Mathematical
Inequalities
& Applications
Volume 22, Number 4 (2019), 1177–1191 doi:10.7153/mia-2019-22-81

INTEGRAL ERROR REPRESENTATION OF HERMITE INTERPOLATING POLYNOMIALS AND RELATED GENERALIZATIONS OF STEFFENSEN'S INEQUALITY

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(*Communicated by J. Jakšetić*)

Abstract. Some representations of Steffensen's inequality are obtained by using Hermite interpolating polynomials. The obtained representations are used to prove new generalizations of Steffensen's inequality for $n-$ convex functions and to give some bounds for integrals in these representations.

1. Introduction

We will first mention some results regarding Hermite interpolation polynomials used in this paper (for details see [1]). Let $-\infty < a \leqslant a_1 < a_2 < \ldots < a_r \leqslant b < \infty$, $(r \geqslant 2)$ be given. For $f \in C^n[a,b]$ there exists a unique polynomial P_H of degree $n-1$, called the Hermite interpolating polynomial of the function f, fulfilling the following Hermite conditions:

$$P_H^{(i)}(a_j) = f^{(i)}(a_j), \quad 0 \leqslant i \leqslant k_j,\ 1 \leqslant j \leqslant r,\ \sum_{j=1}^{r} k_j + r = n.$$

Notice that Hermite conditions include the following particular cases:

Simple Hermite or Osculatory conditions $(n = 2m, r = m, k_j = 1$ for all $j)$

$$P_O(a_j) = f(a_j),\ P_O'(a_j) = f'(a_j),\ 1 \leqslant j \leqslant m,$$

Lagrange conditions $(r = n, k_j = 0$ for all $j)$

$$P_L(a_j) = f(a_j),\ 1 \leqslant j \leqslant n,$$

Mathematics subject classification (2010): 26D15, 26A51.

Keywords and phrases: Steffensen's inequality, Hermite interpolating polynomial, Green's function, $n-$ convex function.

The research of the first author was supported by the Ministry of Education and Science of the Russian Federation (the Agreement number No. 02a03.21.0008).

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Type $(m,n-m)$ **conditions** $(r=2, a_1=a, a_2=b, 1\leqslant m\leqslant n-1,\ k_1=m-1, k_2=n-m-1)$

$$P_{mn}^{(i)}(a)=f^{(i)}(a),\ 0\leqslant i\leqslant m-1,$$
$$P_{mn}^{(i)}(b)=f^{(i)}(b),\ 0\leqslant i\leqslant n-m-1,$$

One-point Taylor conditions $(r=1,k_1=n-1)$

$$P_T^{(i)}(a)=f^{(i)}(a),\ 0\leqslant i\leqslant n-1,$$

Two-point Taylor conditions $(n=2m, r=2, a_1=a, a_2=b, k_1=k_2=m-1)$

$$P_{2T}^{(i)}(a)=f^{(i)}(a),\ P_{2T}^{(i)}(b)=f^{(i)}(b),\ 0\leqslant i\leqslant m-1.$$

In [1] the following result is given:

THEOREM 1. *Let* $f\in C^n[a,b]$, *and let* P_H *be its Hermite interpolating polynomial. Then*

$$\begin{aligned} f(t) &= P_H(t)+e_H(t) \\ &= \sum_{j=1}^{r}\sum_{i=0}^{k_j} H_{ij}(t)f^{(i)}(a_j)+\int_a^b G_{H,n}(t,s)f^{(n)}(s)ds, \end{aligned} \tag{1}$$

where H_{ij} *are fundamental polynomials of the Hermite basis defined by*

$$H_{ij}(t)=\frac{1}{i!}\frac{\omega(t)}{(t-a_j)^{k_j+1-i}}\sum_{k=0}^{k_j-i}\frac{1}{k!}\frac{d^k}{dt^k}\left(\frac{(t-a_j)^{k_j+1}}{\omega(t)}\right)\Big|_{t=a_j}(t-a_j)^k, \tag{2}$$

where

$$\omega(t)=\prod_{j=1}^{r}(t-a_j)^{k_j+1}, \tag{3}$$

and $G_{H,n}$ *is Green's function for Hemite interpolation given by*

$$G_{H,n}(t,s)=\begin{cases} \sum\limits_{j=1}^{\ell}\sum\limits_{i=0}^{k_j}\frac{(a_j-s)^{n-i-1}}{(n-i-1)!}H_{ij}(t),\ s\leqslant t, \\ -\sum\limits_{j=\ell+1}^{r}\sum\limits_{i=0}^{k_j}\frac{(a_j-s)^{n-i-1}}{(n-i-1)!}H_{ij}(t),\ s\geqslant t, \end{cases} \tag{4}$$

for all $a_\ell\leqslant s\leqslant a_{\ell+1}, \ell=0,1,\dots,r$ $(a_0=a,\ a_{r+1}=b)$.

The following lemma describes positivity of Green's function (4) (see Beesack [3] and Levin [6]).

LEMMA 1. *Green's function* $G_{H,n}(t,s)$ *given by* (4) *has the following properties:*

(i) $\dfrac{G_{H,n}(t,s)}{\omega(t)} > 0, \quad \text{for } a_1 \leqslant t \leqslant a_r,\ a_1 < s < a_r;$

(ii) $G_{H,n}(t,s) \leqslant \dfrac{1}{(n-1)!(b-a)}|\omega(t)|;$

(iii) $\displaystyle\int_a^b G_{H,n}(t,s)ds = \frac{\omega(t)}{n!}.$

The aim of this paper is to obtain some new generalizations of Steffensen's inequality for $n-$convex functions using Hermite polynomials. The well-known Steffensen inequality states ([10]) :

THEOREM 2. *Suppose that f is nonincreasing and g is integrable on $[a,b]$ with $0 \leqslant g \leqslant 1$ and $\lambda = \int_a^b g(t)dt$. Then we have*

$$\int_{b-\lambda}^{b} f(t)dt \leqslant \int_a^b f(t)g(t)dt \leqslant \int_a^{a+\lambda} f(t)dt. \tag{5}$$

The inequalities are reversed for f nondecreasing.

Over the years Steffensen's inequality has been generalized in many ways. Extensive overviews of generalizations of Steffensen's inequality can be found in [7] and [9] (see also [2], [8]).

2. Generalizations of Steffensen's inequality by Hermite polynomial

Using Hermite polynomials we obtain the following representations of Steffensen's inequality.

THEOREM 3. *Let $-\infty < a \leqslant a_1 < a_2 \ldots < a_r \leqslant b < \infty$, $(r \geqslant 2)$ be given points and $f \in C^n[a,b]$. Let $g,p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive, $0 \leqslant g \leqslant 1$ and $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$. Let the function $\mathcal{G}_1$ be defined by*

$$\mathcal{G}_1(x) = \begin{cases} \int_a^x (1-g(t))p(t)dt, & x \in [a,a+\lambda], \\ \int_x^b g(t)p(t)dt, & x \in [a+\lambda,b]. \end{cases} \tag{6}$$

Then

$$\begin{aligned} &\int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt + \sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i+1)}(a_j)\int_a^b \mathcal{G}_1(x)H_{ij}(x)dx \\ &= -\int_a^b \left(\int_a^b \mathcal{G}_1(x)G_{H,n-1}(x,s)dx\right) f^{(n)}(s)ds \end{aligned} \tag{7}$$

where H_{ij} are defined on $[a,b]$ by (2) and $G_{H,n-1}$ is Green's function defined by (4).

Proof. Using identity

$$\int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt = \int_a^{a+\lambda} f(t)(1-g(t))p(t)dt - \int_{a+\lambda}^b f(t)g(t)p(t)dt$$

and integration by parts we have

$$\begin{aligned}
&\int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt \\
&= \int_a^{a+\lambda} [f(t)-f(a+\lambda)][1-g(t)]p(t)dt + \int_{a+\lambda}^b [f(a+\lambda)-f(t)]g(t)p(t)dt \\
&= -\int_a^{a+\lambda} \left[\int_a^x (1-g(t))p(t)dt\right] df(x) - \int_{a+\lambda}^b \left[\int_x^b g(t)p(t)dt\right] df(x) \\
&= -\int_a^b \mathcal{G}_1(x)df(x) = -\int_a^b \mathcal{G}_1(x)f'(x)dx.
\end{aligned}$$

By Theorem 1 $f'(x)$ can be expressed as

$$f'(x) = \sum_{j=1}^{r}\sum_{i=0}^{k_j} H_{ij}(x)f^{(i+1)}(a_j) + \int_a^b G_{H,n}(x,s)f^{(n+1)}(s)ds. \tag{8}$$

Replacing n with $n-1$ in (8) and using that result we obtain

$$\begin{aligned}
\int_a^b \mathcal{G}_1(x)f'(x)dx = &\sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i+1)}(a_j)\int_a^b \mathcal{G}_1(x)H_{ij}(x)dx \\
&+ \int_a^b \mathcal{G}_1(x)\left(\int_a^b G_{H,n-1}(x,s)f^{(n)}(s)ds\right)dx.
\end{aligned} \tag{9}$$

After applying Fubini's theorem on the last term in (9) we obtain (7). □

THEOREM 4. *Let* $-\infty < a \leqslant a_1 < a_2 \ldots < a_r \leqslant b < \infty$, $(r \geqslant 2)$ *be given points and* $f \in C^n[a,b]$. *Let* $g,p : [a,b] \to \mathbb{R}$ *be integrable functions such that* p *is positive,* $0 \leqslant g \leqslant 1$ *and* $\int_{b-\lambda}^b p(t)dt = \int_a^b g(t)p(t)dt$. *Let the function* $\mathcal{G}_2$ *be defined by*

$$\mathcal{G}_2(x) = \begin{cases} \int_a^x g(t)p(t)dt, & x \in [a,b-\lambda], \\ \int_x^b (1-g(t))p(t)dt, & x \in [b-\lambda,b]. \end{cases} \tag{10}$$

Then

$$\begin{aligned}
&\int_a^b f(t)g(t)p(t)dt - \int_{b-\lambda}^b f(t)p(t)dt + \sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i+1)}(a_j)\int_a^b \mathcal{G}_2(x)H_{ij}(x)dx \\
&= -\int_a^b \left(\int_a^b \mathcal{G}_2(x)G_{H,n-1}(x,s)dx\right) f^{(n)}(s)ds,
\end{aligned} \tag{11}$$

where H_{ij} *are defined on* $[a,b]$ *by* (2) *and* $G_{H,n-1}$ *is Green's function defined by* (4).

Proof. Similar to the proof of Theorem 3 using identity

$$\int_a^b f(t)g(t)p(t)dt - \int_{b-\lambda}^b f(t)p(t)dt = \int_a^{b-\lambda} f(t)g(t)p(t)dt - \int_{b-\lambda}^b f(t)(1-g(t))p(t)dt. \quad \square$$

Using Theorems 3 and 4 we can obtain the following generalizations of Steffensen's inequality by Hermite polynomials.

THEOREM 5. *Let* $-\infty < a \leqslant a_1 < a_2 \ldots < a_r \leqslant b < \infty$, $(r \geqslant 2)$ *be given points and* $f \in C^n[a,b]$. *Let* $g, p : [a,b] \to \mathbb{R}$ *be integrable functions such that* p *is positive,* $0 \leqslant g \leqslant 1$ *and* $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$. *Let the function* $\mathscr{G}_1$ *be defined by* (6). *If* f *is* $n-$*convex and*

$$\int_a^b \mathscr{G}_1(x) G_{H,n-1}(x,s)dx \geqslant 0, \quad s \in [a,b], \tag{12}$$

then

$$\int_a^b f(t)g(t)p(t)dt \geqslant \int_a^{a+\lambda} f(t)p(t)dt + \sum_{j=1}^{r} \sum_{i=0}^{k_j} f^{(i+1)}(a_j) \int_a^b \mathscr{G}_1(x) H_{ij}(x)dx, \tag{13}$$

where H_{ij} *are defined on* $[a,b]$ *by* (2) *and* $G_{H,n-1}$ *is Green's function defined by* (4). *If the reverse inequality in* (12) *holds, then the reverse inequality in* (13) *holds.*

Proof. If the function f is n-convex, without loss of generality we can assume that f is $n-$times differentiable and $f^{(n)} \geqslant 0$ see [7, p. 16 and p. 293]. Now we can apply Theorem 3 to obtain (13). $\square$

THEOREM 6. *Let* $-\infty < a \leqslant a_1 < a_2 \ldots < a_r \leqslant b < \infty$, $(r \geqslant 2)$ *be given points and* $f \in C^n[a,b]$. *Let* $g, p : [a,b] \to \mathbb{R}$ *be integrable functions such that* p *is positive,* $0 \leqslant g \leqslant 1$ *and* $\int_{b-\lambda}^b p(t)dt = \int_a^b g(t)p(t)dt$. *Let the function* $\mathscr{G}_2$ *be defined by* (10). *If* f *is* $n-$*convex and*

$$\int_a^b \mathscr{G}_2(x) G_{H,n-1}(x,s)dx \geqslant 0, \quad s \in [a,b], \tag{14}$$

then

$$\int_a^b f(t)g(t)p(t)dt \leqslant \int_{b-\lambda}^b f(t)p(t)dt - \sum_{j=1}^{r} \sum_{i=0}^{k_j} f^{(i+1)}(a_j) \int_a^b \mathscr{G}_2(x) H_{ij}(x)dx, \tag{15}$$

where H_{ij} *are defined on* $[a,b]$ *by* (2) *and* $G_{H,n-1}$ *is Green's function defined by* (4). *If the reverse inequality in* (14) *holds, then the reverse inequality in* (15) *holds.*

Proof. Similar to the proof of Theorem 5. $\square$

REMARK 1. Note that functions $\mathscr{G}_i$, $i = 1,2$ defined by (6) and (10) are nonnegative. If all $k_1,\dots,k_r$ are odd then $\omega(x) = \prod_{j=1}^{r}(x-a_j)^{k_j+1} \geqslant 0$ and according to (i)-part of Lemma 1 $G_{H,n}(x,s) \geqslant 0$. Therefore, in Theorems 5 and 6 it is enough to assume that the function f is $n-$convex. For the case when only one k_j is even and others are odd we have $\omega(x) = \prod_{j=1}^{r}(x-a_j)^{k_j+1} \leqslant 0$ and by Lemma 1, $G_{H,n}(x,s) \leqslant 0$. Hence, integrals in (12) and (14) are nonpositive and the reverse inequalities in (13) and (15) hold.

2.1. Related results for type $(m,n-m)$ conditions

Let $r=2$, $a_1=a$, $a_2=b$, $1 \leqslant m \leqslant n-1$, $k_1 = m-1$ and $k_2 = n-m-1$. In this case

$$f(x) = \sum_{i=0}^{m-1} \tau_i(x) f^{(i)}(a) + \sum_{i=0}^{n-m-1} \eta_i(x) f^{(i)}(b) + \int_a^b G_{m,n}(x,s) f^{(n)}(s)ds,$$

where

$$\tau_i(x) = \frac{1}{i!}(x-a)^i \left(\frac{x-b}{a-b}\right)^{n-m} \sum_{k=0}^{m-1-i} \binom{n-m+k-1}{k} \left(\frac{x-a}{b-a}\right)^k, \tag{16}$$

$$\eta_i(x) = \frac{1}{i!}(x-b)^i \left(\frac{x-a}{b-a}\right)^{m} \sum_{k=0}^{n-m-1-i} \binom{m+k-1}{k} \left(\frac{x-b}{a-b}\right)^k, \tag{17}$$

and Green's function $G_{m,n}$ is of the form

$$G_{m,n}(x,s) = \begin{cases} \sum_{j=0}^{m-1} \left[\sum_{p=0}^{m-1-j} \binom{n-m+p-1}{p} \left(\frac{x-a}{b-a}\right)^p\right] \frac{(x-a)^j (a-s)^{n-j-1}}{j!(n-j-1)!} \left(\frac{b-x}{b-a}\right)^{n-m}, & s \leqslant x, \\ -\sum_{i=0}^{n-m-1} \left[\sum_{q=0}^{n-m-1-i} \binom{m+q-1}{q} \left(\frac{b-x}{b-a}\right)^q\right] \frac{(x-b)^i (b-s)^{n-i-1}}{i!(n-i-1)!} \left(\frac{x-a}{b-a}\right)^{m}, & s \geqslant x. \end{cases} \tag{18}$$

The following corollaries are representations of Steffensen's inequality by Hermite polynomials for type $(m,n-m)$ conditions.

COROLLARY 1. *Let $-\infty < a < b < \infty$ be given points and $f \in C^n[a,b]$. Let $g,p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive, $0 \leqslant g \leqslant 1$ and $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$. Let the function $\mathscr{G}_1$ be defined by (6) and τ_i, η_i be defined by (16) and (17), respectively. Then*

$$\int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt + \sum_{i=0}^{m-1} f^{(i+1)}(a) \int_a^b \mathscr{G}_1(x)\tau_i(x)dx$$
$$+ \sum_{i=0}^{n-m-2} f^{(i+1)}(b) \int_a^b \mathscr{G}_1(x)\eta_i(x)dx = -\int_a^b \left(\int_a^b \mathscr{G}_1(x) G_{m,n-1}(x,s)dx\right) f^{(n)}(s)ds,$$

where $G_{m,n-1}$ is Green's function defined by (18).

COROLLARY 2. *Let* $-\infty < a < b < \infty$ *be given points and* $f \in C^n[a,b]$. *Let* $g,p : [a,b] \to \mathbb{R}$ *be integrable functions such that* p *is positive,* $0 \leqslant g \leqslant 1$ *and* $\int_{b-\lambda}^{b} p(t)dt = \int_a^b g(t)p(t)dt$. *Let the function* $\mathscr{G}_2$ *be defined by* (10) *and* τ_i, η_i *be defined by* (16) *and* (17), *respectively. Then*

$$\int_a^b f(t)g(t)p(t)dt - \int_{b-\lambda}^{b} f(t)p(t)dt + \sum_{i=0}^{m-1} f^{(i+1)}(a) \int_a^b \mathscr{G}_2(x)\tau_i(x)dx$$
$$+ \sum_{i=0}^{n-m-2} f^{(i+1)}(b) \int_a^b \mathscr{G}_2(x)\eta_i(x)dx = -\int_a^b \left(\int_a^b \mathscr{G}_2(x)G_{m,n-1}(x,s)dx \right) f^{(n)}(s)ds,$$

where $G_{m,n-1}$ *is Green's function defined by* (18).

By using type $(m,n-m)$ conditions we obtain the following generalizations of Steffensen's inequality.

COROLLARY 3. *Let* $-\infty < a < b < \infty$ *be given points and* $f \in C^n[a,b]$. *Let* $g,p : [a,b] \to \mathbb{R}$ *be integrable functions such that* p *is positive,* $0 \leqslant g \leqslant 1$ *and* $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$. *Let the function* $\mathscr{G}_1$ *be defined by* (6) *and* τ_i, η_i *be defined by* (16) *and* (17), *respectively. If* f *is* $n-$*convex and*

$$\int_a^b \mathscr{G}_1(x)G_{m,n-1}(x,s)dx \geqslant 0, \quad s \in [a,b],$$

then

$$\int_a^b f(t)g(t)p(t)dt \geqslant \int_a^{a+\lambda} f(t)p(t)dt + \sum_{i=0}^{m-1} f^{(i+1)}(a) \int_a^b \mathscr{G}_1(x)\tau_i(x)dx$$
$$+ \sum_{i=0}^{n-m-2} f^{(i+1)}(b) \int_a^b \mathscr{G}_1(x)\eta_i(x)dx,$$

where $G_{m,n-1}$ *is Green's function defined by* (18).

COROLLARY 4. *Let* $-\infty < a < b < \infty$ *be given points and* $f \in C^n[a,b]$. *Let* $g,p : [a,b] \to \mathbb{R}$ *be integrable functions such that* p *is positive,* $0 \leqslant g \leqslant 1$ *and* $\int_{b-\lambda}^{b} p(t)dt = \int_a^b g(t)p(t)dt$. *Let the function* $\mathscr{G}_2$ *be defined by* (10) *and* τ_i, η_i *be defined by* (16) *and* (17), *respectively. If* f *is* $n-$*convex and*

$$\int_a^b \mathscr{G}_2(x)G_{m,n-1}(x,s)dx \geqslant 0, \quad s \in [a,b],$$

then

$$\int_a^b f(t)g(t)p(t)dt \leqslant \int_{b-\lambda}^{b} f(t)p(t)dt - \sum_{i=0}^{m-1} f^{(i+1)}(a) \int_a^b \mathscr{G}_2(x)\tau_i(x)dx$$
$$- \sum_{i=0}^{n-m-2} f^{(i+1)}(b) \int_a^b \mathscr{G}_2(x)\eta_i(x)dx,$$

where $G_{m,n-1}$ *is Green's function defined by* (18).

3. Ostrowski-type inequalities

In this section we present the Ostrowski-type inequalities related to generalizations obtained in the previous section.

Here, the symbol $L_p[a,b]$ $(1 \leqslant p < \infty)$ denotes the space of p-power integrable functions on the interval $[a,b]$ equipped with the norm

$$\|f\|_p = \left(\int_a^b |f(t)|^p \, dt\right)^{\frac{1}{p}}$$

and $L_\infty[a,b]$ denotes the space of essentially bounded functions on $[a,b]$ with the norm

$$\|f\|_\infty = \text{ess} \sup_{t\in[a,b]} |f(t)|.$$

THEOREM 7. *Suppose that all assumptions of Theorem 3 hold. Assume also that (p,q) is a pair of conjugate exponents, that is $1 \leqslant p,q \leqslant \infty$, $1/p+1/q=1$ and $f^{(n)} \in L_p[a,b]$ for some $n \geqslant 2$. Then we have*

$$\begin{aligned} &\left|\int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt + \sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i+1)}(a_j)\int_a^b \mathscr{G}_1(x)H_{ij}(x)dx\right| \\ &\leqslant \left\|f^{(n)}\right\|_p \left\|\int_a^b \mathscr{G}_1(x)G_{H,n-1}(x,\cdot)dx\right\|_q . \end{aligned} \tag{19}$$

The constant on the right-hand side of (19) is sharp for $1 < p \leqslant \infty$ and the best possible for $p = 1$.

Proof. Let's denote

$$K(s) = \int_a^b \mathscr{G}_1(x)G_{H,n-1}(x,s)dx.$$

By taking the modulus of (7) and applying Hölder's inequality we obtain

$$\begin{aligned} &\left|\int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt + \sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i+1)}(a_j)\int_a^b \mathscr{G}_1(x)H_{ij}(x)dx\right| \\ &= \left|\int_a^b K(s)f^{(n)}(s)ds\right| \leqslant \left\|f^{(n)}\right\|_p \|K\|_q . \end{aligned}$$

For the proof of the sharpness of the constant $\|K\|_q$ let us find a function f for which the equality in (19) is obtained.
For $1 < p < \infty$ take f to be such that

$$f^{(n)}(s) = \operatorname{sgn} K(s) \, |K(s)|^{\frac{1}{p-1}} .$$

For $p=1$ we prove that

$$\left|\int_a^b K(s) f^{(n)}(s) ds\right| \leqslant \max_{s\in[a,b]} |K(s)| \left(\int_a^b \left|f^{(n)}(s)\right| ds\right) \tag{20}$$

is the best possible inequality. $K(\cdot)$ is a continuous function on $[a,b]$ and so is $|K(\cdot)|$. Suppose that $|K(\cdot)|$ attains its maximum at $s_0 \in [a,b]$. First we assume that $K(s_0) > 0$. For $\varepsilon > 0$ small enough we define $f_\varepsilon(s)$ by

$$f_\varepsilon(s) = \begin{cases} 0, & a \leqslant s \leqslant s_0, \\ \frac{1}{\varepsilon n!}(s-s_0)^n, & s_0 \leqslant s \leqslant s_0+\varepsilon, \\ \frac{1}{n!}(s-s_0)^{n-1}, & s_0+\varepsilon \leqslant s \leqslant b. \end{cases}$$

Then

$$\left|\int_a^b K(s) f_\varepsilon^{(n)}(s) ds\right| = \left|\int_{s_0}^{s_0+\varepsilon} K(s)\frac{1}{\varepsilon} ds\right| = \frac{1}{\varepsilon}\int_{s_0}^{s_0+\varepsilon} K(s) ds.$$

Now from the inequality (20) we have

$$\frac{1}{\varepsilon}\int_{s_0}^{s_0+\varepsilon} K(s) ds \leqslant \frac{1}{\varepsilon} K(s_0) \int_{s_0}^{s_0+\varepsilon} ds = K(s_0).$$

Since,

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_{s_0}^{s_0+\varepsilon} K(s) ds = K(s_0)$$

the statement follows. In the case $K(s_0) < 0$, we define $f_\varepsilon(s)$ by

$$f_\varepsilon(s) = \begin{cases} \frac{1}{n!}(s-s_0-\varepsilon)^{n-1}, & a \leqslant s \leqslant s_0, \\ -\frac{1}{\varepsilon n!}(s-s_0-\varepsilon)^n, & s_0 \leqslant s \leqslant s_0+\varepsilon, \\ 0, & s_0+\varepsilon \leqslant s \leqslant b, \end{cases}$$

and the rest of the proof is the same as above. □

Using identity (11) we obtain the following result.

THEOREM 8. *Suppose that all assumptions of Theorem 4 hold. Assume also that (p,q) is a pair of conjugate exponents, that is $1 \leqslant p,q \leqslant \infty$, $1/p+1/q = 1$. Let $f^{(n)} \in L_p[a,b]$ for some $n \geqslant 2$. Then we have*

$$\begin{aligned} &\left|\int_a^b f(t)g(t)p(t)dt - \int_{b-\lambda}^b f(t)p(t)dt + \sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i+1)}(a_j)\int_a^b \mathscr{G}_2(x)H_{ij}(x)dx\right| \\ &\leqslant \left\|f^{(n)}\right\|_p \left\|\int_a^b \mathscr{G}_2(x)G_{H,n-1}(x,\cdot)dx\right\|_q . \end{aligned} \tag{21}$$

The constant on the right-hand side of (21) is sharp for $1 < p \leqslant \infty$ and the best possible *for* $p=1$.

Proof. Similar to the proof of Theorem 7. □

By using $(m,n-m)$ conditions we obtain the following results.

COROLLARY 5. *Suppose that all assumptions of Corollary 1 hold. Assume also that* (p,q) *is a pair of conjugate exponents, that is* $1 \leqslant p,q \leqslant \infty$, $1/p+1/q=1$ *and* $f^{(n)} \in L_p[a,b]$ *for some* $n \geqslant 2$. *Then we have*

$$\left|\int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt + \sum_{i=0}^{m-1} f^{(i+1)}(a)\int_a^b \mathscr{G}_1(x)\tau_i(x)dx \right. \\ \left. + \sum_{i=0}^{n-m-2} f^{(i+1)}(b)\int_a^b \mathscr{G}_1(x)\eta_i(x)dx\right| \leqslant \left\|f^{(n)}\right\|_p \left\|\int_a^b \mathscr{G}_1(x)G_{m,n-1}(x,\cdot)dx\right\|_q. \tag{22}$$

The constant on the right-hand side of (22) *is sharp for* $1<p\leqslant\infty$ *and the best possible for* $p=1$.

COROLLARY 6. *Suppose that all assumptions of Corollary 2 hold. Assume also that* (p,q) *is a pair of conjugate exponents, that is* $1 \leqslant p,q \leqslant \infty$, $1/p+1/q=1$ *and* $f^{(n)} \in L_p[a,b]$ *for some* $n \geqslant 2$. *Then we have*

$$\left|\int_a^b f(t)g(t)p(t)dt - \int_{b-\lambda}^b f(t)p(t)dt + \sum_{i=0}^{m-1} f^{(i+1)}(a)\int_a^b \mathscr{G}_2(x)\tau_i(x)dx \right. \\ \left. + \sum_{i=0}^{n-m-2} f^{(i+1)}(b)\int_a^b \mathscr{G}_2(x)\eta_i(x)dx\right| \leqslant \left\|f^{(n)}\right\|_p \left\|\int_a^b \mathscr{G}_2(x)G_{m,n-1}(x,\cdot)dx\right\|_q. \tag{23}$$

The constant on the right-hand side of (23) *is sharp for* $1<p\leqslant\infty$ *and the best possible for* $p=1$.

4. Inequalities related to the bounds for the Čebyšev functional

For two Lebesgue integrable functions $f,h:[a,b]\to\mathbb{R}$ we define the Čebyšev functional $T(f,h)$ by

$$T(f,h) := \frac{1}{b-a}\int_a^b f(t)h(t)dt - \frac{1}{b-a}\int_a^b f(t)dt \cdot \frac{1}{b-a}\int_a^b h(t)dt.$$

In 1882, Čebyšev in proved that

$$|T(f,h)| \leqslant \frac{1}{12}\left\|f'\right\|_\infty \left\|h'\right\|_\infty (b-a)^2,$$

provided that f',h' exist and are continuous on $[a,b]$ and $\|f'\|_\infty = \sup_{t\in[a,b]}|f'(t)|$. It also holds if $f,h:[a,b]\to\mathbb{R}$ are absolutely continuous and $f',g' \in L_\infty[a,b]$ while $\|f'\|_\infty = ess\sup_{t\in[a,b]}|f(t)|$.

In 1934, Grüss in his paper [5] proved that

$$|T(f,h)| \leqslant \frac{1}{4}(M-m)(N-n),$$

provided that there exist real numbers m, M, n, N such that

$$m \leqslant f(t) \leqslant M, \quad n \leqslant h(t) \leqslant N$$

for a.e. $t \in [a,b]$. The constant $1/4$ is the best possible.

In [4] Cerone and Dragomir proved the following theorems:

THEOREM 9. *Let $f:[a,b] \to \mathbb{R}$ be a Lebesgue integrable function and $h:[a,b] \to \mathbb{R}$ be an absolutely continuous function with $(\cdot - a)(b - \cdot)[h']^2 \in L_1[a,b]$. Then we have the inequality*

$$|T(f,h)| \leqslant \frac{1}{\sqrt{2}}[T(f,f)]^{\frac{1}{2}} \frac{1}{\sqrt{b-a}} \left(\int_a^b (x-a)(b-x)[h'(x)]^2 dx \right)^{\frac{1}{2}}. \tag{24}$$

The constant $\frac{1}{\sqrt{2}}$ in (24) is the best possible.

THEOREM 10. *Assume that $h:[a,b] \to \mathbb{R}$ is monotonic nondecreasing on $[a,b]$ and $f:[a,b] \to \mathbb{R}$ is absolutely continuous with $f' \in L_\infty[a,b]$. Then we have the inequality*

$$|T(f,h)| \leqslant \frac{1}{2(b-a)} \|f'\|_\infty \int_a^b (x-a)(b-x)dh(x). \tag{25}$$

The constant $\frac{1}{2}$ in (25) is the best possible.

Now, using the above theorems we obtain some new bounds for integrals on the left hand side in the perturbed versions of identities obtained in Theorems 3 and 4.

Firstly, let us denote

$$\Omega_i(s) = \int_a^b \mathscr{G}_i(x) G_{H,n-1}(x,s)dx, \quad i = 1,2 \tag{26}$$

and

$$\Phi_i(s) = \int_a^b \mathscr{G}_i(x) G_{m,n-1}(x,s)dx, \quad i = 1,2, \tag{27}$$

for $\mathscr{G}_i$ defined by (6) and (10) and $G_{H,n-1}$, $G_{m,n-1}$ defined by (4) and (18), respectively.

THEOREM 11. *Let $-\infty < a \leqslant a_1 < a_2 \ldots < a_r \leqslant b < \infty$, $(r \geqslant 2)$ be given points, $f \in C^{n+1}[a,b]$ and $(\cdot - a)(b - \cdot)[f^{(n+1)}]^2 \in L_1[a,b]$. Let $g, p:[a,b] \to \mathbb{R}$ be integrable* *functions such that* p *is positive,* $0 \leqslant g \leqslant 1$ *and* $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$. *Let* $\mathscr{G}_1$ *and* Ω_1 *be defined by* (6) *and* (26), *respectively. Then*

$$\begin{aligned}&\int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt + \sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i+1)}(a_j)\int_a^b \mathscr{G}_1(x)H_{ij}(x)dx \\ &+\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\int_a^b \Omega_1(s)ds = S_n^1(f;a,b),\end{aligned} \tag{28}$$

where the remainder $S_n^1(f;a,b)$ *satisfies the estimation*

$$\left|S_n^1(f;a,b)\right| \leqslant \frac{\sqrt{b-a}}{\sqrt{2}}\left[T(\Omega_1,\Omega_1)\right]^{\frac{1}{2}}\left(\int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2 ds\right)^{\frac{1}{2}}. \tag{29}$$

Proof. Applying Theorem 9 for $f \to \Omega_1$ and $h \to f^{(n)}$ we obtain

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b \Omega_1(s)f^{(n)}(s)ds - \frac{1}{b-a}\int_a^b \Omega_1(s)ds\cdot\frac{1}{b-a}\int_a^b f^{(n)}(s)ds\right| \\ \leqslant &\frac{1}{\sqrt{2}}\left[T(\Omega_1,\Omega_1)\right]^{\frac{1}{2}}\frac{1}{\sqrt{b-a}}\left(\int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2 ds\right)^{\frac{1}{2}}.\end{aligned} \tag{30}$$

If we add

$$\frac{1}{(b-a)}\int_a^b \Omega_1(s)ds\int_a^b f^{(n)}(s)ds = \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)}\int_a^b \Omega_1(s)ds$$

to both sides of identity (7) and use inequality (30) we obtain representation (28) and bound (29). □

Similarly, using identity (11) we obtain the following result:

THEOREM 12. *Let* $-\infty < a \leqslant a_1 < a_2 \ldots < a_r \leqslant b < \infty$, $(r \geqslant 2)$ *be given points,* $f \in C^{n+1}[a,b]$ *and* $(\cdot - a)(b-\cdot)[f^{(n+1)}]^2 \in L_1[a,b]$. *Let* $g,p : [a,b] \to \mathbb{R}$ *be integrable functions such that* p *is positive,* $0 \leqslant g \leqslant 1$ *and* $\int_{b-\lambda}^b p(t)dt = \int_a^b g(t)p(t)dt$. *Let* $\mathscr{G}_2$ *and* Ω_2 *be defined by* (10) *and* (26), *respectively. Then*

$$\begin{aligned}&\int_a^b f(t)g(t)p(t)dt - \int_{b-\lambda}^b f(t)p(t)dt + \sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i+1)}(a_j)\int_a^b \mathscr{G}_2(x)H_{ij}(x)dx \\ &+\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\int_a^b \Omega_2(s)ds = S_n^2(f;a,b),\end{aligned} \tag{31}$$

where the remainder $S_n^2(f;a,b)$ *satisfies the estimation*

$$\left|S_n^2(f;a,b)\right| \leqslant \frac{\sqrt{b-a}}{\sqrt{2}}\left[T(\Omega_2,\Omega_2)\right]^{\frac{1}{2}}\left(\int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2 ds\right)^{\frac{1}{2}}.$$

Proof. Similar to the proof of Theorem 11. □

Using Theorem 10 we obtain the following Grüss type inequalities.

THEOREM 13. *Let* $-\infty < a \leqslant a_1 < a_2 \ldots < a_r \leqslant b < \infty$, $(r \geqslant 2)$ *be given points,* $f \in C^{n+1}[a,b]$ *and* $f^{(n+1)} \geqslant 0$ *on* $[a,b]$. *Let functions* Ω_i, $i = 1,2$ *be defined by* (26).

(a) *Let* $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$. *Then the representation* (28) *holds and the remainder* $S_n^1(f;a,b)$ *satisfies the bound*

$$\left|S_n^1(f;a,b)\right| \leqslant (b-a)\|\Omega_1'\|_\infty\left\{\frac{f^{(n-1)}(b)+f^{(n-1)}(a)}{2} - \frac{f^{(n-2)}(b)-f^{(n-2)}(a)}{b-a}\right\}. \tag{32}$$

(b) *Let* $\int_{b-\lambda}^b p(t)dt = \int_a^b g(t)p(t)dt$. *Then the representation* (31) *holds and the remainder* $S_n^2(f;a,b)$ *satisfies the bound*

$$\left|S_n^2(f;a,b)\right| \leqslant (b-a)\|\Omega_2'\|_\infty\left\{\frac{f^{(n-1)}(b)+f^{(n-1)}(a)}{2} - \frac{f^{(n-2)}(b)-f^{(n-2)}(a)}{b-a}\right\}.$$

Proof.

(a) Applying Theorem 10 for $f \to \Omega_1$, $h \to f^{(n)}$ and multiplying by $(b-a)$ we obtain

$$\begin{aligned}&\left|\int_a^b \Omega_1(s)f^{(n)}(s)ds - \int_a^b \Omega_1(s)ds\cdot\frac{1}{b-a}\int_a^b f^{(n)}(s)ds\right|\\ \leqslant&\frac{1}{2}\|\Omega_1'\|_\infty\int_a^b (s-a)(b-s)f^{(n+1)}(s)ds.\end{aligned} \tag{33}$$

Since

$$\begin{aligned}&\int_a^b (s-a)(b-s)f^{(n+1)}(s)ds = \int_a^b [2s-(a+b)]f^{(n)}(s)ds\\ =&(b-a)\left[f^{(n-1)}(b)+f^{(n-1)}(a)\right] - 2\left(f^{(n-2)}(b)-f^{(n-2)}(a)\right).\end{aligned}$$

Using representation (7) and inequality (33) we deduce (32).

(b) Similar to the (a)-part. □

Similary, using the $(m,n-m)$ conditions we obtain the following results.

COROLLARY 7. *Let $-\infty < a < b < \infty$ be given points, $f \in C^{n+1}[a,b]$ and $(\cdot - a)(b-\cdot)[f^{(n+1)}]^2 \in L_1[a,b]$. Let $g,p:[a,b]\to\mathbb{R}$ be integrable functions such that p is positive, $0 \leqslant g \leqslant 1$ and $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$. Let $\mathscr{G}_1$, Φ_1, τ_i and η_i be defined by (6), (27),(16) and (17) respectively. Then*

$$\int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt + \sum_{i=0}^{m-1} f^{(i+1)}(a)\int_a^b \mathscr{G}_1(x)\tau_i(x)dx + \sum_{i=0}^{n-m-2} f^{(i+1)}(b)\int_a^b \mathscr{G}_1(x)\eta_i(x)dx + \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\int_a^b \Phi_1(s)ds = S_n^3(f;a,b), \tag{34}$$

where the remainder $S_n^3(f;a,b)$ satisfies the estimation

$$\left|S_n^3(f;a,b)\right| \leqslant \frac{\sqrt{b-a}}{\sqrt{2}}[T(\Phi_1,\Phi_1)]^{\frac12}\left(\int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2ds\right)^{\frac12}.$$

COROLLARY 8. *Let $-\infty < a < b < \infty$ be given points, $f \in C^{n+1}[a,b]$ and $(\cdot - a)(b-\cdot)[f^{(n+1)}]^2 \in L_1[a,b]$. Let $g,p:[a,b]\to\mathbb{R}$ be integrable functions such that p is positive, $0 \leqslant g \leqslant 1$ and $\int_{b-\lambda}^b p(t)dt = \int_a^b g(t)p(t)dt$. Let $\mathscr{G}_2$, Φ_2, τ_i and η_i be defined by (10), (27),(16) and (17) respectively. Then*

$$\int_a^b f(t)g(t)p(t)dt - \int_{b-\lambda}^b f(t)p(t)dt + \sum_{i=0}^{m-1} f^{(i+1)}(a)\int_a^b \mathscr{G}_2(x)\tau_i(x)dx + \sum_{i=0}^{n-m-2} f^{(i+1)}(b)\int_a^b \mathscr{G}_2(x)\eta_i(x)dx + \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\int_a^b \Phi_2(s)ds = S_n^4(f;a,b), \tag{35}$$

where the remainder $S_n^4(f;a,b)$ satisfies the estimation

$$\left|S_n^4(f;a,b)\right| \leqslant \frac{\sqrt{b-a}}{\sqrt{2}}[T(\Phi_2,\Phi_2)]^{\frac12}\left(\int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2ds\right)^{\frac12}.$$

COROLLARY 9. *Let $-\infty < a < b < \infty$ be given points, $f \in C^{n+1}[a,b]$ and $f^{(n+1)} \geqslant 0$ on $[a,b]$. Let functions Φ_i, $i=1,2$ be defined by (27).*

(a) Let $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$. Then the representation (34) holds and the remainder $S_n^3(f;a,b)$ satisfies the bound

$$\left|S_n^3(f;a,b)\right| \leqslant (b-a)\|\Phi_1'\|_\infty\left\{\frac{f^{(n-1)}(b)+f^{(n-1)}(a)}{2} - \frac{f^{(n-2)}(b)-f^{(n-2)}(a)}{b-a}\right\}.$$

(b) Let $\int_{b-\lambda}^{b} p(t)dt = \int_a^b g(t)p(t)dt$. *Then the representation* (35) *holds and the remainder* $S_n^4(f;a,b)$ *satisfies the bound*

$$\left|S_n^4(f;a,b)\right| \leqslant (b-a)\|\Phi_2'\|_\infty \left\{ \frac{f^{(n-1)}(b)+f^{(n-1)}(a)}{2} - \frac{f^{(n-2)}(b)-f^{(n-2)}(a)}{b-a} \right\}.$$

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(Received September 21, 2018)

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