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SOME INEQUALITIES ON w^*UR MODULUS OF CONVEXITY AND GEOMETRIC PROPERTIES OF BANACH SPACES X AND X^*

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Abstract. Let X be a Banach space. In this paper, we study the properties of w^*UR modulus of convexity of X^* respect to x, $\delta_X{}^*(\varepsilon,x)$, where $0 \leqslant \varepsilon \leqslant 2$ and $x \in S(X)$, and the relationship between the values of w^*UR modulus and reflexivity, uniform non-squareness and normal structure respectively. Among other results, we proved that if $\delta_X{}^*(\varepsilon,x) > \frac{1}{2} - \frac{\varepsilon}{4}$ for all $x \in S(X)$, and any $0 < \varepsilon < 2$ then both X and X^* have uniform normal structure.

1. Introduction and preliminaries

Let X be a normed linear space. Let $B(X) = \{x \in X : \|x\| \leqslant 1\}$ and $S(X) = \{x \in X : \|x\| = 1\}$ be the unit ball, and the unit sphere of X, respectively. Let X^* be the dual space of X, and X^{**} be the dual space of X^* respectively.

The concept of normal structure was defined by Brodskiĭ and Mil'man:

DEFINITION 1.1. [1] A bounded and convex subset K of a Banach space X is said to have normal structure if every convex subset H of K that contains more than one point contains a point $x_0 \in H$, such that $\sup\{\|x_0 - y\| : y \in H\} < d(H)$, where $d(H) = \sup\{\|x - y\| : x, y \in H\}$ denotes the diameter of H.

A Banach space X is said to have normal structure if every bounded and convex subset of X has normal structure.

A Banach space X is said to have weak normal structure if for each weakly compact convex set K of X has normal structure.

A Banach space X is said to have uniform normal structure if there exists $0 < c < 1$ such that for any bounded closed convex subset K of X that contains more than one point, there exists $x_0 \in K$ such that $\sup\{\|x_0 - y\| : y \in K\} \leqslant c \cdot d(K)$.

For a reflexive Banach space, the normal structure and weak normal structure coincide.

Let D be a nonempty subset of a Banach space X. A mapping $T : D \to D$ is called to be non-expensive whenever $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in D$. A Banach space is said to have fixed point property if for every bounded closed and convex subset D of

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X and for each non-expansive mapping $T: D \to D$, there is a point $x \in D$ such that $x = Tx$ [11].

Kirk proved that if a Banach space X has weak normal structure then it has weak fixed point property, that is, every non-expansive mapping from a weakly compact and convex subset of X into itself has a fixed point [11].

DEFINITION 1.2. [9] A Banach space X is called uniformly non-square if there exists $\delta > 0$ such that if $x, y \in S(X)$, then either $\frac{\|x+y\|}{2} \leqslant 1-\delta$ or $\frac{\|x-y\|}{2} \leqslant 1-\delta$.

DEFINITION 1.3. [3] Let X and Y be Banach spaces. We say that Y is *finitely representable in* X if for any $\varepsilon > 0$ and any finite dimensional subspace $N \subseteq Y$ there is an isomorphism $T: N \to X$ such that for any $y \in N$, $(1-\varepsilon)\|y\| \leqslant \|Ty\| \leqslant (1+\varepsilon)\|y\|$.

The Banach space X is called *super-reflexive* if any space Y which is finitely representable in X is reflexive.

REMARK 1.4. It is well known that:

(a) if X is uniformly non-square then X is supper-reflexive and therefore X is reflexive;

(b) X is super-reflexive if and only if X^* is supper-reflexive.

The concept of modulus of uniformly rotund or uniformly convex was defined by Clarkson:

DEFINITION 1.5. [2] Let X be a Banach space, the modulus of convexity is a function from [0, 2] to [0, 1] defined by the formula:

$$\delta_X(\varepsilon) = \inf\{1 - \frac{1}{2}\|x+y\| : x, y \in S(X), \|x-y\| \geqslant \varepsilon\},$$

where $0 \leqslant \varepsilon \leqslant 2$.

If $\delta_X(\varepsilon) > 0$ for any $0 < \varepsilon \leqslant 2$, X is called uniformly rotund or uniformly convex. The abbreviation UR is used for this space.

Gao proved that if there exists $0 \leqslant \varepsilon \leqslant 1$ such that $\delta_X(1+\varepsilon) > \frac{\varepsilon}{2}$, then X has uniform normal structure [4].

The concept of modulus of weakly uniformly rotund or weakly uniformly convex was defined by Smulain:

DEFINITION 1.6. [14] Let X be a Banach space and $f \in S(X^*)$, the modulus of convexity of X with respect to f, is a function from $[0,2] \times S(X^*)$ to $[0,1]$ defined by the formula:

$$\delta_X(\varepsilon, f) = \inf\{\{1\} \cup \{1 - \frac{1}{2}\|x+y\| : x, y \in S(X), |<x-y, f>| \geqslant \varepsilon\}\},$$

where $0 \leqslant \varepsilon \leqslant 2$.

If $\delta_X(\varepsilon, f) > 0$ for all $f \in S(X^*)$ and $0 < \varepsilon \leqslant 2$, X is called weakly uniformly rotund or weakly uniformly convex. The abbreviation wUR is used for this space.

The reason for specifically including 1 in the set whose infimum defines the wUR modulus is to avoid the following particular situation: when f is a non-norm attaining functional, so there are no points x, y in $S(X)$ such that $| < x-y, f > | \geqslant 2$. Therefore $\delta_X(2, f)$ would not be well defined.

The following results were proved for $\delta_X(\varepsilon, f)$ by Gao [5]:

THEOREM 1.7. *For a Banach space X, if $\delta_X(\varepsilon, f) > 1-\varepsilon$ for all $f \in S(X^*)$ and $0 < \varepsilon < 1$ then X is reflexive.*

THEOREM 1.8. *For a Banach space X, if $\delta_X(1, f) > 0$ for all $f \in S(X^*)$, then X has weak normal structure.*

THEOREM 1.9. *For a Banach space X, if $\delta_X(\varepsilon, f) > \frac{1}{2} - \frac{\varepsilon}{4}$ for all $f \in S(X^*)$ and $0 < \varepsilon < 2$ then X is uniform non-square and has uniform normal structure.*

THEOREM 1.10. [15] *For any $f \in X^*$, $\frac{\delta_X(\varepsilon, f)}{\varepsilon}$ is an increasing function of ε in $(0, 2]$, and $\delta_X(\varepsilon, f)$ is a continuous function in $0 \leqslant \varepsilon < 2$.*

2. Normal structure and inequalities on w^*UR modulus

The concept of the modulus of weakly* uniformly rotund or weakly* uniformly convex was defined by Smulian too.

DEFINITION 2.1. [14] Let X be a Banach space and $x \in S(X)$, the modulus of convexity of X^* with respect to x, is a function from $[0, 2] \times S(X)$ to $[0, 1]$ defined by the formula:

$$\delta_{X^*}(\varepsilon, x) = \inf\{1 - \frac{1}{2}\|f+g\| : f, g \in S(X^*), | < x, f-g > | \geqslant \varepsilon\},$$

where $0 \leqslant \varepsilon \leqslant 2$.

If $\delta_{X^*}(\varepsilon, x) > 0$ for all $x \in S(X)$ and $0 < \varepsilon \leqslant 2$, X^* is called weakly* uniformly rotund or weakly* uniformly convex. The abbreviation w^*UR is used for this space.

THEOREM 2.2. [8] *Let X be a Banach space. Then X is not reflexive if and only if for any $0 < \varepsilon < 1$ there are a sequence $\{x_n\} \subseteq S(X)$ and a sequence $\{f_n\} \subseteq S(X^*)$ such that*

(a) $\langle x_m, f_n \rangle = \varepsilon$ whenever $n \leqslant m$; and

(b) $\langle x_m, f_n \rangle = 0$ whenever $n > m$.

PROPOSITION 2.3. *Let X be a Banach space, then:*

(a) $\delta_{(X^*)^*}(\varepsilon, f) \leqslant \delta_X(\varepsilon, f)$, *where* $f \in S(X^*)$.

(b) $\delta_{X^*}(\varepsilon, x)$ *is a non-decreasing function of* ε *for any* $x \in S(X)$.

(c) For a Banach space X, *if* X^* *is wUR then* X^* *is* w^*UR.

(d) Let X_2^* *be a 2-dimensional subspace of* X^*, *then* $\delta_{X^*}(\varepsilon, x) = \inf_{X_2^* \subseteq X^*}(\delta_{X_2^*}(\varepsilon, x))$.

(e) $\delta_{X^*}(\varepsilon, x)$ *is a continuous function of* $\varepsilon \in [0,2)$ *for any* $x \in S(X)$.

Proof.

(a) Since $S(X) \subseteq S(X^{**})$.

(b) From definition of w^*UR.

(c) It is a direct result of (a).

(d) From definition of w^*UR.

(e) First we show that: Let X_2 be a 2 dimensional Banach space, $x \in S(X_2)$, and $u, v \in S(X_2^*)$ are independent. Let $f_2 - f_1 = au, g_2 - g_1 = bu, h_2 - h_1 = cu, \frac{f_1+f_2}{2} = dv$, $\frac{g_1+g_2}{2} = ev$, and $\frac{h_1+h_2}{2} = fv$, where all a, b, c, d, e, and $f > 0$, and all f_1, f_2, g_1, g_2, h_1 and $h_2 \in S(X^*)$. If $< x, f_2 - f_1 > + < x, h_2 - h_1 >= 2 < x, g_2 - g_1 >$, then $a + c = 2b$, therefore from convexity of $S(X_2^*)$, we have $d + f \leqslant 2e$. Let $\delta_{X_2^*}^{u,v}(\varepsilon, x) = \inf\{1 - \frac{1}{2}\|f + g\| : f, g \in S(X_2^*), f - g = \alpha u, f + g = \beta v, | < x, f - g > | \geqslant \varepsilon\}$, where $0 \leqslant \varepsilon \leqslant 2$. This means that for any $x \in S(X_2)$ and $u, v \in S(X_2^*)$, $\delta_{X_2^*}^{u,v}(\varepsilon, x)$ is a convex function of ε. The following proof is similar to the proof of Lemma 5.1 in [7]: Let $0 \leqslant \varepsilon_1 < 2$. For $0 < \varepsilon \leqslant 2$, the convexity of $\delta_{X_2^*}^{u,v}(\varepsilon, x)$ implies that: $\frac{\delta_{X_2^*}^{u,v}(\varepsilon,x) - \delta_{X_2^*}^{u,v}(\varepsilon_1,x)}{\varepsilon - \varepsilon_1} \leqslant \frac{\delta_{X_2^*}^{u,v}(2,x) - \delta_{X_2^*}^{u,v}(\varepsilon_1,x)}{2 - \varepsilon_1} \leqslant \frac{1}{2-\varepsilon_1}$. Therefore, $\delta_{X_2^*}^{u,v}(\varepsilon, x) - \delta_{X_2^*}^{u,v}(\varepsilon_1, x) \leqslant \frac{\varepsilon - \varepsilon_1}{2 - \varepsilon_1}$. From (d) of above Proposition 2.3, we have $\delta_{X^*}(\varepsilon, x) - \delta_{X^*}(\varepsilon_1, x) \leqslant \frac{\varepsilon - \varepsilon_1}{2 - \varepsilon_1}$. This proved that $\delta_{X^*}(\varepsilon, x)$ is a continuous function of $\varepsilon \in [0,2)$ for any $x \in S(X)$. □

THEOREM 2.4. *For a Banach space* X, *if* $\delta_{X^*}(\varepsilon, x) > 1 - \varepsilon$ *for all* $x \in S(X)$ *and* $0 < \varepsilon < 1$, *then* X *is reflexive.*

Proof. The idea of the proof is similar to the proof of Theorem 2.1 of [5]. Suppose X is not reflexive. For any $0 < \varepsilon < 1$, let the sequence $\{x_m\} \subseteq S(X)$ and the sequence $\{f_n\} \subseteq S(X^*)$ satisfy the two conditions in Theorem 2.2. Let $n_1 < m < n_2$, we have $< x_m, f_{n_1} - f_{n_2} >= \varepsilon$. Let $n_1 < n_2 < m_1$, we have $< x_{m_1}, f_{n_1} + f_{n_2} >= 2\varepsilon$, therefore $\|f_{n_1} + f_{n_2}\| \geqslant 2\varepsilon, 1 - \frac{\|f_{n_1} + f_{n_2}\|}{2} \leqslant 1 - \varepsilon$. This implies $\delta_{X^*}(\varepsilon, x_m) = \inf\{1 - \frac{\|f+g\|}{2}, < x_m, f - g >\geqslant \varepsilon\} \leqslant 1 - \frac{\|f_{n_1} + f_{n_2}\|}{2} \leqslant 1 - \varepsilon$, for this fixed $x_m \in S(X)$. □

LEMMA 2.5. [12] *If X is a Banach space with $B(X^*)$ is weak* sequentially compact (for example, X is reflexive or separable, or has an equivalent smooth norm) and fails to have weak normal structure, then for any $\varepsilon > 0$ there are a sequence $\{x_n\} \subseteq S(X)$ and a sequence $\{f_n\} \subseteq S(X^*)$ such that:*

(a) $|\|x_i - x_j\| - 1| < \varepsilon$, *where* $i \neq j$;

(b) $\langle x_i, f_i \rangle = 1$, *where* $1 \leqslant i \leqslant \infty$;

(c) $|\langle x_j, f_i \rangle| < \varepsilon$, *where* $i \neq j$; *and*

(d) $\|f_i - f_j\| > 2 - \varepsilon$, *where* $i \neq j$.

THEOREM 2.6. *For a Banach space X which satisfies one of any condition in above Lemma 2.5, if $\delta_{X^*}(\varepsilon, x) > \frac{1}{2} - \frac{\varepsilon}{2}$ for all $x \in S(X)$ and $0 < \varepsilon < 1$ then X has weak normal structure.*

Proof. Suppose X satisfies one of any condition in above Lemma 2.5 but fails to have weak normal structure, for any $0 < \varepsilon < 1$, let $f = f_i \in S(X^*), g = f_j \in S(X^*)$ and $x = x_i \in S(X)$ where $i \neq j$ be chosen as in above Lemma 2.5. We have $\|f - g\| \geqslant 2 - \varepsilon$, so $1 - \frac{\|f-g\|}{2} \leqslant \frac{\varepsilon}{2}$ and $< x, f + g > \geqslant 1 - \varepsilon$. From definition of $\delta_{X^*}(\varepsilon, x)$ we have $\delta_{X^*}(1 - \varepsilon, x) \leqslant \frac{\varepsilon}{2}$ for this $x \in S(X)$. This implies that if $\delta_{X^*}(1 - \varepsilon, x) > \frac{\varepsilon}{2}$ for all $x \in S(X)$ and $0 < \varepsilon < 1$, then X has weak normal structure. It is equivalent to the condition $\delta_{X^*}(\varepsilon, x) > \frac{1}{2} - \frac{\varepsilon}{2}$ for all $x \in S(X)$ and $0 < \varepsilon < 1$. $\square$

REMARK 2.7. Since $\delta_{X^*}(\varepsilon, x)$ is a non-decreasing function of ε for any $x \in S(X)$, to use Theorem 2.4 and Theorem 2.6 we only need to check those ε that arbitrarily close to 1.

LEMMA 2.8. [6] *If $x_1, x_2 \in B(X)$ and $0 < \varepsilon < 1$ are such that $\frac{\|x_1+x_2\|}{2} > 1 - \varepsilon$, then for all $0 \leqslant c \leqslant 1$ and $z = cx_1 + (1-c)x_2 \in [x_1, x_2]$, the line segment connecting x_1 and x_2, it follows that $\|z\| > 1 - 2\varepsilon$.*

The following characteristic of reflexivity is given by James:

LEMMA 2.9. [8] *The Banach space is reflexive if and only if each bounded linear functional on X is norm -attaining.*

THEOREM 2.10. *For a Banach space X, if $\delta_{X^*}(\varepsilon, x) > \frac{1}{2} - \frac{\varepsilon}{4}$ for all $x \in S(X)$ and $0 < \varepsilon < 2$ then X^* is uniform non-square.*

Proof. Suppose X^* is not uniform non-square. For any $0 < \varepsilon < 2$, let $f, g \in S(X^*)$ such that both $\|f + g\| \geqslant 1 + \frac{\varepsilon}{2}$ and $\|f - g\| \geqslant 1 + \frac{\varepsilon}{2}$. So we have $\frac{\|f+g\|}{2} \geqslant \frac{1}{2} + \frac{\varepsilon}{4}$, and $\frac{\|f-g\|}{2} \geqslant \frac{1}{2} + \frac{\varepsilon}{4}$. This implies $1 - \frac{\|f-g\|}{2} \leqslant \frac{1}{2} - \frac{\varepsilon}{4}$. $\delta_{X^*}(\varepsilon, x) > \frac{1}{2} - \frac{\varepsilon}{4}$ for all $x \in S(X)$ and $0 < \varepsilon < 2$ implies $\delta_{X^*}(\varepsilon, x) > 1 - \varepsilon$ for all $x \in S(X)$ and $\frac{2}{3} < \varepsilon < 1$. From Remark 2.7 and Theorem 2.4, X^*, hence X is reflexive, and therefore from Lemma 2.9, there exist

$x,y\in S(X)$ such that $<x,f>=<y,g>=1$. The following proof is similar to the proof of Theorem 2.3 of [5]. Consider the 2-dimensional subspace X_2 of X spanned by x and y, and x and y are clockwise located on $\widetilde{x,y}\subseteq S(X_2)$, and the 2-dimensional subspace X_2^* of X^* spanned by f and g, and f and g are clockwise located on $\widetilde{f,g}\subseteq S(X_2^*)$. Since $<x,f-g>\geqslant 0$, and $<y,f-g>\leqslant 0$, and $<t,f-g>$ is a continuous function for $t\in\widetilde{x,y}\subseteq S(X_2)$, there must be a $z\in\widetilde{x,y}\subseteq S(X_2)$, such that $<t,f-g>=0$. Let $<z,f>=<z,g>=l$, then for all $0\leqslant\alpha\leqslant 1$, $<z,\alpha f+(1-\alpha)g>=l$. Taking $0<\alpha_1<1$ such that $h=\frac{\alpha_1 f+(1-\alpha_1)g}{\|\alpha_1 f+(1-\alpha_1)g\|}\in S(X^*)$ with $<z,h>=<z,\frac{\alpha_1 f+(1-\alpha_1)g}{\|\alpha_1 f+(1-\alpha_1)g\|}>=1$, then $\|\alpha_1 f+(1-\alpha_1)g\|=<z,\alpha_1 f+(1-\alpha_1)g>=l$. Since $\frac{\|f+g\|}{2}\geqslant\frac{1}{2}+\frac{\varepsilon}{4}=1-(\frac{1}{2}-\frac{\varepsilon}{4})$, from Lemma 2.8, we have $<z,\frac{f+g}{2}>=l=\|\alpha_1 f+(1-\alpha_1)g\|\geqslant 1-2(\frac{1}{2}-\frac{\varepsilon}{4})=\frac{\varepsilon}{2}$. Therefore, $<z,f+g>\geqslant\varepsilon$. By using Hahn-Banach Theorem to extend z from X_2 to X, from definition of $\delta_{X^*}(\varepsilon,x)$, we have $\delta_{X^*}(\varepsilon,z)\leqslant\frac{1}{2}-\frac{\varepsilon}{4}$ for this $z\in S(X)$, and any $0<\varepsilon<2$. □

REMARK 2.11. Since $\delta_{X^*}(\varepsilon,x)$ is a non-decreasing function of ε for any $x\in S(X)$, to use Theorem 2.10 we only need to check those ε that arbitrarily close to 2.

THEOREM 2.12. *For a Banach space X, if $\delta_{X^*}(\varepsilon,x)>\frac{1}{2}-\frac{\varepsilon}{4}$ for all $x\in S(X)$, and any $0<\varepsilon<2$ then X has normal structure and X^* is uniform non-square.*

Proof. Since $\delta_{X^*}(\varepsilon,x)>\frac{1}{2}-\frac{\varepsilon}{4}$, for all $x\in S(X)$, and $0<\varepsilon<2$ implies $\delta_{X^*}(\varepsilon,x)>1-\varepsilon$ for all $x\in S(X)$, and $\varepsilon>\frac{2}{3}$, from Theorem 2.4, X is reflexive. Since $\delta_{X^*}(\varepsilon,x)>\frac{1}{2}-\frac{\varepsilon}{4}$, for all $x\in S(X)$, and $0<\varepsilon<2$ implies $\delta_{X^*}(\varepsilon,x)>\frac{1}{2}-\frac{\varepsilon}{2}$ for all $x\in S(X)$, and $0<\varepsilon<1$, from Theorem 2.6, X has weak normal structure. So the Theorem 2.12 is a direct result of Theorem 2.4, Theorem 2.6 and Theorem 2.10. □

3. Uniform normal structure and inequalities on w^*UR modulus

We consider the uniform normal structure.

Let $\mathscr{F}$ be a filter of an index set I, and let $\{x_i\}_{i\in I}$ be a subset in a Hausdorff topological space X, $\{x_i\}_{i\in I}$ is said to *converge to x with respect to $\mathscr{F}$*, denoted by $\lim_{\mathscr{F}} x_i=x$, if for each neighborhood U of x, $\{i\in I: x_i\in U\}\in\mathscr{F}$.

A filter $\mathscr{U}$ on I is called an *ultrafilter* if it is maximal with respect to the ordering of the set inclusion. An ultrafilter is called *trivial* if it is of the form $\{A:A\subseteq I, i_0\in A\}$ for some $i_0\in I$. We will use the fact that if $\mathscr{U}$ is an ultrafilter, then:

(i) for any $A\subseteq I$, either $A\subseteq U$ or $I-A\subseteq U$;

(ii) if $\{x_i\}_{i\in I}$ has a cluster point x, then $\lim_{\mathscr{U}} x_i$ exists and equals to x.

Let $\{X_i\}_{i\in I}$ be a family of Banach spaces and let $l_\infty(I,X_i)$ denote the subspace of the product space equipped with the norm $\|(x_i)\|=\sup_{i\in I}\|x_i\|<\infty$.

DEFINITION 3.1. [13]. Let $\mathscr{U}$ be an ultrafilter on I and let $N_U=\{(x_i)\in l_\infty(I,X_i):\lim_{\mathscr{U}}\|x_i\|=0\}$. The *ultra-product* of $\{X_i\}_{i\in I}$ is the quotient space $l_\infty(I,X_i)/N_{\mathscr{U}}$ equipped with the quotient norm.

We will use $(x_i)_{\mathcal{U}}$ to denote the element of the ultra-product. It follows from Remark (ii) above, and the definition of quotient norm that

$$\|(x_i)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_i\|. \tag{3.1}$$

In the following we will restrict our index set I to be $\mathbb{N}$, the set of natural numbers, and let $X_i = X, i \in \mathbb{N}$ for some Banach space X. For an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we use $X_{\mathcal{U}}$ to denote the ultra-product. Note that if $\mathcal{U}$ is nontrivial, then X can be embedded into $X_{\mathcal{U}}$ isometrically.

LEMMA 3.2. [13]. *Suppose that $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$ and X is a Banach space. Then $(X^*)_{\mathcal{U}} \cong (X_{\mathcal{U}})^*$ if and only if X is super-reflexive; and in this case, the mapping J defined by*

$$\langle (x_i)_{\mathcal{U}}, J((f_i)_{\mathcal{U}})\rangle = \lim_{\mathcal{U}} \langle x_i, f_i\rangle, \quad \text{for all } (x_i)_{\mathcal{U}} \in X_{\mathcal{U}}$$

is the canonical isometric isomorphism from $(X^*)_{\mathcal{U}}$ onto $(X_{\mathcal{U}})^*$.

THEOREM 3.3. [5] *Let X be a super-reflexive Banach space. Then for any nontrivial ultrafilter $\mathcal{U}$ on $\mathbb{N}$, and for any $0 < \varepsilon < 2$, we have $\delta_{X_{\mathcal{U}}}(\varepsilon, (f_i)_{\mathcal{U}}) > a$ for all $(f_i)_{\mathcal{U}} \in S(X^*_{\mathcal{U}})$ if and only if $\delta_X(\varepsilon, f) > a$ for all $f \in S(X^*)$.*

THEOREM 3.4. *Let X be a super-reflexive Banach space. Then for any nontrivial ultrafilter $\mathcal{U}$ on $\mathbb{N}$, and for any $0 < \varepsilon < 2$, we have $\delta_{X^*_{\mathcal{U}}}(\varepsilon, (x_i)_{\mathcal{U}}) > a$ for all $(x_i)_{\mathcal{U}} \in S(X_{\mathcal{U}})$ if and only if $\delta_{X^*}(\varepsilon, x) > a$ for all $x \in S(X)$.*

Proof. From Remark 1.4, X is super-reflexive if and only if X^* is super-reflexive. So X is isomorphic and isometry to X^{**}, therefore $X_{\mathcal{U}}$ is isomorphic and isometry to $X^{**}_{\mathcal{U}}$. The Theorem 3.4 is a direct result of Theorem 3.3. □

LEMMA 3.5. [10] *If X is a super-reflexive Banach space, then X has uniform normal structure if and only if $X_{\mathcal{U}}$ has normal structure.*

THEOREM 3.6. *For a Banach space X, if $\delta_{X^*}(\varepsilon, x) > \frac{1}{2} - \frac{\varepsilon}{4}$ for all $x \in S(X)$, and any $0 < \varepsilon < 2$ then X^* is uniform non-square, X is super-reflexive and both X and X^* have uniform normal structure.*

Proof. $\delta_{X^*}(\varepsilon, x) > \frac{1}{2} - \frac{\varepsilon}{4}$ for all $x \in S(X)$, and any $0 < \varepsilon < 2$ implies that X has weak normal structure from Theorem 2.6, and X^* is uniformly non-square from Theorem 2.10. So, X^*, hence X is super-reflexive. Then the result follows directly from Theorems 3.4 and Lemma 3.5. □

Let $sgn(x) = \begin{cases} -1, & if\ \ x < 0 \\ 0, & if\ \ x = 0 \\ 1, & if\ \ x > 0 \end{cases}$ be the sign function of x.

EXAMPLE 3.7. Let $X = l_1$, and $X^* = l_\infty$, then $\delta_{l_\infty}(\varepsilon, x) = 0$ for all $x \in S(l_1)$ and $0 \leqslant \varepsilon < 2$.

Proof. For any $x = \{x_1, x_2, x_3, ..., x_n, x_{n+1}, x_{n+2}, ...\} \in S(l_1)$, we have $\sum_{i=1}^{\infty} |x_i| = 1$. For any $\delta > 0$ take n such that $\Sigma_{i=1}^{n} |x_i| > 1 - \delta$, we have $|x_j| < \delta$ for all $j > n$. Let $f_1 = (sgn(x_1), sgn(x_2), sgn(x_3), ...sgn(x_n), 1, 0, 0, ...) \in S(l_\infty)$, and $f_2 = (-sgn(x_1), -sgn(x_2), -sgn(x_3), ... - sgn(x_n), 1, 0, 0, ...) \in S(l_\infty)$. We have $< x, f_1 - f_2 >= \Sigma_{i=1}^{n} x_i(2sgn(x_i)) = 2\Sigma_{i=1}^{n} |x_i| > 2 - 2\delta$. But $\|f_1 + f_2\|_{l_\infty} = \|(0, 0, 0, ..., 2, 0, 0, 0, ...)\|_{l_\infty} = 2$, so $1 - \frac{\|f_1 + f_2\|_{l_\infty}}{2} = 0$. We have $\delta_{l_\infty}(\varepsilon, x) = 0$ for all $x \in S(l_1)$ and $0 \leqslant \varepsilon < 2 - 2\delta$. Since δ can be arbitrarily small we have $\delta_{l_\infty}(\varepsilon, x) = 0$ for all $x \in S(l_1)$ and $0 \leqslant \varepsilon < 2$. □

EXAMPLE 3.8. Let $X = c_0$, and $X^* = l_1$, then $\delta_{l_1}(\varepsilon, x) = 0$ for all $x \in S(c_0)$ and $0 \leqslant \varepsilon < 1$.

Proof. For any $x = \{x_1, x_2, x_3, ..., x_n, x_{n+1}, x_{n+2}, ...\} \in S(c_0)$, there exists an i, such that $|x_i| = 1$, and for any $\eta > 0$ there exists a j such that $|x_j| < \eta$, where $i < j$. Let $f_1 = (0, 0, 0, ..., sgn(x_i), ..., 0, 0, 0, 0, ...) \in S(l_1)$, where i-th position of f_1 is $sgn(x_i)$ and others are 0; and $f_2 = (0, 0, 0, ..., 0, ..., 0, 1, 0, 0, ...) \in S(l_1)$, where j-th position of f_2 is 1 and others are 0. We have $< x, f_1 - f_2 >> 1 - \eta$. But $\|f_1 + f_2\|_{l_1} = \|(0, 0, 0, ..., sgn(x_i), ..., 1, 0, 0, ...)\|_{l_1} = 2$, so $1 - \frac{\|f_1 + f_2\|_{l_1}}{2} = 0$. We have $\delta_{l_1}(\varepsilon, x) = 0$ for all $x \in S(l_1)$ and $0 \leqslant \varepsilon < 1 - \eta$. Since η can be arbitrarily small we have $\delta_{l_1}(\varepsilon, x) = 0$ for all $x \in S(c_0)$ and $0 \leqslant \varepsilon < 1$. □

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