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INEQUALITIES OF THE EDMUNDSON–LAH–RIBARIČ TYPE FOR SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract. By exploiting some scalar inequalities obtained via Hermite's interpolating polynomial, we will obtain lower and upper bounds for the difference in Jensen's inequality and in the Edmundson-Lah-Ribarič inequality for selfadjoint operators in Hilbert space that hold for the class of n-convex functions. As an application, main results are applied to quasi-arithmetic operator means, with a particular emphasis to power operator means.

1. Introduction

Let H be a Hilbert space and let $\mathscr{B}(H)$ be the C^*-algebra of all bounded (i.e., continuous) linear operators on H. A bounded linear operator A on a Hilbert space H is said to be selfadjoint if $A = A^*$.

An operator $A \in \mathscr{B}(H)$ is selfadjoint if and only if $\langle Ax, x\rangle \in \mathbb{R}$ for every $x \in H$. We denote by $\mathscr{B}_h(H)$ a semi-space of all selfadjoint operators in $\mathscr{B}(H)$.
If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geqslant 0$ for every $t \in Sp(A)$ implies that $f(A) \geqslant 0$, i.e., $f(A)$ is a positive operator on H.
Equivalently, if both f and g are real valued continuous function on $Sp(A)$, then the following property holds:

$$f(t) \geqslant g(t) \text{ for any } t \in Sp(A) \text{ implies that } f(A) \geqslant g(A) \tag{1}$$

in the operator order of $\mathscr{B}(H)$.

Since it was proved, the famous Jensen inequality and its converses have been extensively studied by many authors and have been generalized in numerous directions. The following result that provides an operator version for Jensen's inequality is due to Mond and Pečarić [11] (see also [3] and [5]):

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THEOREM 1. [11] (Mond – Pečarić) *Let $A \in \mathscr{B}_h(H)$ be a selfadjoint operator with $Sp(A) \subseteq [a,b]$ for some scalars a¡b. If f is a convex function on $[a,b]$, then*

$$f(\langle Ax,x\rangle) \leqslant \langle f(A)x,x\rangle \tag{2}$$

holds for each unit vector x in H.

We also need the following converse for the Mond-Pečarić inequality that generalizes the scalar Edmundson-Lah-Ribarič inequality for convex functions found in [10] (see also [5]):

THEOREM 2. ([10]) *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [a,b]$ for some scalars a and b with a¡b. If f is a convex function on $[a,b]$, then*

$$\langle f(A)x,x\rangle \leqslant \frac{b-\langle Ax,x\rangle}{b-a}f(a)+\frac{\langle Ax,x\rangle - a}{b-a}f(b) \tag{3}$$

holds for each unit vector x in H.

For some recent results on the converses of the Jensen inequality, the reader is referred to [2], [4], [5], [6], [7] and [8]. Most of the mentioned results require operator convexity or convexity in the classical sense of the involved functions.

Definition of n-convex functions is characterized by n-th order divided differences. The n-th order divided difference of a function $f\colon [a,b] \to \mathbb{R}$ at mutually distinct points $t_0,t_1,\dots,t_n \in [a,b]$ is defined recursively by

$$f[t_i] = f(t_i), \quad i = 0,\dots,n,$$
$$f[t_0,\dots,t_n] = \frac{f[t_1,\dots,t_n]-f[t_0,\dots,t_{n-1}]}{t_n - t_0}.$$

The value $f[t_0,\dots,t_n]$ is independent of the order of the points $t_0,\dots,t_n$.

Definition of divided differences can be extended to include the cases in which some or all the points coincide (see e.g. [1], [12]):

$$f[\underbrace{a,\dots,a}_{n\ times}] = \frac{1}{(n-1)!}f^{(n-1)}(a), \quad n \in \mathbb{N}.$$

A function $f\colon [a,b] \to \mathbb{R}$ is said to be n-convex ($n \geqslant 0$) if and only if for all choices of $(n+1)$ distinct points $t_0,t_1,\dots,t_n \in [a,b]$, we have $f[t_0,\dots,t_n] \geqslant 0$.

Following representations of the left side in the scalar Edmundson-Lah-Ribarič inequality are obtained in [9] by using Hermite's interpolating polynomials in terms of divided differences. For more details about the Hermite interpolating polynomial, the reader is referred to [1].

LEMMA 1. ([9]) Let a,b be real numbers such that $a<b$. For a function $f \in \mathscr{C}^n([a,b])$, $n \geqslant 3$, the following identities hold:

- $$f(t) - \frac{b-t}{b-a}f(a) - \frac{t-a}{b-a}f(b) = \sum_{k=2}^{n-1} f[a;\underbrace{b,\dots,b}_{k\ times}](t-a)(t-b)^{k-1} + R_1(t) \tag{4}$$

$$\bullet\; f(t)-\frac{b-t}{b-a}f(a)-\frac{t-a}{b-a}f(b)=f[a,a;b](t-a)(t-b)+\sum_{k=2}^{n-2} f[a,a;\underbrace{b,\dots,b}_{k\ times}](t-a)^2(t-b)^{k-1}+R_2(t) \quad (5)$$

where

$$R_m(t)=(t-a)^m(t-b)^{n-m}f[t;\underbrace{a,\dots,a}_{m\ times};\underbrace{b,b,\dots,b}_{(n-m)\ times}]. \quad (6)$$

Additionally, if $n>m\geqslant 3$, then we have

$$\bullet\; f(t)-\frac{b-t}{b-a}f(a)-\frac{t-a}{b-a}f(b)=(t-a)\left(f[a,a]-f[a,b]\right)+\sum_{k=2}^{m-1}\frac{f^{(k)}(a)}{k!}(t-a)^k+\sum_{k=1}^{n-m} f[\underbrace{a,\dots,a}_{m\ times};\underbrace{b,\dots,b}_{k\ times}](t-a)^m(t-b)^{k-1}+R_m(t). \quad (7)$$

LEMMA 2. ([9]) Let a,b be real numbers such that $a<b$. For a function $f\in\mathscr{C}^n([a,b])$, $n\geqslant 3$, the following identities hold:

$$\bullet\; f(t)-\frac{b-t}{b-a}f(a)-\frac{t-a}{b-a}f(b)=\sum_{k=2}^{n-1} f[b;\underbrace{a,\dots,a}_{k\ times}](t-b)(t-a)^{k-1}+R_1^*(t) \quad (8)$$

$$\bullet\; f(t)-\frac{b-t}{b-a}f(a)-\frac{t-a}{b-a}f(b)=f[b,b;a](t-b)(t-a)+\sum_{k=2}^{n-2} f[b,b;\underbrace{a,\dots,a}_{k\ times}](t-b)^2(t-a)^{k-1}+R_2^*(t) \quad (9)$$

where

$$R_m^*(t)=f[t;\underbrace{b,\dots,b}_{m\ times};\underbrace{a,a,\dots,a}_{(n-m)\ times}](t-b)^m(t-a)^{n-m}. \quad (10)$$

Also, if $n>m\geqslant 3$, then

$$\bullet\; f(t)-\frac{b-t}{b-a}f(a)-\frac{t-a}{b-a}f(b)=(b-t)\left(f[a,b]-f[b,b]\right)+\sum_{k=2}^{m-1}\frac{f^{(k)}(b)}{k!}(t-b)^k+\sum_{k=1}^{n-m} f[\underbrace{b,\dots,b}_{m\ times};\underbrace{a,\dots,a}_{k\ times}](t-b)^m(t-a)^{k-1}+R_m^*(t). \quad (11)$$

In this paper, the goal is to exploit identities from Lemma 1 and Lemma 2 in order to obtain inequalities of the Jensen and Edmundson-Lah-Ribarič type that hold for n-convex functions, that is, to find lower and upper bounds for the difference in the Jensen and Edmundson-Lah-Ribarič inequality which are valid for the class of n-convex functions.

2. Results

Throughout this paper, whenever mentioning the interval $[a,b]$, we assume that a,b are finite real numbers such that $a<b$. Let $A \in \mathscr{B}_h(H)$ be a selfadjoint operator with $Sp(A) \subseteq [a,b]$. We can write the Edmundson-Lah-Ribarič operator inequality (3) in the form

$$\alpha_f \langle Ax,x\rangle + \beta_f - \langle f(A)x,x\rangle \geqslant 0 \tag{12}$$

with standard notation

$$\alpha_f = \frac{f(b)-f(a)}{b-a} \quad \text{and} \quad \beta_f = \frac{bf(a)-af(b)}{b-a}.$$

An inequality of the Edmundson-Lah-Ribarič type for scalar product obtained from Lemma 1 is given in the following theorem.

THEOREM 3. *Let $A \in \mathscr{B}_h(H)$ be a selfadjoint operator with $Sp(A) \subseteq [a,b]$ and let $f \in \mathscr{C}^n([a,b])$. If the function f is n-convex and if $n > m \geqslant 3$ are of different parity, then*

$$\begin{aligned} f(A) - \alpha_f A - \beta_f \mathbf{1} \leqslant & (A-a\mathbf{1})\left(f'(a) - f[a,b]\right) + \sum_{k=2}^{m-1} \frac{f^{(k)}(a)}{k!}(A-a\mathbf{1})^k \\ & + \sum_{k=1}^{n-m} f[\underbrace{a,\dots,a}_{m\ times};\underbrace{b,\dots,b}_{k\ times}](A-a\mathbf{1})^m (A-b\mathbf{1})^{k-1}. \end{aligned} \tag{13}$$

Inequality (13) also holds when the function f is n-concave and n and m are of equal parity. In case when the function f is n-convex and n and m are of equal parity, or when the function f is n-concave and n and m are of different parity, the inequality sign in (13) is reversed.

Proof. Because $f \in \mathscr{C}^n([a,b])$, it is continuous and its n-th order divided difference $f_n(t) = f[t;\underbrace{a,\dots,a}_{m\ times};\underbrace{b,b,\dots,b}_{(n-m)\ times}]$ is continuous, so consequently the function $R_m(t)$ defined in (6) is also continuous. Now, due to continuous functional calculus we can replace t with operator A in (7) and obtain:

$$f(A) - \alpha_f A - \beta_f \mathbf{1} = (A-a\mathbf{1})\left(f[a,a] - f[a,b]\right) + \sum_{k=2}^{m-1} \frac{f^{(k)}(a)}{k!}(A-a\mathbf{1})^k$$

$$+\sum_{k=1}^{n-m} f[\underbrace{a,\dots,a}_{m\ times};\underbrace{b,\dots,b}_{k\ times}](A-a\mathbf{1})^m(A-b\mathbf{1})^{k-1}+R_m(A). \quad (14)$$

Next, we set our focus on positivity and negativity of the term $R_m(A)$. Due to property (1), it is enough to study positivity and negativity of the function:

$$R_m(t)=(t-a)^m\,(t-b)^{n-m}\,f[t;\underbrace{a,\dots,a}_{m\ times};\ \underbrace{b,b,\dots,b}_{(n-m)\ times}].$$

Since $a\leqslant t\leqslant b$, we have $(t-a)^m\geqslant 0$ for any choice of m. For the same reason we have $(t-b)\leqslant 0$. Trivially it follows that $(t-b)^{n-m}\leqslant 0$ when n and m are of different parity, and $(t-b)^{n-m}\geqslant 0$ when n and m are of equal parity.

If the function f is n-convex, then $f[t;\underbrace{a,\dots,a}_{m\ times};\ \underbrace{b,b,\dots,b}_{(n-m)\ times}]\geqslant 0$, and if the function f is n-concave, then $f[t;\underbrace{a,\dots,a}_{m\ times};\ \underbrace{b,b,\dots,b}_{(n-m)\ times}]\leqslant 0$ for any $t\in[a,b]$.

Now (13) easily follows from (14). □

Next result is another inequality of the Edmundson-Lah-Ribarič type in terms of divided differences, but obtained from Lemma 2.

THEOREM 4. *Let $A\in\mathscr{B}_h(H)$ be a selfadjoint operator with $Sp(A)\subseteq[a,b]$ and let $f\in\mathscr{C}^n([a,b])$. If the function f is n-convex and if $m\geqslant 3$ is odd and $n>m$, then*

$$\begin{aligned} f(A)-\alpha_f A-\beta_f\mathbf{1}\leqslant&(b\mathbf{1}-A)\left(f[a,b]-f'(b)\right)+\sum_{k=2}^{m-1}\frac{f^{(k)}(b)}{k!}(A-b\mathbf{1})^k\\ &+\sum_{k=1}^{n-m} f[\underbrace{b,\dots,b}_{m\ times};\underbrace{a,\dots,a}_{k\ times}](A-b\mathbf{1})^m(A-a\mathbf{1})^{k-1}. \end{aligned} \quad (15)$$

Inequality (15) also holds when the function f is n-concave and m is even. In case when the function f is n-convex and m is even, or when the function f is n-concave and m is odd, the inequality sign in (15) is reversed.

Proof. In a similar manner as in the proof of the previous theorem, since all the involved functions are continuous, we can replace t with operator A in (11) . In that way we get

$$\begin{aligned} f(A)-\alpha_f A-\beta_f\mathbf{1}=&(b\mathbf{1}-A)\left(f[a,b]-f'(b)\right)+\sum_{k=2}^{m-1}\frac{f^{(k)}(b)}{k!}(A-b\mathbf{1})^k\\ &+\sum_{k=1}^{n-m} f[\underbrace{b,\dots,b}_{m\ times};\underbrace{a,\dots,a}_{k\ times}](A-b\mathbf{1})^m(A-a\mathbf{1})^{k-1}+R_m^*(A). \end{aligned} \quad (16)$$

Now, we study positivity and negativity of the term $R_m^*(A)$. Again, due to property (1), it is enough to study positivity and negativity of the function:

$$R_m^*(t) = (t-b)^m (t-a)^{n-m} f[t; \underbrace{b,\dots,b}_{m\ times}; \underbrace{a,a,\dots,a}_{(n-m)\ times}].$$

Since $t \in [a,b]$, we have $(t-a)^{n-m} \geqslant 0$ for every t and any choice of m. For the same reason we have $(t-b) \leqslant 0$. Trivially it follows that $(t-b)^m \leqslant 0$ when m is odd, and $(t-b)^m \geqslant 0$ when m is even. If the function f is n-convex, then its n-th order divided differences are greater of equal to zero, and if the function f is n-concave, then its n-th order divided differences are less or equal to zero.

Now (15) easily follows from Lemma 2. □

When we combine the results from Theorem 3 and Theorem 4, we get lower and upper bounds for the difference in the Edmundson-Lah-Ribarič inequality that hold for the class of n-convex functions.

COROLLARY 1. *Let $A \in \mathscr{B}_h(H)$ be a selfadjoint operator with $Sp(A) \subseteq [a,b]$, let n be an odd number and let $f \in \mathscr{C}^n([a,b])$. If the function f is n-convex, $m \geqslant 3$ is odd and $m < n$, then*

$$\begin{aligned}
&(A-a\mathbf{1})\left(f'(a)-f[a,b]\right)+\sum_{k=2}^{m-1}\frac{f^{(k)}(a)}{k!}(A-a\mathbf{1})^k \\
&+\sum_{k=1}^{n-m} f[\underbrace{a,\dots,a}_{m\ times};\underbrace{b,\dots,b}_{k\ times}](A-a\mathbf{1})^m(A-b\mathbf{1})^{k-1} \leqslant f(A)-\alpha_f A-\beta_f\mathbf{1} \qquad (17)\\
\leqslant &(b\mathbf{1}-A)\left(f[a,b]-f'(b)\right)+\sum_{k=2}^{m-1}\frac{f^{(k)}(b)}{k!}(A-b\mathbf{1})^k \\
&+\sum_{k=1}^{n-m} f[\underbrace{b,\dots,b}_{m\ times};\underbrace{a,\dots,a}_{k\ times}](A-b\mathbf{1})^m(A-a\mathbf{1})^{k-1}.
\end{aligned}$$

Inequality (17) also holds when the function f is n-concave and m is even. In case when the function f is n-convex and m is even, or when the function f is n-concave and m is odd, the inequality signs in (17) are reversed.

The result that follows also provides us with a lower and upper bound for the difference in the Edmundson-Lah-Ribarič inequality, and it is obtained from Lemma 1.

THEOREM 5. *Let $A \in \mathscr{B}_h(H)$ be a selfadjoint operator with $Sp(A) \subseteq [a,b]$ and let $f \in \mathscr{C}^n([a,b])$. If the function f is n-convex and if $n \geqslant 3$ is odd, then*

$$\sum_{k=2}^{n-1} f[a;\underbrace{b,\dots,b}_{k\ times}](A-a\mathbf{1})(A-b\mathbf{1})^{k-1} \leqslant f(A)-\alpha_f A-\beta_f\mathbf{1} \qquad (18)$$

$$\leqslant f[a,a;b](A-a\mathbf{1})(A-b\mathbf{1})+\sum_{k=2}^{n-2} f[a,a;\underbrace{b,\ldots,b}_{k\ times}](A-a\mathbf{1})^2(A-b\mathbf{1})^{k-1}.$$

Inequalities (18) also hold when the function f is n-concave and n is even. In case when the function f is n-convex and n is even, or when the function f is n-concave and n is odd, the inequality signs in (18) are reversed.

Proof. For the reasons stated in proofs of the previous theorems, we can replace t with operator A in (4) and (5) then apply scalar product to the obtained relations . In that way we get

$$f(A)-\alpha_f A-\beta_f\mathbf{1}=\sum_{k=2}^{n-1} f[a;\underbrace{b,\ldots,b}_{k\ times}](A-a\mathbf{1})(A-b\mathbf{1})^{k-1}+R_1(A) \tag{19}$$

and

$$\begin{aligned}f(A)-\alpha_f A-\beta_f\mathbf{1}=&f[a,a;b](A-a\mathbf{1})(A-b\mathbf{1})\\&+\sum_{k=2}^{n-2} f[a,a;\underbrace{b,\ldots,b}_{k\ times}](A-a\mathbf{1})^2(A-b\mathbf{1})^{k-1}+R_2(A).\end{aligned} \tag{20}$$

From the discussion about positivity and negativity of the term $R_m(A)$ in the proof of Theorem 3, for $m=1$ it follows that

* $R_1(A)\geqslant\mathbf{0}$ when the function f is n-convex and n is odd, or when f is n-concave and n even;
* $R_1(A)\leqslant\mathbf{0}$ when the function f is n-concave and n is odd, or when f is n-convex and n even.

Now the relation (19) gives us

$$f(A)-\alpha_f A-\beta_f\mathbf{1}\geqslant\sum_{k=2}^{n-1} f[a;\underbrace{b,\ldots,b}_{k\ times}](A-a\mathbf{1})(A-b\mathbf{1})^{k-1}$$

for $R_1(A)\geqslant\mathbf{0}$, and in case $R_1(A)\leqslant\mathbf{0}$ the inequality sign is reversed.

In the same manner, for $m=2$ it follows that

* $R_2(A)\leqslant\mathbf{0}$ when the function f is n-convex and n is odd, or when f is n-concave and n even;
* $R_2(A)\geqslant\mathbf{0}$ when the function f is n-concave and n is odd, or when f is n-convex and n even.

In this case the relation (20) for $R_2(A) \leqslant \mathbf{0}$ gives us

$$\begin{aligned} &f(A)-\alpha_f A-\beta_f \mathbf{1} \\ \leqslant &f[a,a;b](A-a\mathbf{1})(A-b\mathbf{1})+\sum_{k=2}^{n-2} f[a,a;\underbrace{b,\ldots,b}_{k\ times}](A-a\mathbf{1})^2(A-b\mathbf{1})^{k-1}, \end{aligned}$$

and when $R_2(A) \geqslant \mathbf{0}$ the inequality sign is reversed.

When we combine the two inequalities obtained above, we get exactly (18). □

By utilizing Lemma 2 we can get similar lower and upper bounds for the difference in the Edmundson-Lah-Ribarič operator inequality that hold for all $n \in \mathbb{N}$, not only the odd ones.

THEOREM 6. *Let $A \in \mathscr{B}_h(H)$ be a selfadjoint operator with $Sp(A) \subseteq [a,b]$ and let $f \in \mathscr{C}^n([a,b])$. If the function f is n-convex, $n \geqslant 3$, then*

$$\begin{aligned} &f[b,b;a](A-b\mathbf{1})(A-a\mathbf{1})+\sum_{k=2}^{n-2} f[b,b;\underbrace{a,\ldots,a}_{k\ times}](A-b\mathbf{1})^2(A-a\mathbf{1})^{k-1} \\ \leqslant &f(A)-\alpha_f A-\beta_f \mathbf{1} \leqslant \sum_{k=2}^{n-1} f[b;\underbrace{a,\ldots,a}_{k\ times}](A-b\mathbf{1})(A-a\mathbf{1})^{k-1}. \end{aligned} \tag{21}$$

If the function f is n-concave, the inequality signs in (21) are reversed.

Proof. For already stated reasons we can replace t with operator A in (8) and (9), and get

$$f(A)-\alpha_f A-\beta_f \mathbf{1} = \sum_{k=2}^{n-1} f[b;\underbrace{a,\ldots,a}_{k\ times}](A-b\mathbf{1})(A-a\mathbf{1})^{k-1}+R_1^*(A) \tag{22}$$

and

$$\begin{aligned} f(A)-\alpha_f A-\beta_f \mathbf{1} = &f[b,b;a](A-b\mathbf{1})(A-a\mathbf{1}) \\ &+\sum_{k=2}^{n-2} f[b,b;\underbrace{a,\ldots,a}_{k\ times}](A-b\mathbf{1})^2(A-a\mathbf{1})^{k-1}+R_2^*(A). \end{aligned} \tag{23}$$

Now we need to return to the discussion about positivity and negativity of the term $R_m^*(A)$ in the proof of Theorem 4. For $m=1$ we have

$$(t-b)^1(t-a)^{n-1} \leqslant 0 \ \text{ for every } t \in [a,b],$$

so $R_1^*(A) \geqslant \mathbf{0}$ when the function f is n-concave, and $R_1^*(A) \leqslant \mathbf{0}$ when the function f is n-convex. The relation (22) for a n-convex function f gives us

$$f(A)-\alpha_f A-\beta_f \mathbf{1} \leqslant \sum_{k=2}^{n-1} f[b;\underbrace{a,\ldots,a}_{k\ times}](A-b\mathbf{1})(A-a\mathbf{1})^{k-1},$$

and if the function f is n-concave, the inequality sign is reversed.

Similarly, for $m = 2$ we have

$$(t-b)^2(t-a)^{n-2} \geqslant 0 \text{ for every } t \in [a,b],$$

so $R_2^*(A) \geqslant \mathbf{0}$ when the function f is n-convex, and $R_2^*(A) \leqslant \mathbf{0}$ when the function f is n-concave. In this case the identity (23) for a n-convex function f gives us

$$\begin{aligned} f(A) - \alpha_f A - \beta_f \mathbf{1} \geqslant & f[b,b;a](A-b\mathbf{1})(A-a\mathbf{1}) \\ & + \sum_{k=2}^{n-2} f[b,b;\underbrace{a,\ldots,a}_{k \text{ times}}](A-b\mathbf{1})^2(A-a\mathbf{1})^{k-1}, \end{aligned}$$

and if the function f is n-concave, the inequality sign is reversed.

When we combine the two results from above, we get exactly (21). □

3. Jensen-type inequalities

In this section we will utilize the results from the previous section, as well Lemma 1 and Lemma 2, in order to obtain some Jensen-type inequalities that hold for n-convex functions.

Our first result is a consequence of Corollary 1, and it provides us with a lower and an upper bound for the difference in the Mond-Pečarić inequality (2).

THEOREM 7. *Let $A \in \mathscr{B}_h(H)$ be a selfadjoint operator with $Sp(A) \subseteq [a,b]$, let n be an odd number and let $f \in \mathscr{C}^n([a,b])$. If the function f is n-convex, $n > m$, and if $m \geqslant 3$ is odd, then*

$$\begin{aligned} & f(a) - f(b) + bf'(b) - af'(a) + (f'(a) - f'(b))\langle Ax,x\rangle \\ & + \sum_{k=2}^{m-1} \left(\frac{f^{(k)}(a)}{k!} \langle (A-a\mathbf{1})^k x,x\rangle - \frac{f^{(k)}(b)}{k!} (\langle Ax,x\rangle - b)^k \right) \\ & + \sum_{k=1}^{n-m} f[\underbrace{a,\ldots,a}_{m \text{ times}};\underbrace{b,\ldots,b}_{k \text{ times}}]\langle (A-a\mathbf{1})^m (A-b\mathbf{1})^{k-1} x,x\rangle \\ & - \sum_{k=1}^{n-m} f[\underbrace{b,\ldots,b}_{m \text{ times}};\underbrace{a,\ldots,a}_{k \text{ times}}](\langle Ax,x\rangle - b)^m (\langle Ax,x\rangle - a)^{k-1} \leqslant \langle f(A)x,x\rangle - f(\langle Ax,x\rangle) \end{aligned} \tag{24}$$

$$\begin{aligned} \leqslant & f(b) - f(a) + af'(a) - bf'(b) + (f'(b) - f'(a))\langle Ax,x\rangle \\ & + \sum_{k=2}^{m-1} \left(\frac{f^{(k)}(b)}{k!} \langle (A-b\mathbf{1})^k x,x\rangle - \frac{f^{(k)}(a)}{k!} (\langle Ax,x\rangle - a)^k \right) \\ & + \sum_{k=1}^{n-m} f[\underbrace{b,\ldots,b}_{m \text{ times}};\underbrace{a,\ldots,a}_{k \text{ times}}]\langle (A-b\mathbf{1})^m (A-a\mathbf{1})^{k-1} x,x\rangle \end{aligned}$$

$$-\sum_{k=1}^{n-m} f[\underbrace{a,\ldots,a}_{m\ times};\underbrace{b,\ldots,b}_{k\ times}](\langle Ax,x\rangle-a)^m(\langle Ax,x\rangle-b)^{k-1}$$

where $x\in H$ is a unit vector. Inequalities (24) also hold when the function f is n-concave and m is even. In case when the function f is n-convex and m is even, or when the function f is n-concave and m is odd, the inequality signs in (24) are reversed.

Proof. Because $Sp(A)\subseteq[a,b]$, we have $\langle Ax,x\rangle\in[a,b]$ for each unit vector x in H, so we can substitute t with $\langle Ax,x\rangle$ in (7) and obtain

$$\begin{aligned} f(\langle Ax,x\rangle)-\alpha_f\langle Ax,x\rangle-\beta_f =&(\langle Ax,x\rangle-a)\,(f[a,a]-f[a,b])+\sum_{k=2}^{m-1}\frac{f^{(k)}(a)}{k!}(\langle Ax,x\rangle-a)^k\\ &+\sum_{k=1}^{n-m} f[\underbrace{a,\ldots,a}_{m\ times};\underbrace{b,\ldots,b}_{k\ times}](\langle Ax,x\rangle-a)^m(\langle Ax,x\rangle-b)^{k-1}\\ &+R_m(\langle Ax,x\rangle). \end{aligned} \tag{25}$$

Next, we study positivity and negativity of the term:

$$R_m(\langle Ax,x\rangle)=(\langle Ax,x\rangle-a)^m\,(\langle Ax,x\rangle-b)^{n-m}\,f[\langle Ax,x\rangle;\underbrace{a,\ldots,a}_{m\ times};\ \underbrace{b,b,\ldots,b}_{(n-m)\ times}].$$

Since $\langle Ax,x\rangle\in[a,b]$ for any unit vector $x\in H$, we have $(\langle Ax,x\rangle-a)^m\geqslant 0$ for any choice of m, and $(\langle Ax,x\rangle-b)^{n-m}\leqslant 0$ when n and m are of different parity, and $(\langle Ax,x\rangle-b)^{n-m}\geqslant 0$ when n and m are of equal parity.

If the function f is n-convex, then $f[\langle Ax,x\rangle;\underbrace{a,\ldots,a}_{m\ times};\ \underbrace{b,b,\ldots,b}_{(n-m)\ times}]\geqslant 0$ for any $t\in[a,b]$, and if the function f is n-concave, then the inequality sign is reversed.

Now the relation (25) for n-convex function f and n and $m\geqslant 3$ of different parity, or n-concave function f and n and $m\geqslant 3$ of the same parity, becomes

$$\begin{aligned} f(\langle Ax,x\rangle)-\alpha_f\langle Ax,x\rangle-\beta_f \leqslant&(\langle Ax,x\rangle-a)\,(f[a,a]-f[a,b])+\sum_{k=2}^{m-1}\frac{f^{(k)}(a)}{k!}(\langle Ax,x\rangle-a)^k\\ &+\sum_{k=1}^{n-m} f[\underbrace{a,\ldots,a}_{m\ times};\underbrace{b,\ldots,b}_{k\ times}](\langle Ax,x\rangle-a)^m(\langle Ax,x\rangle-b)^{k-1}, \end{aligned} \tag{26}$$

and for n-convex function f and n and $m\geqslant 3$ of the same parity, or n-concave function f and n and $m\geqslant 3$ of different parity, the inequality sign is reversed.

In the same way we can replace t with $\langle Ax,x\rangle$ in (11) and get

$$f(\langle Ax,x\rangle)-\alpha_f\langle Ax,x\rangle-\beta_f =(b-\langle Ax,x\rangle)\,(f[a,b]-f[b,b])+\sum_{k=2}^{m-1}\frac{f^{(k)}(b)}{k!}(\langle Ax,x\rangle-b)^k$$

$$+\sum_{k=1}^{n-m} f[\underbrace{b,\dots,b}_{m\ times};\underbrace{a,\dots,a}_{k\ times}](\langle Ax,x\rangle-b)^m(\langle Ax,x\rangle-a)^{k-1}$$
$$+R_m^*(\langle Ax,x\rangle). \tag{27}$$

Now, we study positivity and negativity of the term $R_m^*(\langle Ax,x\rangle)$:

$$R_m^*(\langle Ax,x\rangle)=(\langle Ax,x\rangle-b)^m(\langle Ax,x\rangle-a)^{n-m}f[\langle Ax,x\rangle;\underbrace{b,\dots,b}_{m\ times};\ \underbrace{a,a,\dots,a}_{(n-m)\ times}].$$

Again, since $\langle Ax,x\rangle\in[a,b]$, we have $(\langle Ax,x\rangle-a)^{n-m}\geqslant 0$ for any choice of m, and $(\langle Ax,x\rangle-b)^m\leqslant 0$ when m is odd, and $(\langle Ax,x\rangle-b)^m\geqslant 0$ when m is even. If the function f is n-convex, then its n-th order divided differences are greater of equal to zero, and if the function f is n-concave, then its n-th order divided differences are less or equal to zero.

Equality (27) now turns into

$$f(\langle Ax,x\rangle)-\alpha_f\langle Ax,x\rangle-\beta_f\leqslant(b-\langle Ax,x\rangle)(f[a,b]-f[b,b])+\sum_{k=2}^{m-1}\frac{f^{(k)}(b)}{k!}(\langle Ax,x\rangle-b)^k$$
$$+\sum_{k=1}^{n-m} f[\underbrace{b,\dots,b}_{m\ times};\underbrace{a,\dots,a}_{k\ times}](\langle Ax,x\rangle-b)^m(\langle Ax,x\rangle-a)^{k-1} \tag{28}$$

for n-convex function f and an odd number $m\geqslant 3$ or n-concave function f and an even number $m\geqslant 3$. If f is n-convex and m is even, or if f is n-concave and m is odd, the inequality is reversed.

By combining inequalities (26) and (28) we get that

$$(\langle Ax,x\rangle-a)(f[a,a]-f[a,b])+\sum_{k=2}^{m-1}\frac{f^{(k)}(a)}{k!}(\langle Ax,x\rangle-a)^k$$
$$+\sum_{k=1}^{n-m} f[\underbrace{a,\dots,a}_{m\ times};\underbrace{b,\dots,b}_{k\ times}](\langle Ax,x\rangle-a)^m(\langle Ax,x\rangle-b)^{k-1}$$
$$\leqslant f(\langle Ax,x\rangle)-\alpha_f\langle Ax,x\rangle-\beta_f \tag{29}$$
$$\leqslant(b-\langle Ax,x\rangle)(f[a,b]-f[b,b])+\sum_{k=2}^{m-1}\frac{f^{(k)}(b)}{k!}(\langle Ax,x\rangle-b)^k$$
$$+\sum_{k=1}^{n-m} f[\underbrace{b,\dots,b}_{m\ times};\underbrace{a,\dots,a}_{k\ times}](\langle Ax,x\rangle-b)^m(\langle Ax,x\rangle-a)^{k-1}$$

holds if n is odd and f is n-convex and m is odd, or f is n-concave and m is even. If f is n-convex and m is even, or f is n-concave and m is odd, then the inequality signs are reversed.

Scalar product is linear in the first argument and $\|x\| = 1$, so when we apply it to (17) and then add series of inequalities (29) multiplied by -1, we get exactly (24), and the proof is complete. $\square$

Next result also provides us with a lower and upper bound for the difference in the Mond-Pečarić inequality, and it is obtained from Theorem 5 and Lemma 1.

THEOREM 8. *Let $A \in \mathscr{B}_h(H)$ be a selfadjoint operator with $Sp(A) \subseteq [a,b]$ and let $f \in \mathscr{C}^n([a,b])$. If the function f is n-convex and if $n \geqslant 3$ is odd, then*

$$
\begin{aligned}
&f[a,a;b](b-\langle Ax,x\rangle)(\langle Ax,x\rangle - a) + \sum_{k=2}^{n-1} f[a;\underbrace{b,\ldots,b}_{k\ times}]\langle (A-a\mathbf{1})(A-b\mathbf{1})^{k-1}x,x\rangle \\
&- (\langle Ax,x\rangle - a)^2 \sum_{k=2}^{n-2} f[a,a;\underbrace{b,\ldots,b}_{k\ times}](\langle Ax,x\rangle - b)^{k-1} \leqslant \langle f(A)x,x\rangle - f(\langle Ax,x\rangle) \qquad (30)\\
\leqslant & f[a,a;b]\langle (A-a\mathbf{1})(A-b\mathbf{1})x,x\rangle - (\langle Ax,x\rangle - a)\sum_{k=2}^{n-1} f[a;\underbrace{b,\ldots,b}_{k\ times}](\langle Ax,x\rangle - b)^{k-1} \\
&+ \sum_{k=2}^{n-2} f[a,a;\underbrace{b,\ldots,b}_{k\ times}]\langle (A-a\mathbf{1})^2(A-b\mathbf{1})^{k-1}x,x\rangle,
\end{aligned}
$$

where $x \in H$ is a unit vector. Inequalities (30) also hold when the function f is n-concave and n is even. In case when the function f is n-convex and n is even, or when the function f is n-concave and n is odd, the inequality signs in (30) are reversed.

Proof. By following a similar procedure as in the proof of the previous theorem, we start by replacing t with $\langle Ax,x\rangle$ in with relations (4) and (5) from Lemma 1. We get

$$
f(\langle Ax,x\rangle) - \alpha_f\langle Ax,x\rangle - \beta_f = \sum_{k=2}^{n-1} f[a;\underbrace{b,\ldots,b}_{k\ times}](\langle Ax,x\rangle - a)(\langle Ax,x\rangle - b)^{k-1} + R_1(\langle Ax,x\rangle) \qquad (31)
$$

and

$$
\begin{aligned}
f(\langle Ax,x\rangle) - \alpha_f\langle Ax,x\rangle - \beta_f =& f[a,a;b](\langle Ax,x\rangle - a)(\langle Ax,x\rangle - b) \\
&+ \sum_{k=2}^{n-2} f[a,a;\underbrace{b,\ldots,b}_{k\ times}](\langle Ax,x\rangle - a)^2(\langle Ax,x\rangle - b)^{k-1} \\
&+ R_2(\langle Ax,x\rangle) \qquad (32)
\end{aligned}
$$

respectively. After discussing the positivity an negativity of terms $R_1(\langle Ax,x\rangle)$ and $R_2(\langle Ax,x\rangle)$ in the same way as in the proof Theorem 7, from relations (31) and (32) we get a series of inequalities

$$
\begin{aligned}
&(\langle Ax,x\rangle - a)\sum_{k=2}^{n-1} f[a;\underbrace{b,\ldots,b}_{k\ times}](\langle Ax,x\rangle - b)^{k-1} \leqslant f(\langle Ax,x\rangle) - \alpha_f\langle Ax,x\rangle - \beta_f \qquad (33)\\
&\leqslant f[a,a;b](\langle Ax,x\rangle - a)(\langle Ax,x\rangle - b) + (\langle Ax,x\rangle - a)^2\sum_{k=2}^{n-2} f[a,a;\underbrace{b,\ldots,b}_{k\ times}](\langle Ax,x\rangle - b)^{k-1}
\end{aligned}
$$

that holds when n is odd and f is n-convex, or when n is even and f is n-concave. If n is odd and f is n-concave, or if n is even and f is n-convex, then the inequality signs in (33) are reversed.

By applying scalar product to (18) with $\|x\| = 1$, and then adding (33) multiplied by -1, we get exactly (30), which completes the proof. □

In the analogue way as described in the proof of the previous theorem, but with utilizing Lemma 2 and Theorem 6, we can get a similar lower and upper bound for the difference in the Mond-Pečarić inequality (2) that holds for all $n \in \mathbb{N}$, not only the odd ones.

THEOREM 9. *Let $A \in \mathscr{B}_h(H)$ be a selfadjoint operator with $Sp(A) \subseteq [a,b]$ and let $f \in \mathscr{C}^n([a,b])$. If the function f is n-convex, $n \geqslant 3$, then*

$$
\begin{aligned}
&f[b,b;a]\langle (A-b\mathbf{1})(A-a\mathbf{1})x,x\rangle - (\langle Ax,x\rangle - b)\sum_{k=2}^{n-1} f[b;\underbrace{a,\ldots,a}_{k\ times}](\langle Ax,x\rangle - a)^{k-1}\\
&+\sum_{k=2}^{n-2} f[b,b;\underbrace{a,\ldots,a}_{k\ times}]\langle (A-b\mathbf{1})^2(A-a\mathbf{1})^{k-1}x,x\rangle \leqslant \langle f(A)x,x\rangle - f(\langle Ax,x\rangle) \qquad (34)\\
&\leqslant f[b,b;a](b-\langle Ax,x\rangle)(\langle Ax,x\rangle - a) + \sum_{k=2}^{n-1} f[b;\underbrace{a,\ldots,a}_{k\ times}]\langle (A-b\mathbf{1})(A-a\mathbf{1})^{k-1}x,x\rangle\\
&-(\langle Ax,x\rangle - b)^2\sum_{k=2}^{n-2} f[b,b;\underbrace{a,\ldots,a}_{k\ times}](\langle Ax,x\rangle - a)^{k-1},
\end{aligned}
$$

where $x \in H$ is a unit vector. If the function f is n-concave, the inequality signs in (34) are reversed.

4. Applications to quasi-arithmetic means

Let A be a positive invertible operator on a Hilbert space such that $Sp(A) \subseteq [a,b]$ for some scalars $a < b$ and x a unit vector in H. Let f be a strictly monotone continuous function on $[a,b]$. Quasi-arithmetic mean of the operator A with respect to f is defined by

$$M_f(A,x) = f^{-1}\langle f(A)x,x\rangle. \qquad (35)$$

In papers [2] and [8] the authors have obtained some bounds for quasi-arithmetic operator means for convex and operator convex functions via Jensen's inequality.

Now, our intention is to derive mutual bounds for quasi-arithmetic means for a wider class of functions. In such a way, we will obtain some new reverse relations for quasi-arithmetic means that correspond to n-convex functions.

Before we state such results, we have to introduce some notations arising from this particular setting. Throughout this section we denote

$$F = f \circ g^{-1}, \quad \alpha_F = \frac{f(b)-f(a)}{g(b)-g(a)}, \quad \beta_F = \frac{g(b)f(a)-g(a)f(b)}{g(b)-g(a)}$$

and

$$g_a = \min\{g(a), g(b)\}, \quad g_b = \max\{g(a), g(b)\},$$

where f and g are strictly monotone functions. It is obvious that if the function g is increasing, then $g_a = g(a)$, $g_b = g(b)$, and if g is decreasing, then $g_a = g(b)$, $g_b = g(a)$.

Operator $g(A)$ is selfadjoint, and its spectrum is contained in the interval $[g_a, g_b]$, so all of the results from previous sections can be exploited in establishing some new reverses of Jensen's inequality and the Edmundson-Lah-Ribarič inequality for selfadjoint operators related to quasi-arithmetic means by substituting f with $F = f \circ g^{-1}$ and A with $g(A)$.

We start with some Edmundson-Lah-Ribarič type inequalities for quasi-arithmetic means which arise from the results from Section 2. The first result in this section is carried out by virtue of our Theorem 3.

COROLLARY 2. *Let $A \in \mathscr{B}_h(H)$ be a selfadjoint operator with $Sp(A) \subseteq [a,b]$, and let f,g be strictly monotone continuous functions such that $F = f \circ g^{-1} \in \mathscr{C}^n([a,b])$. If the function F is n-convex and if $n > m \geqslant 3$ are of different parity, then*

$$\begin{aligned} & f(M_f(A,x)) - \alpha_F g(M_g(A,x)) - \beta_F \\ \leqslant & (g(M_g(A,x)) - g_a)\left(F'(g_a) - F[g_a,g_b]\right) + \sum_{k=2}^{m-1} \frac{F^{(k)}(g_a)}{k!} \langle (g(A) - g_a\mathbf{1})^k x, x\rangle \\ & + \sum_{k=1}^{n-m} F[\underbrace{g_a,\dots,g_a}_{m \text{ times}};\underbrace{g_b,\dots,g_b}_{k \text{ times}}] \langle (g(A)-g_a\mathbf{1})^m (g(A)-g_b\mathbf{1})^{k-1} x, x\rangle, \end{aligned} \tag{36}$$

where $x \in H$ is a unit vector. Inequality (36) also holds when the function F is n-concave and n and m are of equal parity. In case when the function F is n-convex and n and m are of equal parity, or when the function F is n-concave and n and m are of different parity, the inequality sign in (36) is reversed.

The following result is a direct consequence of Theorem 4.

COROLLARY 3. *Let $A \in \mathscr{B}_h(H)$ be a selfadjoint operator with $Sp(A) \subseteq [a,b]$ and let f,g be strictly monotone continuous functions such that $F = f \circ g^{-1} \in \mathscr{C}^n([a,b])$. If the function F is n-convex and if $m \geqslant 3$ is odd, $n > m$, then*

$$\begin{aligned} & f(M_f(A,x)) - \alpha_F g\left(M_g(A,x)\right) - \beta_F \\ \leqslant & (g_b - g\left(M_g(A,x)\right))\left(F[g_a,g_b] - F'(g_b)\right) + \sum_{k=2}^{m-1} \frac{F^{(k)}(g_b)}{k!} \langle (g(A) - g_b\mathbf{1})^k x,x\rangle \\ & + \sum_{k=1}^{n-m} F[\underbrace{g_b,\ldots,g_b}_{m \text{ times}};\underbrace{g_a,\ldots,g_a}_{k \text{ times}}]\langle (g(A)-g_b\mathbf{1})^m (g(A)-g_a\mathbf{1})^{k-1}x,x\rangle, \end{aligned} \tag{37}$$

where $x \in H$ is a unit vector. Inequality (37) also holds when the function F is n-concave and m is even. In case when the function F is n-convex and m is even, or when the function f is n-concave and m is odd, the inequality sign in (37) is reversed.

Our next result arises from Theorem 5.

COROLLARY 4. *Let $A \in \mathscr{B}_h(H)$ be a selfadjoint operator with $Sp(A) \subseteq [a,b]$ and let f,g be strictly monotone continuous functions such that $F = f \circ g^{-1} \in \mathscr{C}^n([a,b])$. If the function F is n-convex and if $n \geqslant 3$ is odd, then*

$$\begin{aligned} & \sum_{k=2}^{n-1} F[g_a;\underbrace{g_b,\ldots,g_b}_{k \text{ times}}]\langle (g(A)-g_a\mathbf{1})(g(A)-g_b\mathbf{1})^{k-1}x,x\rangle \\ \leqslant & f(M_f(A,x)) - \alpha_F g\left(M_g(A,x)\right) - \beta_F \leqslant F[g_a,g_a;g_b]\langle (g(A)-g_a\mathbf{1})(g(A)-g_b\mathbf{1})x,x\rangle \\ & + \sum_{k=2}^{n-2} F[g_a,g_a;\underbrace{g_b,\ldots,g_b}_{k \text{ times}}]\langle (g(A)-g_a\mathbf{1})^2(g(A)-g_b\mathbf{1})^{k-1}x,x\rangle, \end{aligned} \tag{38}$$

where $x \in H$ is a unit vector. Inequalities (38) also hold when the function F is n-concave and n is even. In case when the function F is n-convex and n is even, or when the function F is n-concave and n is odd, the inequality signs in (38) are reversed.

As a consequence of Theorem 6, we have the following result.

COROLLARY 5. *Let $A \in \mathscr{B}_h(H)$ be a selfadjoint operator with $Sp(A) \subseteq [a,b]$ and let f,g be strictly monotone continuous functions such that $F = f \circ g^{-1} \in \mathscr{C}^n([a,b])$. If the function F is n-convex, $n \geqslant 3$, then*

$$\begin{aligned} & F[g_b,g_b;g_a]\langle (g(A)-g_b\mathbf{1})(g(A)-g_a\mathbf{1})x,x\rangle \\ & + \sum_{k=2}^{n-2} F[g_b,g_b;\underbrace{g_a,\ldots,g_a}_{k \text{ times}}]\langle (g(A)-g_b\mathbf{1})^2(g(A)-g_a\mathbf{1})^{k-1}x,x\rangle \\ \leqslant & f(M_f(A,x)) - \alpha_F g\left(M_g(A,x)\right) - \beta_F \end{aligned} \tag{39}$$

$$\leqslant \sum_{k=2}^{n-1} F[g_b; \underbrace{g_a, \ldots, g_a}_{k \text{ times}}] \langle (g(A) - g_b \mathbf{1})(g(A) - g_a \mathbf{1})^{k-1} x, x \rangle,$$

where $x \in H$ is a unit vector. If the function F is n-concave, the inequality signs in (39) are reversed.

The corollaries below arise from the results from Section 3 and give us Jensen type inequalities for quasi-arithmetic means. They are obtained from Theorem 7, 8 and 9 respectively.

COROLLARY 6. *Let $A \in \mathscr{B}_h(H)$ be a selfadjoint operator with $Sp(A) \subseteq [a,b]$, let n be an odd number and let f, g be strictly monotone continuous functions such that $F = f \circ g^{-1} \in \mathscr{C}^n([a,b])$. If the function F is n-convex and if $m \geqslant 3$ is odd, $n > m$, then*

$$\begin{aligned}
& F(g_a) - F(g_b) + g_b F'(g_b) - g_a F'(g_a) + (F'(g_a) - F'(g_b)) g(M_g(A,x)) \\
& + \sum_{k=2}^{m-1} \left(\frac{F^{(k)}(g_a)}{k!} \langle (g(A) - g_a \mathbf{1})^k x, x \rangle - \frac{F^{(k)}(g_b)}{k!} (g(M_g(A,x)) - g_b)^k \right) \\
& + \sum_{k=1}^{n-m} F[\underbrace{g_a, \ldots, g_a}_{m \text{ times}}; \underbrace{g_b, \ldots, g_b}_{k \text{ times}}] \langle (g(A) - g_a \mathbf{1})^m (g(A) - g_b \mathbf{1})^{k-1} x, x \rangle \\
& - \sum_{k=1}^{n-m} F[\underbrace{g_b, \ldots, g_b}_{m \text{ times}}; \underbrace{g_a, \ldots, g_a}_{k \text{ times}}] (g(M_g(A,x)) - g_b)^m (g(M_g(A,x)) - g_a)^{k-1} \\
\leqslant & f(M_f(A,x)) - f(M_g(A,x)) \qquad (40) \\
\leqslant & F(g_b) - F(g_a) + g_a F'(g_a) - g_b F'(g_b) + (F'(g_b) - F'(g_a)) g(M_g(A,x)) \\
& + \sum_{k=2}^{m-1} \left(\frac{F^{(k)}(g_b)}{k!} \langle (g(A) - g_b \mathbf{1})^k x, x \rangle - \frac{F^{(k)}(g_a)}{k!} (g(M_g(A,x)) - g_a)^k \right) \\
& + \sum_{k=1}^{n-m} F[\underbrace{g_b, \ldots, g_b}_{m \text{ times}}; \underbrace{g_a, \ldots, g_a}_{k \text{ times}}] \langle (g(A) - g_b \mathbf{1})^m (g(A) - g_a \mathbf{1})^{k-1} x, x \rangle \\
& - \sum_{k=1}^{n-m} F[\underbrace{g_a, \ldots, g_a}_{m \text{ times}}; \underbrace{g_b, \ldots, g_b}_{k \text{ times}}] (g(M_g(A,x)) - g_a)^m (g(M_g(A,x)) - g_b)^{k-1},
\end{aligned}$$

where $x \in H$ is a unit vector. Inequalities (40) also hold when the function F is n-concave and m is even. In case when the function F is n-convex and m is even, or when the function F is n-concave and m is odd, the inequality signs in (40) are reversed.

COROLLARY 7. *Let $A \in \mathscr{B}_h(H)$ be a selfadjoint operator with $Sp(A) \subseteq [a,b]$ and let f, g be strictly monotone continuous functions such that $F = f \circ g^{-1} \in \mathscr{C}^n([a,b])$.*

If the function F is n-convex and if $n \geqslant 3$ is odd, then

$$
\begin{aligned}
& F[g_a, g_a; g_b](g_b - g(M_g(A,x)))(g(M_g(A,x)) - g_a) \\
& \quad + \sum_{k=2}^{n-1} F[g_a; \underbrace{g_b, \dots, g_b}_{k \text{ times}}] \langle (g(A) - g_a \mathbf{1})(g(A) - g_b \mathbf{1})^{k-1} x, x \rangle \\
& \quad - (g(M_g(A,x)) - g_a)^2 \sum_{k=2}^{n-2} F[g_a, g_a; \underbrace{g_b, \dots, g_b}_{k \text{ times}}](g(M_g(A,x)) - g_b)^{k-1} \\
\leqslant & f(M_f(A,x)) - f(M_g(A,x)) \qquad (41) \\
\leqslant & F[g_a, g_a; g_b] \langle (g(A) - g_a \mathbf{1})(g(A) - g_b \mathbf{1}) x, x \rangle \\
& \quad - (g(M_g(A,x)) - g_a) \sum_{k=2}^{n-1} F[g_a; \underbrace{g_b, \dots, g_b}_{k \text{ times}}](g(M_g(A,x)) - g_b)^{k-1} \\
& \quad + \sum_{k=2}^{n-2} F[g_a, g_a; \underbrace{g_b, \dots, g_b}_{k \text{ times}}] \langle (g(A) - g_a \mathbf{1})^2 (g(A) - g_b \mathbf{1})^{k-1} x, x \rangle,
\end{aligned}
$$

where $x \in H$ is a unit vector. Inequalities (41) also hold when the function F is n-concave and n is even. In case when the function F is n-convex and n is even, or when the function F is n-concave and n is odd, the inequality signs in (41) are reversed.

COROLLARY 8. *Let $A \in \mathscr{B}_h(H)$ be a selfadjoint operator with $Sp(A) \subseteq [a,b]$ and let f, g be strictly monotone continuous functions such that $F = f \circ g^{-1} \in \mathscr{C}^n([a,b])$. If the function F is n-convex, $n \geqslant 3$, then*

$$
\begin{aligned}
& F[g_b, g_b; g_a] \langle (g(A) - g_b \mathbf{1})(g(A) - g_a \mathbf{1}) x, x \rangle \\
& \quad - (g(M_g(A,x)) - g_b) \sum_{k=2}^{n-1} F[g_b; \underbrace{g_a, \dots, g_a}_{k \text{ times}}](g(M_g(A,x)) - g_a)^{k-1} \\
& \quad + \sum_{k=2}^{n-2} F[g_b, g_b; \underbrace{g_a, \dots, g_a}_{k \text{ times}}] \langle (g(A) - g_b \mathbf{1})^2 (g(A) - g_a \mathbf{1})^{k-1} x, x \rangle \\
\leqslant & f(M_f(A,x)) - f(M_g(A,x)) \qquad (42) \\
\leqslant & F[g_b, g_b; g_a](g_b - g(M_g(A,x)))(g(M_g(A,x)) - g_a) \\
& \quad + \sum_{k=2}^{n-1} F[g_b; \underbrace{g_a, \dots, g_a}_{k \text{ times}}] \langle (g(A) - g_b \mathbf{1})(g(A) - g_a \mathbf{1})^{k-1} x, x \rangle \\
& \quad - (g(M_g(A,x)) - g_b)^2 \sum_{k=2}^{n-2} F[g_b, g_b; \underbrace{g_a, \dots, g_a}_{k \text{ times}}](g(M_g(A,x)) - g_a)^{k-1},
\end{aligned}
$$

where $x \in H$ is a unit vector. If the function F is n-concave, the inequality signs in (42) are reversed.

4.1. Examples with power means

Let A be a positive invertible operator on a Hilbert space and x a unit vector in H. For $r \in \mathbb{R}$, the power mean $M_r(A,x)$ is defined by

$$M_r(A,x) = (\langle A^r x, x\rangle)^{1/r}.$$

In [5] it has been shown that if $r \to 0$, then $(\langle A^r x, x\rangle)^{1/r}$ converges monotone to $\exp\langle \log Ax, x\rangle$, so we can extend the definition of the power mean to the case $r = 0$.

Since power means are a special case of quasi-arithmetic means for particular choices of functions f and g, first let us set $f(t) = t^s$ and $g(t) = t^r$, where s and r are real parameters such that $sr \neq 0$ and $t > 0$.

Now we have

- ▷ for $rs \neq 0$: $F_{r,s}(t) = t^{s/r}$ and $F_{r,s}^{(n)}(t) = \frac{s}{r}\left(\frac{s}{r}-1\right)\left(\frac{s}{r}-2\right)\cdots\left(\frac{s}{r}-n+1\right)t^{\frac{s}{r}-n}$
- ▷ for $r \neq 0$: $F_{r,0}(t) = \frac{1}{r}\log t$ and $F_{r,0}^{(n)}(t) = \frac{1}{r}(-1)^{n-1}(n-1)!\, t^{-n}$
- ▷ for $s \neq 0$: $F_{0,s}(t) = e^{st}$ and $F_{0,s}^{(n)}(t) = s^n\, e^{st}$,

and we see that the function $F_{r,s}(t)$ belongs to the class $\mathscr{C}^n(\mathbb{R})$ for any $n \in \mathbb{N}$ and any $r, s \in \mathbb{R}$.

It is trivial to check under which conditions on parameters r and s the function $F_{r,s}(t)$ is n-convex (or n-concave), so all of the results regarding quasi-arithmetic means from Section 4 can be applied to power means.

Here below we will show only Jensen type inequalities for power means that hold for every $n \in \mathbb{N}$ which are a special case of Corollary 8. All of the other inequalities regarding power means that arise from the previous section are obtained in the same manner. Let $n \in \mathbb{N}$, $n \geqslant 3$.

- If n is even and $s < 0 < r$, or if n is odd and $0 < s < r$, or if $\lfloor s/r \rfloor$ and n are of different parity and $0 < r < s$, then we have

$$\begin{aligned}
&\frac{1}{b^r - a^r}\left(\frac{s}{r}b^{s-r} - \frac{b^s - a^s}{b^r - a^r}\right)\langle (A^r - b^r\mathbf{1})(A^r - a^r\mathbf{1})x, x\rangle \\
&\quad - (M_r(A,x)^r - b^r)\sum_{k=2}^{n-1} F_{r,s}[b^r; \underbrace{a^r, \ldots, a^r}_{k \text{ times}}](M_r(A,x)^r - a^r)^{k-1} \\
&\quad + \sum_{k=2}^{n-2} F_{r,s}[b^r, b^r; \underbrace{a^r, \ldots, a^r}_{k \text{ times}}]\langle (A^r - b^r\mathbf{1})^2 (A^r - a^r\mathbf{1})^{k-1} x, x\rangle \\
&\leqslant M_s(A,x)^s - M_r(A,x)^s \\
&\leqslant \frac{1}{b^r - a^r}\left(\frac{s}{r}b^{s-r} - \frac{b^s - a^s}{b^r - a^r}\right)(b^r - M_r(A,x)^r)(M_r(A,x)^r - a^r) \\
&\quad + \sum_{k=2}^{n-1} F_{r,s}[b^r; \underbrace{a^r, \ldots, a^r}_{k \text{ times}}]\langle (A^r - b^r\mathbf{1})(A^r - a^r\mathbf{1})^{k-1} x, x\rangle
\end{aligned}$$

$$-(M_r(A,x)^r-b^r)^2\sum_{k=2}^{n-2}F_{r,s}[b^r,b^r;\underbrace{a^r,\dots,a^r}_{k\ times}](M_r(A,x)^r-a^r)^{k-1},$$

where $x\in H$ is a unit vector. If n is odd and $s<0<r$, or if n is even and $0<s<r$, or if $\lfloor s/r\rfloor$ and n are of same parity and $0<r<s$, then the inequality signs are reversed.

- If n is even and $r<0<s$, or if n is odd and $r<s<0$, or if $\lfloor s/r\rfloor$ and n are of different parity and $s<r<0$, then we have

$$\begin{aligned}
&\frac{1}{b^r-a^r}\left(\frac{b^s-a^s}{b^r-a^r}-\frac{s}{r}a^{s-r}\right)\langle(A^r-a^r\mathbf{1})(A^r-b^r\mathbf{1})x,x\rangle\\
&-(M_r(A,x)^r-a^r)\sum_{k=2}^{n-1}F_{r,s}[a^r;\underbrace{b^r,\dots,b^r}_{k\ times}](M_r(A,x)^r-b^r)^{k-1}\\
&+\sum_{k=2}^{n-2}F_{r,s}[a^r,a^r;\underbrace{b^r,\dots,b^r}_{k\ times}]\langle(A^r-a^r\mathbf{1})^2(A^r-b^r\mathbf{1})^{k-1}x,x\rangle\\
\leqslant&M_s(A,x)^s-M_r(A,x)^s\\
\leqslant&\frac{1}{b^r-a^r}\left(\frac{b^s-a^s}{b^r-a^r}-\frac{s}{r}a^{s-r}\right)(b^r-M_r(A,x)^r)(M_r(A,x)^r-a^r)\\
&+\sum_{k=2}^{n-1}F_{r,s}[a^r;\underbrace{b^r,\dots,b^r}_{k\ times}]\langle(A^r-a^r\mathbf{1})(A^r-b^r\mathbf{1})^{k-1}x,x\rangle\\
&-(M_r(A,x)^r-a^r)^2\sum_{k=2}^{n-2}F_{r,s}[a^r,a^r;\underbrace{b^r,\dots,b^r}_{k\ times}](M_r(A,x)^r-b^r)^{k-1},
\end{aligned}$$

where $x\in H$ is a unit vector. If n is odd and $r<0<s$, or if n is even and $r<s<0$, or if $\lfloor s/r\rfloor$ and n are of same parity and $s<r<0$, then the inequality signs are reversed.

- If n is odd and $r>0$

$$\begin{aligned}
&\frac{1}{b^r-a^r}\left(\frac{1}{rb^r}-\frac{\log b-\log a}{b^r-a^r}\right)\langle(A^r-b^r\mathbf{1})(A^r-a^r\mathbf{1})x,x\rangle\\
&-(M_r(A,x)^r-b^r)\sum_{k=2}^{n-1}F_{r,0}[b^r;\underbrace{a^r,\dots,a^r}_{k\ times}](M_r(A,x)^r-a^r)^{k-1}\\
&+\sum_{k=2}^{n-2}F_{r,0}[b^r,b^r;\underbrace{a^r,\dots,a^r}_{k\ times}]\langle(A^r-b^r\mathbf{1})^2(A^r-a^r\mathbf{1})^{k-1}x,x\rangle\\
\leqslant&\log M_0(A,x)-\log M_r(A,x) \qquad (43)\\
\leqslant&\frac{1}{b^r-a^r}\left(\frac{1}{rb^r}-\frac{\log b-\log a}{b^r-a^r}\right)(b^r-M_r(A,x)^r)(M_g(A,x)^r-a^r)
\end{aligned}$$

$$+\sum_{k=2}^{n-1} F_{r,0}[b^r;\underbrace{a^r,\dots,a^r}_{k\ times}]\langle (A^r-b^r\mathbf{1})(A^r-a^r\mathbf{1})^{k-1}x,x\rangle$$
$$-(M_r(A,x)^r-b^r)^2\sum_{k=2}^{n-2} F_{r,0}[b^r,b^r;\underbrace{a^r,\dots,a^r}_{k\ times}](M_r(A,x)^r-a^r)^{k-1},$$

where $x\in H$ is a unit vector. If n is even and $r>0$, the inequality signs are reversed.

- If n is even and $r<0$

$$\frac{1}{b^r-a^r}\left(\frac{\log b-\log a}{b^r-a^r}-\frac{1}{ra^r}\right)\langle (A^r-a^r\mathbf{1})(A^r-b^r\mathbf{1})x,x\rangle$$
$$-(M_r(A,x)^r-a^r)\sum_{k=2}^{n-1} F_{r,0}[a^r;\underbrace{b^r,\dots,b^r}_{k\ times}](M_r(A,x)^r-b^r)^{k-1}$$
$$+\sum_{k=2}^{n-2} F_{r,0}[a^r,a^r;\underbrace{b^r,\dots,b^r}_{k\ times}]\langle (A^r-a^r\mathbf{1})^2(A^r-b^r\mathbf{1})^{k-1}x,x\rangle$$
$$\leqslant \log M_0(A,x)-\log M_r(A,x) \tag{44}$$
$$\leqslant \frac{1}{b^r-a^r}\left(\frac{\log b-\log a}{b^r-a^r}-\frac{1}{ra^r}\right)(b^r-M_r(A,x)^r)(M_g(A,x)^r-a^r)$$
$$+\sum_{k=2}^{n-1} F_{r,0}[a^r;\underbrace{b^r,\dots,b^r}_{k\ times}]\langle (A^r-a^r\mathbf{1})(A^r-b^r\mathbf{1})^{k-1}x,x\rangle$$
$$-(M_r(A,x)^r-a^r)^2\sum_{k=2}^{n-2} F_{r,0}[a^r,a^r;\underbrace{b^r,\dots,b^r}_{k\ times}](M_r(A,x)^r-b^r)^{k-1},$$

where $x\in H$ is a unit vector. If n is odd and $r<0$, the inequality signs are reversed.

- If $s>0$, or if n is even and $s<0$, then

$$\frac{1}{\log b-\log a}\left(sb^s-\frac{b^s-a^s}{\log b-\log a}\right)\langle (\log A-\log b\mathbf{1})(\log A-\log a\mathbf{1})x,x\rangle$$
$$-(\log M_0(A,x)-\log b)\sum_{k=2}^{n-1} F_{0,s}[\log b;\underbrace{\log a,\dots,\log a}_{k\ times}](\log M_0(A,x)-\log a)^{k-1}$$
$$+\sum_{k=2}^{n-2} F_{0,s}[\log b,\log b;\underbrace{\log a,\dots,\log a}_{k\ times}]\langle (\log A-\log b\mathbf{1})^2(\log A-\log a\mathbf{1})^{k-1}x,x\rangle$$
$$\leqslant M_s(A,x)^s-M_0(A,x)^s \tag{45}$$
$$\leqslant \frac{1}{\log b-\log a}\left(sb^s-\frac{b^s-a^s}{\log b-\log a}\right)(\log b-\log M_g(A,x))\,(\log M_g(A,x-\log a)$$

$$+\sum_{k=2}^{n-1} F_{0,s}[\log b;\underbrace{\log a,\ldots,\log a}_{k\ times}]\langle(\log A-\log b\mathbf{1})(\log A-\log a\mathbf{1})^{k-1}x,x\rangle$$

$$-(\log M_0(A,x)-\log b)^2$$

$$\sum_{k=2}^{n-2} F_{0,s}[\log b,\log b;\underbrace{\log a,\ldots,\log a}_{k\ times}](\log M_0(A,x)-\log a)^{k-1},$$

where $x\in H$ is a unit vector. If n is odd and $s<0$, the inequality signs are reversed.

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