-q}}, & p\neq q;\\ \exp\left(\frac{\frac{d}{dp}(H_i(p))}{H_i(p)}\right), & p=q.\end{cases}$$

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