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OPERATOR INEQUALITIES VIA GEOMETRIC CONVEXITY

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Abstract. The main goal of this paper is to present new generalizations of some known inequalities for the numerical radius and unitarily invariant norms of Hilbert space operators. These extensions result from a special treatment of both convex and geometrically convex functions. In the end, we present several scalar inequalities for geometrically convex functions similar to those inequalities known for convex functions.

1. Introduction

Let $\mathscr{H}$ be a Hilbert space and let $\mathscr{B}(\mathscr{H})$ denote the C^*–algebra of all bounded linear operators acting on $\mathscr{H}$. An important class of operators in $\mathscr{B}(\mathscr{H})$ is the cone $\mathscr{B}(\mathscr{H})^+$ of positive semidefinite operators; where an operator A is said to be positive semidefinite if $\langle Ax,x\rangle \geqslant 0$ for all $x\in\mathscr{H}$. If $A\in\mathscr{B}(\mathscr{H})^+$, we simply write $A\geqslant 0$. If, in addition to being positive semidefinite, A is invertible, it is said to be strictly positive, and it is denoted as $A>0$.

For decades, inequalities governing strictly positive operators have attracted researchers in the field of operator theory. Among the most basic inequalities in this field is the so called arithmetic-geometric mean inequality (operator Young inequality) stating [1]

$$A\sharp_v B \leqslant A\nabla_v B,\quad 0\leqslant v\leqslant 1, \tag{1}$$

where $A\sharp_v B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^v A^{\frac{1}{2}}$ and $A\nabla_v B=(1-v)A+vB$ are the geometric and arithmetic means of the strictly positive operators A and B, respectively. For the scalars $a,b>0$, we use the same notations. A simple proof of this inequality follows from the scalar Young inequality (i.e., $a\sharp_v b\leqslant a\nabla_v b, 0\leqslant v\leqslant 1$) together with a standard functional calculus argument.

Although this inequality looks very simple, it has attracted numerous researchers, where several variants of this inequality have been obtained. We refer the reader to [2, 7, 23] as a sample of recent studies of this inequality.

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Given a unitarily invariant norm $|||\cdot|||$ on $\mathscr{B}(\mathscr{H})$, for finite dimensional $\mathscr{H}$, the following Hölder inequality holds [10]

$$|||A^{1-v}XB^{v}||| \leqslant |||AX|||^{1-v}|||XB|||^{v}, \quad 0 \leqslant v \leqslant 1, \tag{2}$$

for $A,B \in \mathscr{B}(\mathscr{H})^{+}$ and an arbitrary $X \in \mathscr{B}(\mathscr{H})$.

In [16], it was shown that the function $f(v) = |||A^{1-v}XB^{v}|||$ is log-convex on $\mathbb{R}$. This entails (2) and its reverse when $v \notin [0,1]$.

Searching the literature, we find that convexity and log-convexity have stood behind many celebrated inequalities. This includes (1), (2), the Heinz inequality and almost all their variants. See [8, 16, 17].

In this paper, we present some applications of convex functions and geometrically convex functions to operator inequalities. In the sequel, we use the symbol I as a sub–interval of $(0,\infty)$ and consider the continuous function $f : I \to (0,\infty)$, unless otherwise specified. Recall that f is called geometrically convex [14] if

$$f\left(a^{1-v}b^{v}\right) \leqslant f^{1-v}(a)f^{v}(b), \quad a,b \in I, \quad v \in [0,1]. \tag{3}$$

We shall prove that the function $f(v) = |||A^{v}XB^{v}|||$ is geometrically convex under some conditions on A,B. This adds a new property to the already known properties of this function. Of course, this will imply a new set of operator inequalities.

Another interesting application that we aim to present is how convex functions, geometrically convex functions and the numerical radius are related.

Recall that the numerical radius $\omega(A)$ and the usual operator norm $\|A\|$ of an operator A are defined, respectively, by $\omega(A) = \sup\limits_{\|x\|=1} |\langle Ax,x\rangle|$ and $\|A\| = \sup\limits_{\|x\|=1} \|Ax\|$, where $\|x\| = \sqrt{\langle x,x\rangle}$. Of course, $\omega(A)$ defines a norm on $\mathscr{B}(\mathscr{H})$ and for every $A \in \mathscr{B}(\mathscr{H})$, we have

$$\frac{1}{2}\|A\| \leqslant \omega(A) \leqslant \|A\|. \tag{4}$$

The second inequality in (4) has been improved considerably by Kittaneh in [9] as follows

$$\omega(A) \leqslant \frac{1}{2}\left\|(A^{*}A)^{\frac{1}{2}} + (AA^{*})^{\frac{1}{2}}\right\| \leqslant \frac{1}{2}\left(\|A\| + \left\|A^{2}\right\|^{\frac{1}{2}}\right) \leqslant \|A\|. \tag{5}$$

On the other hand, Dragomir extended (5) to the product of two operators to the following form [4],

$$\omega(B^{*}A)^{r} \leqslant \frac{1}{2}\|(A^{*}A)^{r} + (B^{*}B)^{r}\|, \qquad \text{for all } r \geqslant 1\ . \tag{6}$$

One of main applications is to show that (6) follows as a special case of the following inequality

$$f(\omega(B^{*}A)) \leqslant \frac{1}{2}\|f(A^{*}A) + f(B^{*}B)\|$$

valid for the convex function f, with some additional properties. Many other applications to the numerical radius will be presented too.

Further, we prove that for $0 \leqslant \alpha \leqslant 1$,

$$f\left(\left\|\frac{A+B}{2}\right\|\right) \leqslant \frac{1}{4}\left(\left\|f\left(|A|^{2\alpha}\right)+f\left(|B|^{2\alpha}\right)\right\|+\left\|f\left(|A^*|^{2(1-\alpha)}\right)+f\left(|B^*|^{2(1-\alpha)}\right)\right\|\right) \tag{7}$$

and

$$f\left(\omega\left(\frac{A+B}{2}\right)\right) \leqslant \frac{1}{4}\left\|f\left(|A|^{2\alpha}\right)+f\left(|A^*|^{2(1-\alpha)}\right)+f\left(|B|^{2\alpha}\right)+f\left(|B^*|^{2(1-\alpha)}\right)\right\|, \tag{8}$$

for the convex function f. This provides a considerable generalization of the well known inequality [5]

$$\|A+B\|^r \leqslant 2^{r-2}\left(\left\||A|^{2\alpha r}+|B|^{2\alpha r}\right\|+\left\||A^*|^{2(1-\alpha)r}+|B^*|^{2(1-\alpha)r}\right\|\right), \quad r \geqslant 1, 0 \leqslant \alpha \leqslant 1. \tag{9}$$

Many other related results that generalize well known inequalities will be presented too.

The organization of this paper will be as follows. In the second section, we present several applications including convex and geometrically convex functions when they are applied to the numerical radius and the operator norm of Hilbert space operators. In the third section, we present applications of geometrically convex functions to unitarily invariant norms of matrices and in the end we present several versions of the scalar case (3). This includes reverses, refinements, multidimensional versions and much more.

2. Some numerical radius inequalities

In this section, we present our applications to numerical radius inequalities. We emphasize that such an application to numerical radius inequalities is a new approach that we hope to be useful for researchers in the field.

The results of this section present the general form of some known inequalities in the literature, such as (6), (9) and many other inequalities appearing in [5]. This gives a new perspective to these inequalities.

Our first result in this direction is the general form of (6).

THEOREM 1. *Let $A, B \in \mathscr{B}(\mathscr{H})$ and $f : [0,\infty) \to [0,\infty)$ be an increasing convex function. Then*

$$f\left(\omega\left(B^*A\right)\right) \leqslant \frac{1}{2}\left\|f\left(A^*A\right)+f\left(B^*B\right)\right\|. \tag{10}$$

Proof. We recall the following Jensen's type inequality [6, Theorem 1.2],

$$f\left(\langle Ax, x\rangle\right) \leqslant \langle f(A)x, x\rangle, \tag{11}$$

for any unit vector $x \in \mathscr{H}$, where f is a convex function on I and A is a self-adjoint operator with spectrum contained in I. Now, let $x \in \mathscr{H}$ be a unit vector. We have

$$\begin{aligned}
& f(|\langle B^*Ax,x\rangle|) = f(|\langle Ax,Bx\rangle|) \\
\leqslant & f(\|Ax\|\,\|Bx\|) \quad \text{(by the Cauchy–Schwarz inequality)} \\
= & f\left(\sqrt{\langle Ax,Ax\rangle\,\langle Bx,Bx\rangle}\right) \\
= & f\left(\sqrt{\langle A^*Ax,x\rangle\,\langle B^*Bx,x\rangle}\right) \\
\leqslant & f\left(\frac{\langle A^*Ax,x\rangle + \langle B^*Bx,x\rangle}{2}\right) \quad (f \text{ being increasing and AM–GM inequality}) \\
\leqslant & \frac{1}{2} f(\langle A^*Ax,x\rangle) + \frac{1}{2} f(\langle B^*Bx,x\rangle) \quad (f \text{ being convex}) \\
\leqslant & \frac{1}{2}(\langle f(A^*A)x,x\rangle + \langle f(B^*B)x,x\rangle) \quad \text{(by (11))}.
\end{aligned}$$

Thus, we have shown

$$f(|\langle B^*Ax,x\rangle|) \leqslant \frac{1}{2}\langle f(A^*A) + f(B^*B)x,x\rangle.$$

By taking supremum over $x \in \mathscr{H}$ with $\|x\| = 1$, we get

$$\begin{aligned}
f(\omega(B^*A)) = f&\left(\sup_{\|x\|=1} |\langle B^*Ax,x\rangle|\right) \\
&= \sup_{\|x\|=1} f(|\langle B^*Ax,x\rangle|) \quad (f \text{ is increasing}) \\
&\leqslant \frac{1}{2}\sup_{\|x\|=1} \langle f(A^*A) + f(B^*B)x,x\rangle = \frac{1}{2}\|f(A^*A) + f(B^*B)\|.
\end{aligned}$$

Therefore, (10) holds. □

One can check easily that the function $f(t) = t^r$ $(t > 0, r \geqslant 1)$ satisfies the assumptions in Theorem 1. So, the inequality (10) implies (6).

COROLLARY 1. *Let f as in Theorem 1 and let $A,B,X \in \mathscr{B}(\mathscr{H})$. Then,*

$$f(\omega(A^*XB)) \leqslant \frac{1}{2}\left\|f\left(A^*|X^*|^{2v}A\right) + f\left(B^*|X|^{2(1-v)}B\right)\right\|.$$

Proof. Let $X = U|X|$ be the polar decomposition of X. Then,

$$f(\omega(A^*XB)) = f(\omega(A^*U|X|B)) = f\left(\omega((|X|^vU^*A)^*\left(|X|^{1-v}B\right)\right).$$

By substituting $B = |X|^vU^*A$ and $A = |X|^{1-v}B$ in Theorem 1, we get the desired inequality, noting that when $X = U|X|$, we have $|X^*| = U|X|U^*$ which implies $|X^*|^{2v} = U|X|^{2v}U^*$. □

The function also $f(t)=t^r\ (t>0, r\geqslant 1)$ satisfies the assumptions in Corollary 1. Thus Corollary 1 recovers the inequality given in [21]:

$$\omega^r(A^*XB)\leqslant \frac{1}{2}\left\|\left(A^*|X^*|^{2v}A\right)^r+\left(B^*|X|^{2(1-v)}B\right)^r\right\|,(r\geqslant 1,\ v\in[0,1]).$$

Another interesting inequality for $f(\omega(B^*XA))$ may be obtained as follows. First, notice that if f is a convex function and $0\leqslant\alpha\leqslant 1$, it follows that

$$f(\alpha t)\leqslant \alpha f(t)+(1-\alpha)f(0). \tag{12}$$

For the coming results, we will use the term norm-contractive to mean an operator X whose operator norm satisfies $\|X\|\leqslant 1$. Norm-expansive will mean $\|X\|\geqslant 1$.

PROPOSITION 1. Under the same assumptions as in Theorem 1, the following inequality holds for the norm-contractive $X\in\mathscr{B}(\mathscr{H})$,

$$f(\omega(B^*XA))\leqslant \frac{||X||}{2}\|f(A^*A)+f(B^*B)\|+(1-\|X\|)f(0).$$

In particular, if $f(0)=0$, then

$$f(\omega(B^*XA))\leqslant \frac{||X||}{2}\|f(A^*A)+f(B^*B)\|.$$

Proof. Proceeding as in Theorem 1 and noting (12), we have

$$\begin{aligned} f(|\langle B^*XAx,x\rangle|)&=f(|\langle XAx,Bx\rangle|)\leqslant f(\|XAx\|\,\|Bx\|)\leqslant f(\|X\|\,\|Ax\|\,\|Bx\|)\\ &\leqslant \|X\|f(\|Ax\|\,\|Bx\|)+(1-\|X\|)f(0). \end{aligned}$$

Then an argument similar to Theorem 1 implies the desired inequality. □

In particular, if $f(t)=t^r,\ (t>0,\ r\geqslant 1)$ we obtain the following extension of (6).

COROLLARY 2. *Under the same assumptions as in Proposition 1, we have for* $r\geqslant 1$,

$$\omega^r(B^*XA)\leqslant \frac{\|X\|^r}{2}\|(A^*A)^r+(B^*B)^r\|. \tag{13}$$

Proof. Notice that a direct application of Proposition 1 implies the weaker inequality $\omega^r(B^*XA)\leqslant\frac{\|X\|}{2}\|(A^*A)^r+(B^*B)^r\|$. However, noting the proof of Proposition 1 for the function $f(t)=t^r,\ (t>0,\ r\geqslant 1)$, we have

$$\begin{aligned} f(|\langle B^*XAx,x\rangle|)&=f(|\langle XAx,Bx\rangle|)\leqslant f(\|XAx\|\,\|Bx\|)\leqslant f(\|X\|\,\|Ax\|\,\|Bx\|)\\ &=f(\|X\|)f(\|Ax\|\,\|Bx\|)\quad (\text{Since } f(t)=t^r). \end{aligned}$$

Arguing as before implies the desired inequality. □

When $X=I$, Corollary 2 recovers the inequality (6).

Our next target is to show similar inequalities for geometrically convex functions which are concave, instead of convex. For the purpose of our results, we remind the reader of the following inequality (see, e.g., [12, Theorem 6])

$$f(\langle Ax,x\rangle) \leqslant k(m,M,f)^{-1}\langle f(A)x,x\rangle, \tag{14}$$

valid for a continuous concave function $f:[m,M]\to(0,\infty)$, the unit vector $x\in\mathscr{H}$ and the positive operator A satisfying $m\leqslant A\leqslant M$, for some positive scalars m,M. Here $k(m,M,f)$ is the so called generalized Kantorovich constant and is defined by

$$k(m,M,f)=\min\left\{\frac{1}{f(t)}\left(\frac{M-t}{M-m}f(m)+\frac{t-m}{M-m}f(M)\right):\ t\in[m,M]\right\}. \tag{15}$$

PROPOSITION 2. Let $A,B\in\mathscr{B}(\mathscr{H})$ be such that $0<m\leqslant A,B\leqslant M$ and f be an increasing geometrically convex function. If in addition f is concave, then

$$f\left(\omega\left(A^{\frac{1}{2}}XB^{\frac{1}{2}}\right)\right)\leqslant\frac{\|X\|}{2K}\|f(A)+f(B)\|, \tag{16}$$

for the norm-expansive X (i.e., $\|X\|\geqslant 1$), where $K=k(m,M,f)$.

Proof. Proceeding as in Proposition 1 and noting (14) and the inequality $f(\alpha t)\leqslant\alpha f(t)$ when f is concave and $\alpha\geqslant 1$, we obtain the desired inequality. □

In particular, the function $f(t)=t^r,(t>0,\ 0<r\leqslant 1)$ satisfies the conditions of Proposition 2. Further, noting that $f(\|X\|\,\|AX\|\,\|Bx\|)=f(\|X\|)f(\|Ax\|\,\|Bx\|)$ for the function $f(t)=t^r$, we obtain the inequality

$$\omega\left(A^{\frac{1}{2}}XB^{\frac{1}{2}}\right)\leqslant\left(\frac{\|X\|}{2K(h,r)}\right)^{\frac{1}{r}}\|A^r+B^r\|^{\frac{1}{r}},$$

for the positive operators A,B satisfying $0<m\leqslant A,B\leqslant M$ and the norm-expansive X. The constant $K(h,r)$ is well known by the following formula [6, Definition 2.2]

$$K(h,r)=\frac{(h-h^r)}{(1-r)(h-1)}\left(\frac{1-r}{r}\frac{h^r-1}{h-h^r}\right)^r,\quad h=\frac{M}{m}.$$

Our next result is the generalization of [5, Theorem 6] and the estimate (10) in the same reference; where the sum of two operators is treated.

THEOREM 2. *Let $A,B\in\mathscr{B}(\mathscr{H})$ and f be an increasing convex function. Then for any $\alpha\in[0,1]$,*

$$f\left(\left\|\frac{A+B}{2}\right\|\right)\leqslant\frac{1}{4}\left(\left\|f\left(|A|^{2\alpha}\right)+f\left(|B|^{2\alpha}\right)\right\|+\left\|f\left(|A^*|^{2(1-\alpha)}\right)+f\left(|B^*|^{2(1-\alpha)}\right)\right\|\right) \tag{17}$$

and

$$f\left(\omega\left(\frac{A+B}{2}\right)\right) \leqslant \frac{1}{4}\left\|f\left(|A|^{2\alpha}\right)+f\left(|A^*|^{2(1-\alpha)}\right)+f\left(|B|^{2\alpha}\right)+f\left(|B^*|^{2(1-\alpha)}\right)\right\|. \tag{18}$$

Proof. Before proceeding, we recall the following useful inequality which is known in the literature as the generalized mixed Schwarz inequality (see, e.g., [11]):

$$|\langle Ax,y\rangle| \leqslant \sqrt{\left\langle |A|^{2\alpha}x,x\right\rangle\left\langle |A^*|^{2(1-\alpha)}y,y\right\rangle}, \quad \alpha\in[0,1], \tag{19}$$

where $A\in\mathscr{B}(\mathscr{H})$ and for any $x,y\in\mathscr{H}$. Let $x,y\in\mathscr{H}$ be unit vectors. By the similar way to the proof of Theorem 1, we have

$$\begin{aligned}
&f\left(\frac{1}{2}|\langle(A+B)x,y\rangle|\right) \leqslant f\left(\frac{1}{2}(|\langle Ax,y\rangle|+|\langle Bx,y\rangle|)\right) \leqslant \frac{1}{2}(f(|\langle Ax,y\rangle|)+f(|\langle Bx,y\rangle|))\\
&\leqslant\frac{1}{2}\left(f\left(\sqrt{\left\langle|A|^{2\alpha}x,x\right\rangle\left\langle|A^*|^{2(1-\alpha)}y,y\right\rangle}\right)+f\left(\sqrt{\left\langle|B|^{2\alpha}x,x\right\rangle\left\langle|B^*|^{2(1-\alpha)}y,y\right\rangle}\right)\right)\\
&\leqslant\frac{1}{2}\left(f\left(\frac{1}{2}\left(\left\langle|A|^{2\alpha}x,x\right\rangle+\left\langle|A^*|^{2(1-\alpha)}y,y\right\rangle\right)\right)\right)\\
&\quad+\frac{1}{2}\left(f\left(\frac{1}{2}\left(\left\langle|B|^{2\alpha}x,x\right\rangle+\left\langle|B^*|^{2(1-\alpha)}y,y\right\rangle\right)\right)\right)\\
&\leqslant\frac{1}{4}\left(\left\langle f\left(|A|^{2\alpha}\right)+f\left(|B|^{2\alpha}\right)x,x\right\rangle+\left\langle f\left(|A^*|^{2(1-\alpha)}\right)+f\left(|B^*|^{2(1-\alpha)}\right)y,y\right\rangle\right),
\end{aligned}$$

where the first inequality follows from the triangle inequality and the fact that f is increasing, the second inequality follows from convexity of f, the third inequality follows from (19), the fourth inequality follows from the fact that f is increasing and the arithmetic mean–geometric mean inequality and the last inequality follows from convexity of f and (11).

Now, by taking supremum over $x,y\in\mathscr{H}$ with $\|x\|=\|y\|=1$, we deduce the desired inequality (17).

If we take $x=y$, and apply same procedure as above we get (18). □

The case $f(t)=t^r$, $(t>0,\ r\geqslant 1)$ in Theorem 2 implies the known results due to El-Hadad and Kittaneh (see [5, Theorem 6] and the estimate (10) in [5]):

$$\|A+B\|^r \leqslant 2^{r-2}\left(\left\||A|^{2\alpha r}+|B|^{2\alpha r}\right\|+\left\||A^*|^{2(1-\alpha)r}+|B^*|^{2(1-\alpha)r}\right\|\right),$$

and

$$\omega^r(A+B) \leqslant 2^{r-2}\left\||A|^{2\alpha r}+|B|^{2\alpha r}+|A^*|^{2(1-\alpha)r}+|B^*|^{2(1-\alpha)r}\right\|.$$

Further, letting $A=B$, the above numerical radius inequality reduces to [5, Theorem 1].

Another observation led by Theorem 2 is the following extension; whose proof is identical to that of Theorem 2.

COROLLARY 3. *Let $A,B\in\mathscr{B}(\mathscr{H})$ and f be an increasing convex function. Then for any $\alpha,v\in[0,1]$,*

$$\begin{aligned}&f(\|(1-v)A+vB\|)\\ \leqslant&\frac{1}{2}\left(\left\|(1-v)f\left(|A|^{2\alpha}\right)+vf\left(|B|^{2\alpha}\right)\right\|+\left\|(1-v)f\left(|A^*|^{2(1-\alpha)}\right)+vf\left(|B^*|^{2(1-\alpha)}\right)\right\|\right).\end{aligned}$$

Next, we show the concave version of Theorem 2, which then entails new inequalities for $0\leqslant r\leqslant 1$. However, we will need to impose the extra condition that f is geometrically convex.

THEOREM 3. *Let $A,B\in\mathscr{B}(\mathscr{H}),\alpha\in[0,1]$ and f be an increasing geometrically convex function. Assume that, for positive scalars m,M,*

$$m\leqslant|A|^{2\alpha},|A^*|^{2(1-\alpha)},|B|^{2\alpha},|B^*|^{2(1-\alpha)}\leqslant M.$$

If f is concave, then

$$f(\|A+B\|)\leqslant\frac{1}{2K}\left(\left\|f\left(|A|^{2\alpha}\right)+f\left(|B|^{2\alpha}\right)\right\|+\left\|f\left(|A^*|^{2(1-\alpha)}\right)+f\left(|B^*|^{2(1-\alpha)}\right)\right\|\right)\tag{20}$$

and

$$f(\omega(A+B))\leqslant\frac{1}{2K}\left(\left\|f\left(|A|^{2\alpha}\right)+f\left(|A^*|^{2(1-\alpha)}\right)+f\left(|B|^{2\alpha}\right)+f\left(|B^*|^{2(1-\alpha)}\right)\right\|\right),\tag{21}$$

where $K=k(m,M,f)$.

Proof.

The proof is similar to that of Theorem 2. However, we need to recall that a non–negative concave function f is subadditive, in the sense that $f(a+b)\leqslant f(a)+f(b)$ and to recall (14). These will be needed to obtain the second and fifth inequalities below. All other inequalities follow as in Theorem 2. We have, for the unit vectors x,y,

$$\begin{aligned}&f(|\langle(A+B)x,y\rangle|)\leqslant f((|\langle Ax,y\rangle|+|\langle Bx,y\rangle|))\leqslant f(|\langle Ax,y\rangle|)+f(|\langle Bx,y\rangle|)\\ \leqslant&f\left(\sqrt{\left\langle|A|^{2\alpha}x,x\right\rangle\left\langle|A^*|^{2(1-\alpha)}y,y\right\rangle}\right)+f\left(\sqrt{\left\langle|B|^{2\alpha}x,x\right\rangle\left\langle|B^*|^{2(1-\alpha)}y,y\right\rangle}\right)\\ \leqslant&\sqrt{f\left(\left\langle|A|^{2\alpha}x,x\right\rangle\right)f\left(\left\langle|A^*|^{2(1-\alpha)}y,y\right\rangle\right)}+\sqrt{f\left(\left\langle|B|^{2\alpha}x,x\right\rangle\right)f\left(\left\langle|B^*|^{2(1-\alpha)}y,y\right\rangle\right)}\\ \leqslant&K^{-1}\left(\sqrt{\left\langle f\left(|A|^{2\alpha}\right)x,x\right\rangle\left\langle f\left(|A^*|^{2(1-\alpha)}\right)y,y\right\rangle}\right.\\ &\left.+\sqrt{\left\langle f\left(|B|^{2\alpha}\right)x,x\right\rangle\left\langle f\left(|B^*|^{2(1-\alpha)}\right)y,y\right\rangle}\right)\\ \leqslant&\frac{1}{2K}\left(\left\langle f\left(|A|^{2\alpha}\right)+f\left(|B|^{2\alpha}\right)x,x\right\rangle+\left\langle f\left(|A^*|^{2(1-\alpha)}\right)+f\left(|B^*|^{2(1-\alpha)}\right)y,y\right\rangle\right).\end{aligned}$$

Now, by taking supremum over $x,y \in \mathscr{H}$ with $\|x\| = \|y\| = 1$, we deduce the desired inequalities. $\square$

In particular, if $f(t) = t^r$, $(t > 0,\ 0 \leqslant r \leqslant 1)$, we obtain

$$\|A+B\|^r \leqslant \frac{1}{2K(h,r)}\left(\left\||A|^{2\alpha r}+|B|^{2\alpha r}\right\|+\left\||A^*|^{2(1-\alpha)r}+|B^*|^{2(1-\alpha)r}\right\|\right),$$

where $h = \frac{M}{m}$.

Our next result is the extension of Theorem [5, Theorem 2]. In this result, we will use the concave-version of the inequality (11), where we have

$$f(\langle Ax,x\rangle) \geqslant \langle f(A)x,x\rangle, \tag{22}$$

when $x \in \mathscr{H}$ is a unit vector, $f : I \to \mathbb{R}$ is a concave function and A is self adjoint with spectrum in I.

THEOREM 4. *Let $A \in \mathscr{B}(\mathscr{H}), \alpha \in [0,1]$ and let f be an increasing convex function. Then*

$$f\left(\omega^2(A)\right) \leqslant \left\|\alpha f(|A|^2)+(1-\alpha)f(|A^*|^2)\right\|.$$

Proof. Noting (19) and monotonicity of f, we have for any unit vector $x \in \mathscr{H}$,

$$\begin{aligned}
f\left(|\langle Ax,x\rangle|^2\right) &\leqslant f\left(\left\langle |A|^{2\alpha}x,x\right\rangle\left\langle |A^*|^{2(1-\alpha)}x,x\right\rangle\right)\\
&\leqslant f\left(\left\langle |A|^2x,x\right\rangle^{\alpha}\left\langle |A^*|^2x,x\right\rangle^{1-\alpha}\right) \quad (\text{by concavity of } t^{\alpha} \text{ and } t^{1-\alpha})\\
&\leqslant f\left(\alpha\left\langle |A|^2x,x\right\rangle+(1-\alpha)\left\langle |A^*|^2x,x\right\rangle\right) \quad (\text{by Young's inequality})\\
&\leqslant \alpha f\left(\left\langle |A|^2x,x\right\rangle\right)+(1-\alpha)f\left(\left\langle |A^*|^2x,x\right\rangle\right) \quad (\text{by convexity of } f)\\
&\leqslant \alpha \left\langle f\left(|A|^2\right)x,x\right\rangle+(1-\alpha)\left\langle f\left(|A^*|^2\right)x,x\right\rangle \quad (\text{by } (11))\\
&= \left\langle\left(\alpha f\left(|A|^2\right)+(1-\alpha)f\left(|A^*|^2\right)\right)x,x\right\rangle\\
&\leqslant \left\|\alpha f\left(|A|^2\right)+(1-\alpha)f\left(|A^*|^2\right)\right\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
f\left(\omega^2(A)\right) = f\left(\sup_{\|x\|=1}|\langle Ax,x\rangle|^2\right) &= \sup_{\|x\|=1} f\left(|\langle Ax,x\rangle|^2\right)\\
&\leqslant \left\|\alpha f\left(|A|^2\right)+(1-\alpha)f\left(|A^*|^2\right)\right\|,
\end{aligned}$$

which completes the proof. $\square$

Letting $f(t) = t^r$, $(t > 0,\ r \geqslant 1)$ in Theorem 4 implies

$$\omega^{2r}(A) \leqslant \left\|\ \alpha\ |A|^{2r}+(1-\alpha)|A^*|^{2r}\right\|,$$

which is the result of Theorem [5, Theorem 2].

3. Geometrically convex functions related to matrix norms

Let $\mathscr{M}_n$ denote the algebra of all $n \times n$ complex matrices. In this section, we present some unitarily invariant norm inequalities on $\mathscr{M}_n$ via geometric convexity. It is well known that the function $f(t) = |||A^t XB^t|||$ is log-convex on $\mathbb{R}$, for any unitarily invariant norm $|||\cdot|||$ and positive matrices A, B, see [17]. In this section, we show that the above function $f(t)$ is also geometrically convex, when A, B are expansive. In this context, we say that a matrix A is expansive if $A \geqslant I$ and contractive if $A \leqslant I$.

Notice that if $A, B \leqslant I$, then for any X,

$$|||AXB||| \leqslant \|A\| \; |||X||| \; \|B\| \leqslant |||X|||,$$

by submultiplicativity of unitarily invariant norms. Therefore, if A, B are expansive and $\alpha \leqslant \beta$, then for any X,

$$\left|\left|\left|A^{\alpha-\beta} X B^{\alpha-\beta}\right|\right|\right| \leqslant |||X|||,$$

which gives, upon replacing X with $A^{\beta} X B^{\beta}$

$$|||A^{\alpha} X B^{\alpha}||| \leqslant \left|\left|\left|A^{\beta} X B^{\beta}\right|\right|\right|.$$

In particular, if $t_1, t_2 > 0$, then $\sqrt{t_1 t_2} \leqslant \frac{t_1+t_2}{2}$, and the above inequality implies

$$\left|\left|\left|A^{\sqrt{t_1 t_2}} X B^{\sqrt{t_1 t_2}}\right|\right|\right| \leqslant \left|\left|\left|A^{\frac{t_1+t_2}{2}} X B^{\frac{t_1+t_2}{2}}\right|\right|\right|, \tag{23}$$

when A, B are expansive.

THEOREM 5. *If A, B are expansive and X is arbitrary, then*

$$f(t) = \left|\left|\left|A^t X B^t\right|\right|\right|$$

is geometrically convex on $(0, \infty)$.

Proof. Taking (23) in account, we obtain

$$f(\sqrt{t_1 t_2}) = \left|\left|\left|A^{\sqrt{t_1 t_2}} X B^{\sqrt{t_1 t_2}}\right|\right|\right| \leqslant \left|\left|\left|A^{\frac{t_1+t_2}{2}} X B^{\frac{t_1+t_2}{2}}\right|\right|\right| \leqslant \left|\left|\left|A^{t_1} X B^{t_1}\right|\right|\right|^{\frac{1}{2}} \left|\left|\left|A^{t_2} X B^{t_2}\right|\right|\right|^{\frac{1}{2}},$$

where the last inequality follows from the well known log-convexity of f. $\square$

COROLLARY 4. *If A is expansive and B is contractive, then the function $f(t) = \left|\left|\left|A^t X B^{1-t}\right|\right|\right|$ is geometrically convex on* $(0, \infty)$.

Proof. Notice that if B is contractive, B^{-1} is expansive. Therefore, applying Theorem 5 with X replaced by XB and B replaced by B^{-1}, we obtain the result. $\square$

COROLLARY 5. *Let A,B be expansive and let $X \in \mathscr{M}_n$ be arbitrary. Then*

$$|||X||| \leqslant |||AXB|||,$$

for any unitarily invariant norm $|||\ \ |||$.

Proof. Let $f(t) = |||A^tXB^t|||$. By Theorem 5, f is geometrically convex on $[0,\infty)$. In particular, by letting $t_1 = 0, t_2 = 1$, we obtain

$$f\left(\sqrt{t_1t_2}\right) \leqslant \sqrt{f(t_1)f(t_2)},$$

which implies the desired inequality. □

COROLLARY 6. *Let A,B be expansive and let $X \in \mathscr{M}_n$ be arbitrary. Then*

$$\left|\left|\left|A^{t_1\sharp_v t_2}XB^{t_1\sharp_v t_2}\right|\right|\right| \leqslant |||A^{t_1}XB^{t_1}|||\sharp_v|||A^{t_2}XB^{t_2}|||,$$

for $t_1,t_2 \geqslant 0, 0 \leqslant v \leqslant 1$ and any unitarily invariant norm $|||\ \ |||$.

Proof. The proof follows the same guideline as in Theorem 5, where we have $t_1\sharp_v t_2 \leqslant t_1\nabla_v t_2$. Then

$$f(t_1\sharp_v t_2) \leqslant f(t_1\nabla_v t_2) \leqslant f(t_1)\sharp_v f(t_2),$$

where the last inequality follows from log-convexity of f. □

COROLLARY 7. *Let $a_i,b_i \geqslant 1, t_1,t_2 \geqslant 0$ and $0 \leqslant v \leqslant 1$. Then*

$$\sum_{i=1}^n (a_ib_i)^{t_1\sharp_v t_2} \leqslant \left(\sum_{i=1}^n (a_ib_i)^{t_1}\right)\sharp_v\left(\sum_{i=1}^n (a_ib_i)^{t_2}\right).$$

Proof. For the given parameters, define $A = \mathrm{diag}(a_i)$ and $B = \mathrm{diag}(b_i)$. Then clearly A and B are expansive. Applying Corollary 6 with X being the identity matrix implies the desired inequality. □

Notice that as a special case of Corollary 7 we obtain the Cauchy–Schwarz type inequality

$$\sum_{i=1}^n a_ib_i \leqslant \left(\sum_{i=1}^n (a_ib_i)^{t_1}\right)^{\frac{1}{2}}\left(\sum_{i=1}^n (a_ib_i)^{t_2}\right)^{\frac{1}{2}},$$

for the scalars $a_i,b_i \geqslant 1$ and $t_1,t_2 > 0$ satisfying $t_1t_2 = 1$.

4. Geometrically convex functions and scalar inequalities

In this section, we present several scalar versions of (3). The results in this section are of independent interest. We begin by the following reverse of (3).

LEMMA 1. *Let f be geometrically convex function on the interval $(0,\infty)$ and $a,b>0$. Then for any $v>0$ or $v<-1$,*

$$f(a)^{1+v}f(b)^{-v}\leqslant f\left(a^{1+v}b^{-v}\right). \tag{24}$$

Equivalently,

$$f(a\sharp_v b)\geqslant f(a)\sharp_v f(b), \quad v\notin[0,1].$$

Proof. First assume that $v>0$. We need the following useful identity

$$a=\left(a^{1+v}b^{-v}\right)^{\frac{1}{1+v}}b^{\frac{v}{v+1}}.$$

It follows from (3) that

$$f(a)=f\left(\left(a^{1+v}b^{-v}\right)^{\frac{1}{1+v}}b^{\frac{v}{v+1}}\right)\leqslant f\left(a^{1+v}b^{-v}\right)^{\frac{1}{v+1}}f(b)^{\frac{v}{1+v}}$$

and this implies the desired inequality for this case. For the case $v<-1$, we can use the following identity

$$b=\left(a^{1+v}b^{-v}\right)^{-\frac{1}{v}}a^{\frac{1+v}{v}}. \quad \square$$

It should be noted that Lemma 1 simulates similar behavior of convex and log-convex functions. We refer the reader to [15, Lemma 3.11] for this similarity. In fact, Lemma 1 is needed to prove the following more general reverse and refinement of (3). We remark that Theorem 6 below is the geometric convex version of similar results for convex and log-convex functions; see for example [18].

THEOREM 6. *Let f be a geometrically convex on the interval I and $a,b\in I$. Then for any $v\in[0,1]$,*

$$\left(\frac{f\left(\sqrt{ab}\right)}{\sqrt{f(a)f(b)}}\right)^{2R}f(a)^{1-v}f(b)^{v}\leqslant f\left(a^{1-v}b^{v}\right)\leqslant\left(\frac{f\left(\sqrt{ab}\right)}{\sqrt{f(a)f(b)}}\right)^{2r}f(a)^{1-v}f(b)^{v}, \tag{25}$$

where $r=\min\{v,1-v\}$ and $R=\max\{v,1-v\}$.

Proof. We first assume that $v \in \left[0, \frac{1}{2}\right]$. So,

$$\begin{aligned} f\left(a^{1-v}b^{v}\right) &= f\left(a^{1-2v}\left(\sqrt{ab}\right)^{2v}\right) \\ &\leqslant f(a)^{1-2v} f\left(\sqrt{ab}\right)^{2v} \quad \text{(by (3))} \\ &= \left(\frac{f\left(\sqrt{ab}\right)}{\sqrt{f(a)f(b)}}\right)^{2v} f(a)^{1-v} f(b)^{v}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \left(\frac{f\left(\sqrt{ab}\right)}{\sqrt{f(a)f(b)}}\right)^{2(1-v)} f(a)^{1-v} f(b)^{v} &= f\left(\sqrt{ab}\right)^{2-2v} f(b)^{-(1-2v)} \\ &= f\left(\sqrt{ab}\right)^{1+(1-2v)} f(b)^{-(1-2v)} \\ &\leqslant f\left(\left(\sqrt{ab}\right)^{1+(1-2v)} b^{-(1-2v)}\right) \quad \text{(by Lemma 1)} \\ &= f\left(a^{1-v}b^{v}\right). \end{aligned}$$

Consequently,

$$\left(\frac{f\left(\sqrt{ab}\right)}{\sqrt{f(a)f(b)}}\right)^{2R} f(a)^{1-v} f(b)^{v} \leqslant f\left(a^{1-v}b^{v}\right) \leqslant \left(\frac{f\left(\sqrt{ab}\right)}{\sqrt{f(a)f(b)}}\right)^{2r} f(a)^{1-v} f(b)^{v}.$$

The same procedure also works for the case $v \in \left[\frac{1}{2}, 1\right]$. This completes the proof. □

The first inequality in (25) can be regarded as a reverse of (3). Meanwhile, since $\frac{f\left(\sqrt{ab}\right)}{\sqrt{f(a)f(b)}} \leqslant 1$, the second inequality in (25) provides a refinement of (3).

We can extend (3) to the following form [14]

$$f\left(\prod_{i=1}^{n} x_i^{p_i}\right) \leqslant \prod_{i=1}^{n} f(x_i)^{p_i}, \quad \sum_{i=1}^{n} p_i = 1. \tag{26}$$

On the other hand, (24) can be extended as follows. This extension simulates similar extensions for convex and log-convex functions [19, 20, 22].

COROLLARY 8. *Let* $a, b_i, v_i \geqslant 0$ *and let* $v = \sum_{i=1}^{n} v_i$. *If the function* $f : (0,\infty) \to (0,\infty)$ *is geometrically convex, then*

$$f\left(a^{1+v}\prod_{i=1}^{n} b_i^{-v_i}\right) \geqslant f(a)^{1+v} \prod_{i=1}^{n} f(b)^{-v_i}.$$

Proof. Notice that, for the given parameters,

$$
\begin{aligned}
f\left(a^{1+v}\prod_{i=1}^{n}b_i^{-v_i}\right) &= f\left(a^{1+v}\left(\prod_{i=1}^{n}b_i^{\frac{v_i}{v}}\right)^{-v}\right)\\
&\geqslant f(a)^{1+v} f\left(\prod_{i=1}^{n}b_i^{\frac{v_i}{v}}\right)^{-v} \quad \text{(by (24))}\\
&\geqslant f(a)^{1+v}\left(\prod_{i=1}^{n}f(b_i)^{\frac{v_i}{v}}\right)^{-v} \quad \text{(by (26))}\\
&= f(a)^{1+v}\prod_{i=1}^{n}f(b)^{-v_i},
\end{aligned}
$$

which completes the proof. □

In the following, we aim to improve (26). To this end, we need the following simple lemma which can be proved using (26).

LEMMA 2. *Let f be a geometrically convex on the interval I and $x_1,\dots,x_n \in I$, and $p_1,\dots,p_n$ positive numbers with $P_n=\sum_{i=1}^{n}p_i$, then*

$$
f\left(\left(\prod_{i=1}^{n}x_i^{p_i}\right)^{\frac{1}{P_n}}\right) \leqslant \left(\prod_{i=1}^{n}f(x_i)^{p_i}\right)^{\frac{1}{P_n}}.
$$

THEOREM 7. *Let f be a geometrically convex function on the interval I, $x_1,\dots,x_n \in I$, and $p_1,\dots,p_n$ positive numbers such that $\sum_{i=1}^{n}p_i=1$. Assume $J\subsetneq\{1,2,\dots,n\}$ and $J^c=\{1,2,\dots,n\}\backslash J$, $P_J=\sum\limits_{i\in J}p_i$, $P_{J^c}=1-\sum\limits_{i\in J}p_i$. Then*

$$
f\left(\prod_{i=1}^{n}x_i^{p_i}\right) \leqslant f\left(\left(\prod_{i\in J}x_i^{p_i}\right)^{\frac{1}{P_J}}\right)^{P_J} f\left(\left(\prod_{i\in J^c}x_i^{p_i}\right)^{\frac{1}{P_{J^c}}}\right)^{P_{J^c}} \leqslant \prod_{i=1}^{n}f(x_i)^{p_i}.
$$

Proof. We have

$$
\begin{aligned}
f\left(\prod_{i=1}^{n}x_i^{p_i}\right) &= f\left(\prod_{i\in J}x_i^{p_i}\prod_{i\in J^c}x_i^{p_i}\right) = f\left(\left(\left(\prod_{i\in J}x_i^{p_i}\right)^{\frac{1}{P_J}}\right)^{P_J}\left(\left(\prod_{i\in J^c}x_i^{p_i}\right)^{\frac{1}{P_{J^c}}}\right)^{P_{J^c}}\right)\\
&\leqslant f\left(\left(\prod_{i\in J}x_i^{p_i}\right)^{\frac{1}{P_J}}\right)^{P_J} f\left(\left(\prod_{i\in J^c}x_i^{p_i}\right)^{\frac{1}{P_{J^c}}}\right)^{P_{J^c}} \quad \text{(by (3))}
\end{aligned}
$$

$$\leqslant \left(\left(\prod_{i\in J} f(x_i)^{p_i}\right)^{\frac{1}{P_J}}\right)^{P_J} \left(\left(\prod_{i\in J^c} f(x_i)^{p_i}\right)^{\frac{1}{P_{J^c}}}\right)^{P_{J^c}} \quad \text{(by Lemma 2)}$$

$$= \left(\prod_{i\in J} f(x_i)^{p_i}\right)\left(\prod_{i\in J^c} f(x_i)^{p_i}\right) = \prod_{i=1}^{n} f(x_i)^{p_i}.$$

This completes the proof. $\square$

We refer the reader to [3, Theorem 1] for similar results about convex functions.

It is quite natural to consider the n-tuple version of Theorem 6. Closing this paper, we give the extension for Theorem 6. The convex and log-convex versions of this extension were proved first in [13].

THEOREM 8. *Let f be a geometrically convex function on the interval I and $x_1,\ldots,x_n \in I$, and let $p_1,\ldots,p_n$ be non–negative numbers with $\sum_{i=1}^{n} p_i = 1$. Then*

$$\left(\frac{f\left(\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}\right)}{\left(\prod_{i=1}^{n} f(x_i)\right)^{\frac{1}{n}}}\right)^{nR_n} \prod_{i=1}^{n} f(x_i)^{p_i} \leqslant f\left(\prod_{i=1}^{n} x_i^{p_i}\right) \leqslant \left(\frac{f\left(\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}\right)}{\left(\prod_{i=1}^{n} f(x_i)\right)^{\frac{1}{n}}}\right)^{nr_n} \prod_{i=1}^{n} f(x_i)^{p_i} \tag{27}$$

where $r_n = \min\{p_1,\ldots,p_n\}$ and $R_n = \max\{p_1,\ldots,p_n\}$.

Proof. We first prove the second inequality of (27). We may assume $r_n = p_k$ without loss of generality. For any $k = 1,\ldots,n$, we have

$$\left(\frac{f\left(\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}\right)}{\left(\prod_{i=1}^{n} f(x_i)\right)^{\frac{1}{n}}}\right)^{nr_n} \prod_{i=1}^{n} f(x_i)^{p_i}$$

$$= f\left(\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}\right)^{np_k} \left(\prod_{i=1}^{n} f(x_i)^{\frac{p_i-p_k}{1-np_k}}\right)^{1-np_k} \geqslant f\left(\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}\right)^{np_k} f\left(\prod_{i=1}^{n} x_i^{\frac{p_i-p_k}{1-np_k}}\right)^{1-np_k}$$

$$\geqslant f\left(\left(\prod_{i=1}^{n} x_i\right)^{\frac{np_k}{n}} \prod_{i=1}^{n} x_i^{\left(\frac{p_i-p_k}{1-np_k}\right)(1-np_k)}\right) = f\left(\prod_{i=1}^{n} x_i^{p_i}\right).$$

In the above, the first inequality follows by (26) with $1-np_k \geqslant 0$ and the second inequality follows by (3) with $a = \prod_{i=1}^{n} x_i^{\frac{1}{n}}$, $b = \prod_{i=1}^{n} x_i^{\frac{p_i-p_k}{1-np_k}}$, $1-v = np_k$.

We also assume $R_n = p_l$ and for any $l = 1, \ldots, n$, we have

$$\begin{aligned}&\left(\frac{f\left(\prod\limits_{i=1}^{n} x_i^{p_i}\right)}{\prod\limits_{i=1}^{n} f(x_i)^{p_i}}\right)^{\frac{1}{np_l}}\left(\prod_{i=1}^{n} f(x_i)\right)^{\frac{1}{n}}\\&=f\left(\prod_{i=1}^{n} x_i^{p_i}\right)^{\frac{1}{np_l}}\left(\prod_{i=1}^{n} f(x_i)^{\frac{p_l-p_i}{np_l-1}}\right)^{\frac{np_l-1}{np_l}} \geqslant f\left(\prod_{i=1}^{n} x_i^{p_i}\right)^{\frac{1}{np_l}} f\left(\prod_{i=1}^{n} x_i^{\frac{p_l-p_i}{np_l-1}}\right)^{\frac{np_l-1}{np_l}}\\&\geqslant f\left(\left(\prod_{i=1}^{n} x_i^{p_i}\right)^{\frac{1}{np_l}} \prod_{i=1}^{n} x_i^{\left(\frac{p_l-p_i}{np_l-1}\right)\left(\frac{np_l-1}{np_l}\right)}\right) = f\left(\prod_{i=1}^{n} x_i^{\frac{1}{n}}\right).\end{aligned}$$

In the above, the first inequality follows by (26) with $\frac{np_l-1}{np_l} \geqslant 0$ and the first inequality follows by (3) with $a = \prod_{i=1}^{n} x_i^{p_i}$, $b = \prod_{i=1}^{n} x_i^{\frac{p_l-p_i}{np_l-1}}$, $1 - v = \frac{1}{np_l}$. Thus the first inequality of (27) was proven. $\square$

We easily find that Theorem 8 recovers Theorem 6 when $n = 2$.

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