Mathematical Inequalities & Applications
Volume 22, Number 4 (2019), 1243–1256 doi:10.7153/mia-2019-22-85

POPOVICIU TYPE INEQUALITIES FOR HIGHER ORDER CONVEX FUNCTIONS VIA LIDSTONE INTERPOLATION

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(*Communicated by S. Varošanec*)

Abstract. We use Lidstone's interpolating polynomials to obtain Popoviciu-type inequalities containing sums $\sum_{i=1}^{m} p_i f(x_i)$, where f is an n-convex function with even n.

We also give integral analogues of the results, some related inequalities for n-convex functions at a point and bounds for integral remainders of identities associated with the obtained inequalities.

1. Introduction

Pečarić [4] proved the following result (see also [7, p. 262] and [5]):

PROPOSITION 1. *The inequality*

$$\sum_{i=1}^{m} p_i f(x_i) \geqslant 0 \tag{1}$$

holds for all convex functions f if and only if the $m-$tuples $\mathbf{x} = (x_1, \ldots, x_m)$, $\mathbf{p} = (p_1, \ldots, p_m) \in \mathbb{R}^m$ satisfy

$$\sum_{i=1}^{m} p_i = 0 \quad \text{and} \quad \sum_{i=1}^{m} p_i |x_i - x_k| \geqslant 0 \text{ for } k \in \{1, \ldots, m\}. \tag{2}$$

Since

$$\sum_{i=1}^{m} p_i |x_i - x_k| = 2\sum_{i=1}^{m} p_i (x_i - x_k)_+ - \sum_{i=1}^{m} p_i (x_i - x_k),$$

where $y_+ = \max(y, 0)$, it is easy to see that condition (2) is equivalent to

$$\sum_{i=1}^{m} p_i = 0, \quad \sum_{i=1}^{m} p_i x_i = 0 \quad \text{and} \quad \sum_{i=1}^{m} p_i (x_i - x_k)_+ \geqslant 0 \text{ for } k \in \{1, \ldots, m-1\}. \tag{3}$$

Mathematics subject classification (2010): 26D15, 26D10.
Keywords and phrases: n-convex functions, Lidstone interpolation, Čebyšev functional.
The research of Josip Pečarić was supported by the Ministry of Education and Science of the Russian Federation (the Agreement number No. 02.a03.21.0008.).
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Let A denote the linear operator $A(f) = \sum_{i=1}^{m} p_i f(x_i)$, let $w(x,t) = (x-t)_+$ and $x_{(1)} \leqslant x_{(2)} \leqslant \ldots \leqslant x_{(m)}$ be the sequence $\mathbf{x}$ in ascending order. Notice that $A(w(\cdot,x_k)) = \sum_{i=1}^{m} p_i(x_i - x_k)_+$. For $t \in [x_{(k)}, x_{(k+1)}]$ we have

$$A(w(\cdot,t)) = A(w(\cdot,x_{(k)})) + (x_{(k)} - t) \sum_{\{i: x_i > x_{(k)}\}} p_i,$$

so the mapping $t \mapsto A(w(\cdot,t))$ is linear on $[x_{(k)}, x_{(k+1)}]$. Furthermore, $A(w(\cdot,x_{(m)}) = 0$, so condition (3) is equivalent to

$$\sum_{i=1}^{m} p_i = 0, \quad \sum_{i=1}^{m} p_i x_i = 0 \quad \text{and} \quad \sum_{i=1}^{m} p_i(x_i - t)_+ \geqslant 0 \text{ for every } t \in [x_{(1)}, x_{(m-1)}]. \tag{4}$$

It turns out that condition (4) is appropriate for extension of Proposition 1 to the integral case and the more general class of n-convex functions.

DEFINITION 1. The n-th order divided difference of a function $f : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$, at distinct points $x_0, \ldots, x_n \in I$ is defined recursively (see [7]) by

$$f[x_i] = f(x_i), \quad (i = 0, \ldots, n)$$

and

$$f[x_0, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0}.$$

The function f is said to be n-convex on I, $n \geqslant 0$, if for all choices of $(n+1)$ distinct points in I, the n-th order divided difference of f satisfies

$$f[x_0, \ldots, x_n] \geqslant 0.$$

The value $f[x_0, \ldots, x_n]$ is independent of the order of the points $x_0, \ldots, x_n$. If $f^{(n)}$ exists, then f is n-convex if and only if $f^{(n)} \geqslant 0$. For $1 \leqslant k \leqslant n-2$, a function f is n-convex if and only if $f^{(k)}$ exists and is $(n-k)$-convex.

The following result is due to Popoviciu [8, 9] (see [11, 7, 6] also).

PROPOSITION 2. *Let $n \geqslant 2$. Inequality (1) holds for all n-convex functions $f : [a,b] \to \mathbb{R}$ if and only if the $m-$tuples $\mathbf{x} \in [a,b]^m$, $\mathbf{p} \in \mathbb{R}^m$ satisfy*

$$\sum_{i=1}^{m} p_i x_i^k = 0, \quad \textit{for all } k = 0, 1, \ldots, n-1 \tag{5}$$

$$\sum_{i=1}^{m} p_i (x_i - t)_+^{n-1} \geqslant 0, \quad \textit{for every } t \in [a,b]. \tag{6}$$

In fact, Popoviciu proved a stronger result - it is enough to assume that (6) holds for every $t \in [x_{(1)}, x_{(m-n+1)}]$ and then, due to (5), it is automatically satisfied for every $t \in [a,b]$. The integral analogue (see [10, 7, 6]) is given in the next proposition.

PROPOSITION 3. *Let* $n \geqslant 2$, $p : [\alpha,\beta] \to \mathbb{R}$ *and* $g : [\alpha,\beta] \to [a,b]$. *Then, the inequality*

$$\int_{\alpha}^{\beta} p(x) f(g(x))\, dx \geqslant 0 \tag{7}$$

holds for all n*-convex functions* $f : [a,b] \to \mathbb{R}$ *if and only if*

$$\begin{aligned} \int_{\alpha}^{\beta} p(x) g(x)^k\, dx &= 0, \quad \text{for all } k = 0,1,\ldots,n-1 \\ \int_{\alpha}^{\beta} p(x)\,(g(x)-t)_+^{n-1}\, dx &\geqslant 0, \quad \text{for every } t \in [a,b]. \end{aligned} \tag{8}$$

In this paper we will prove inequalities of type (1) and (7) for n-convex functions by making use of the Lidstone interpolation. Lidstone's series is a generalization of Taylor's series and it approximates a given function in the neighborhood of two points (instead of one). For $f \in C^{(2n)}([0,1])$ there exists a unique polynomial P_L of degree $2n-1$ such that

$$P_L^{(2i)}(0) = f^{(2i)}(0), \quad P_L^{(2i)}(1) = f^{(2i)}(1), \quad 0 \leqslant i \leqslant n-1.$$

The polynomial P_L can be expressed with the Lidstone polynomials. The Lidstone polynomials Λ_n are polynomials of degree $2n+1$ defined by the relations

$$\begin{aligned} &\Lambda_0(t) = t, \\ &\Lambda_n''(t) = \Lambda_{n-1}(t), \\ &\Lambda_n(0) = \Lambda_n(1) = 0, \quad n \geqslant 1. \end{aligned} \tag{9}$$

Some explicit expressions of the Lidstone polynomials are (see [1])

$$\begin{aligned} \Lambda_n(t) &= (-1)^n \frac{2}{\pi^{2n+1}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{2n+1}} \sin k\pi t, \\ \Lambda_n(t) &= \frac{1}{6}\left[\frac{6t^{2n+1}}{(2n+1)!} - \frac{t^{2n-1}}{(2n-1)!}\right] - \sum_{k=0}^{n-2} \frac{2(2^{2k+3}-1)}{(2k+4)!} B_{2k+4} \frac{t^{2n-2k-3}}{(2n-2k-3)!}, \\ \Lambda_n(t) &= \frac{2^{2n+1}}{(2n+1)!} B_{2n+1}\left(\frac{1+t}{2}\right), \end{aligned}$$

where B_{2k+4} is the $(2k+4)$-th Bernoulli number and $B_{2n+1}\left(\frac{1+t}{2}\right)$ is the Bernoulli polynomial. The error term $e_L(t) = f(t) - P_L(t)$ of the interpolation can be expressed in the integral form using Green's function. Widder [12] proved the following lemma.

LEMMA 1. *If* $f \in C^{(2n)}([0,1])$, *then*

$$\begin{aligned} f(t) &= P_L(t) + e_L(t) \\ &= \sum_{k=0}^{n-1} \left[f^{(2k)}(0)\Lambda_k(1-t) + f^{(2k)}(1)\Lambda_k(t)\right] + \int_0^1 G_n(t,s) f^{(2n)}(s)\, ds, \end{aligned} \tag{10}$$

where

$$G_1(t,s) = G(t,s) = \begin{cases} (t-1)s, & \text{if } s \leqslant t, \\ (s-1)t, & \text{if } t \leqslant s. \end{cases} \tag{11}$$

is homogeneous Green's function of the differential operator $\frac{d^2}{ds^2}$ *on* $[0,1]$, *and with the successive iterates of* $G(t,s)$

$$G_n(t,s) = \int_0^1 G_1(t,u)G_{n-1}(u,s)\,du, \quad n \geqslant 2. \tag{12}$$

The Lidstone polynomial can be expressed in terms of $G_n(t,s)$ as

$$\Lambda_n(t) = \int_0^1 G_n(t,s)s\,ds. \tag{13}$$

For more on the Lidstone polynomials and interpolation see [1].

The outline of the paper is as follows: in Section 2 we will use Lidstone's interpolation (10) and properties of Green's function (12) to obtain inequalities of type (1) and (7) for n-convex functions. In Section 3 we will give related inequalities for n-convex functions at a point, a generalization of the class of n-convex functions introduced in [6]. In Section 4 we will give bounds for the integral remainders of identities obtained in earlier sections by using Čebyšev type inequalities.

2. Main results

THEOREM 1. *Let* $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}$ *be* $2n$*-convex and let* $\mathbf{x} \in [a,b]^m$ *and* $\mathbf{p} \in \mathbb{R}^m$ *be* m*-tuples such that*

$$\sum_{i=1}^m p_i G_n\left(\frac{x_i-a}{b-a}, \frac{s-a}{b-a}\right) \geqslant 0, \quad \text{for every } s \in [a,b], \tag{14}$$

where G_n *is Green's function given by (12). Then*

$$\sum_{i=1}^m p_i f(x_i) \geqslant \sum_{i=1}^m \sum_{k=0}^{n-1} (b-a)^{2k} \left[p_i f^{(2k)}(a)\Lambda_k\left(\frac{b-x_i}{b-a}\right) + p_i f^{(2k)}(b)\Lambda_k\left(\frac{x_i-a}{b-a}\right)\right]. \tag{15}$$

If the inequality in (14) is reversed, then (15) holds with the reversed sign of inequality.

Proof. Let us first assume $f \in C^{(2n)}([a,b])$. By Widder's lemma we have

$$f(x) = \sum_{k=0}^{n-1} (b-a)^{2k} \left[f^{(2k)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right) + f^{(2k)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right)\right] + (b-a)^{2n-1} \int_a^b G_n\left(\frac{x-a}{b-a}, \frac{s-a}{b-a}\right) f^{(2n)}(s)ds. \tag{16}$$

Applying (16) at x_i, multiplying the obtained identity by p_i and adding up we get

$$\sum_{i=1}^{m} p_i f(x_i) = \sum_{i=1}^{m}\sum_{k=0}^{n-1}(b-a)^{2k}\left[p_i f^{(2k)}(a)\Lambda_k\left(\frac{b-x_i}{b-a}\right)+p_i f^{(2k)}(b)\Lambda_k\left(\frac{x_i-a}{b-a}\right)\right]$$
$$+(b-a)^{2n-1}\int_a^b \sum_{i=1}^{m} p_i G_n\left(\frac{x_i-a}{b-a},\frac{s-a}{b-a}\right) f^{(2n)}(s)\,ds. \quad (17)$$

Assumption (14) and $f^{(2n)} \geqslant 0$ yield the stated inequality. The inequality for general f follows since every $2n$-convex function can be obtained, by making use of Bernstein polynomials, as a uniform limit of $2n$-convex functions with a continuous $2n$-th derivative (see [7]). $\square$

COROLLARY 1. *Let $j,n \in \mathbb{N}$, $1 \leqslant j \leqslant n$, let $f:[a,b]\to\mathbb{R}$ be $2n$-convex and let m-tuples $\mathbf{x}\in[a,b]^m$ and $\mathbf{p}\in\mathbb{R}^m$ satisfy (5) and (6) with n replaced by $2j$. If $n-j$ is even, then*

$$\sum_{i=1}^{m} p_i f(x_i) \geqslant \sum_{i=1}^{m}\sum_{k=j}^{n-1}(b-a)^{2k}\left[p_i f^{(2k)}(a)\Lambda_k\left(\frac{b-x_i}{b-a}\right)+p_i f^{(2k)}(b)\Lambda_k\left(\frac{x_i-a}{b-a}\right)\right], \quad (18)$$

while the reversed inequality holds if $n-j$ is odd.

Proof. From (11) and (12) by induction one can conclude that $(-1)^n G_n \geqslant 0$. Furthermore, from (12) one can get $\frac{\partial^2}{\partial t^2}G_n(t,s) = G_{n-1}(t,s)$ and, hence, by induction $\frac{\partial^{2i}}{\partial t^{2i}}G_n(t,s) = G_{n-i}(t,s)$ for $0 \leqslant i \leqslant n-1$. Therefore, the function $t \mapsto G_n(t,s)$ is $2j$-convex if $n-j$ is even and $2j$-concave if $n-j$ is odd for $0 \leqslant j \leqslant n-1$, while the statement for $j=n$ follows since $t \mapsto G_1(t,s)$ is convex.

By Proposition 2, assumption (14) in Theorem 1 is satisfied, so (15) holds. Moreover, due to assumption (5), $\sum_{i=1}^{m} p_i P(x_i) = 0$ for every polynomial P of degree $\leqslant 2j-1$ and since Λ_k is a polynomial of degree $2k+1$, the first j terms in the inner sum in (15) vanish, *i. e.*, the right hand side of (15) under the assumptions of this corollary is equal to the right hand side of (18). $\square$

When $j=n$ in (18), the notation means that the inner sum is void, *i. e.*, $\sum_{k=n}^{n-1}\cdots = 0$. In particular, inequality (18) with $j=n$ is inequality (1).

COROLLARY 2. *Let $j,n \in \mathbb{N}$, $1 \leqslant j \leqslant n$, let $f:[a,b]\to\mathbb{R}$ be $2n$-convex, let m-tuples $\mathbf{x}\in[a,b]^m$ and $\mathbf{p}\in\mathbb{R}^m$ satisfy (5) and (6) with n replaced by $2j$ and denote*

$$H(x) = \sum_{k=j}^{n-1}(b-a)^{2k}\left[f^{(2k)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right)+f^{(2k)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right)\right]. \quad (19)$$

If $n-j$ is even and H is $2j$-convex, then

$$\sum_{i=1}^{m} p_i f(x_i) \geqslant 0,$$

while the reversed inequality holds if $n-j$ is odd and H is $2j$-concave.

Proof. Applying Proposition 2 we conclude that the right hand side of (18) is nonnegative for $2j$-convex H and nonpositive for $2j$-concave H. □

REMARK 1. Due to (9) we have $\Lambda_k^{(2l)} = \Lambda_{k-l}$ and, furthermore, $(-1)^n\Lambda_n \geqslant 0$ due to (13). Therefore, if the function f satisfies $(-1)^{k-j}f^{(2k)}(a) \geqslant 0$ and $(-1)^{k-j}f^{(2k)}(b) \geqslant 0$ for $j \leqslant k \leqslant n-1$, then the function H given by (19) is $2j$-convex, while if $(-1)^{k-j}f^{(2k)}(a) \leqslant 0$ and $(-1)^{k-j}f^{(2k)}(b) \leqslant 0$ for $j \leqslant k \leqslant n-1$, then H is $2j$-concave.

As already mentioned before, the inequality in Corollaries 1 and 2 with $j=n$ is the same as the inequality in Proposition 2. Of course, in the proof of Corollary 1 we have used Proposition 2 to prove that assumption (14) holds, so, due to circularity, we didn't obtain another proof of Popoviciu's result. But, it is possible, as we will show in the next lemma, to prove directly that conditions (5) and (6) imply (14), *i. e.*, it is possible to prove Corollary 1 independently of Proposition 2 and, thus, provide a new proof of Popoviciu's result for even n.

LEMMA 2. *Let $n \geqslant 2$ and let m-tuples $\mathbf{x} \in [a,b]^m$ and $\mathbf{p} \in \mathbb{R}^m$ satisfy*

$$\sum_{i=1}^{m} p_i x_i^k = 0, \quad \text{for all } k = 0,1,\ldots,2n-1 \tag{20}$$

$$\sum_{i=1}^{m} p_i (x_i - t)_+^{2n-1} \geqslant 0, \quad \text{for every } t \in [a,b]. \tag{21}$$

Then (14) holds.

Proof. Let $s \in [a,b]$ be fixed and $y = (s-a)/(b-a)$. We will show, by induction, that G_n is of the form

$$G_n(x,y) = P_{s,2n-1}(x) + \frac{1}{(2n-1)!}(x-y)_+^{2n-1}, \tag{22}$$

where $P_{s,2n-1}$ is a polynomial of degree $2n-1$. Hence, similarly as in the proof of Corollary 1, from (20) we can conclude that

$$\sum_{i=1}^{m} p_i P_{s,2n-1}\left(\frac{x_i - a}{b-a}\right) = 0,$$

while (21) yields

$$\sum_{i=1}^{m} \frac{p_i}{(2n-1)!}\left(\frac{x_i-a}{b-a} - \frac{s-a}{b-a}\right)_+^{2n-1} = \frac{1}{(2n-1)!(b-a)^{2n-1}} \sum_{i=1}^{m} p_i (x_i - s)_+^{2n-1} \geqslant 0.$$

Therefore, it is enough to show that (22) holds. From (11) we have

$$G_1(x,y) = xy - \min(x,y) = x(y-1) + (x-y)_+,$$

so (22) holds for $n=1$. Now, assume that (22) holds. Then (12) yields

$$\begin{aligned}G_{n+1}(x,y) &= \int_0^1 (x(u-1)+(x-u)_+)\left(P_{s,2n-1}(u)+\frac{1}{(2n-1)!}(u-y)_+^{2n-1}\right)du\\ &= I+II+III,\end{aligned}$$

where

$$I = x\int_0^1 (u-1)G_n(u,y)\,du = x\cdot \text{ constant}$$

$$II = \int_0^1 (x-u)_+ P_{s,2n-1}(u)\,du \tag{23}$$

$$III = \frac{1}{(2n-1)!}\int_0^1 (x-u)_+(u-y)_+^{2n-1}\,du \tag{24}$$

Integration by parts yields

$$\begin{aligned}II &= \int_0^x (x-u)P_{s,2n-1}(u)\,du\\ &= (x-u)\int_0^u P_{s,2n-1}(z)\,dz\Big|_{u=0}^{u=x} + \int_0^x\int_0^u P_{s,2n-1}(z)\,dz\,du\\ &= \tilde{P}_{s,2n+1}(x)\end{aligned}$$

where $\tilde{P}_{s,2n+1}$ is a polynomial of degree $2n+1$. Notice that

$$I+II = P_{s,2n+1}$$

is a polynomial of degree $2n+1$ in the variable x. Clearly $III=0$ for $x\leqslant y$, while for $x>y$

$$\begin{aligned}III &= \frac{1}{(2n-1)!}\int_y^x (x-u)(u-y)^{2n-1}\,du\\ &= \frac{1}{(2n)!}(x-u)(u-y)\Big|_{u=y}^{u=x} + \frac{1}{(2n)!}\int_y^x (u-y)^{2n}\,du = \frac{1}{(2n+1)!}(x-y)^{2n+1}.\end{aligned}$$

Therefore, $III=(x-y)_+^{2n+1}/(2n+1)!$, so (22) holds for $n+1$ as well, which finishes the proof. $\square$

Lemma 2 together with Theorem 1 gives the "if" part of Proposition 2. On the other hand, the "only if" part is straightforward: since the functions $e_k(x)=x^k$ are both $2n$-convex and $2n$-concave for $k=0,1,\ldots,2n-1$, inequality (1) yields that $\sum_{i=1}^m p_i e_k(x_i)$ is both $\geqslant 0$ and $\leqslant 0$, so (20) holds. Similarly, the function $w_{2n}(x)=(x-t)_+^{2n-1}$ is $2n$-convex and inequality (1) applied to w_{2n} yields (21).

In the remainder of this section we will give integral versions of the results. The proofs are analogous to the discrete case and we will omit them.

THEOREM 2. *Let $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}$ be $2n$-convex and let the functions $p : [\alpha,\beta] \to \mathbb{R}$ and $g : [\alpha,\beta] \to [a,b]$ be such that*

$$\int_\alpha^\beta p(x) G_n\left(\frac{g(x)-a}{b-a}, \frac{s-a}{b-a}\right) dx \geqslant 0, \quad \text{for every } s \in [a,b], \tag{25}$$

where G_n is Green's function given by (12). Then

$$\int_\alpha^\beta p(x) f(g(x))\, dx \geqslant \int_\alpha^\beta p(x) \sum_{k=0}^{n-1} (b-a)^{2k} \left[f^{(2k)}(a) \Lambda_k\left(\frac{b-g(x)}{b-a}\right) + f^{(2k)}(b) \Lambda_k\left(\frac{g(x)-a}{b-a}\right)\right] dx. \tag{26}$$

If the inequality in (25) is reversed, then (26) holds with the reversed sign of inequality.

COROLLARY 3. *Let $j,n \in \mathbb{N}$, $1 \leqslant j \leqslant n$, let $f : [a,b] \to \mathbb{R}$ be $2n$-convex and let the functions $p : [\alpha,\beta] \to \mathbb{R}$ and $g : [\alpha,\beta] \to [a,b]$ satisfy (8) with n replaced by $2j$. If $n-j$ is even, then*

$$\int_\alpha^\beta p(x) f(g(x))\, dx \geqslant \int_\alpha^\beta p(x) \sum_{k=j}^{n-1} (b-a)^{2k} \left[f^{(2k)}(a) \Lambda_k\left(\frac{b-g(x)}{b-a}\right) + f^{(2k)}(b) \Lambda_k\left(\frac{g(x)-a}{b-a}\right)\right] dx,$$

while the reversed inequality holds if $n-j$ is odd.

COROLLARY 4. *Let j,n,f,p and g be as in Corollary 3 and let H be given by (19). If $n-j$ is even and H is $2j$-convex, then*

$$\int_\alpha^\beta p(x) f(g(x))\, dx \geqslant 0,$$

while the reversed inequality holds if $n-j$ is odd and H is $2j$-concave.

LEMMA 3. *Let $n \geqslant 2$ and let the functions $p : [\alpha,\beta] \to \mathbb{R}$ and $g : [\alpha,\beta] \to [a,b]$ satisfy*

$$\int_\alpha^\beta p(x) g(x)^k\, dx = 0, \quad \text{for all } k = 0,1,\ldots,2n-1$$
$$\int_\alpha^\beta p(x)\, (g(x)-t)_+^{2n-1}\, dx \geqslant 0, \quad \text{for every } t \in [a,b].$$

Then (25) holds.

3. Related inequalities for n-convex functions at a point

In this section we will give related results for the class of n-convex functions at a point introduced in [6].

DEFINITION 2. Let I be an interval in $\mathbb{R}$, c a point in the interior of I and $n \in \mathbb{N}$. A function $f : I \to \mathbb{R}$ is said to be n-convex at point c if there exists a constant K such that the function

$$F(x) = f(x) - \frac{K}{(n-1)!}x^{n-1} \tag{27}$$

is $(n-1)$-concave on $I \cap (-\infty, c]$ and $(n-1)$-convex on $I \cap [c, \infty)$. A function f is said to be n-concave at point c if the function $-f$ is n-convex at point c.

A property that explains the name of the class is the fact that a function is n-convex on an interval if and only if it is n-convex at every point of the interval (see [2, 6]). Pečarić, Praljak and Witkowski in [6] studied necessary and sufficient conditions on two linear functionals $A : C([a,c]) \to \mathbb{R}$ and $B : C([c,b]) \to \mathbb{R}$ so that the inequality $A(f) \leqslant B(f)$ holds for every function f that is n-convex at c. In this section we will give inequalities of this type for particular linear functionals related to the inequalities obtained in the previous section.

Let e_i denote the monomials $e_i(x) = x^i$, $i \in \mathbb{N}_0$. For the rest of this section, A and B will denote the linear functionals obtained as the difference of the left and right hand sides of inequality (15) applied to the intervals $[a,c]$ and $[c,b]$, respectively, *i. e.*, for $\mathbf{x} \in [a,c]^m$, $\mathbf{p} \in \mathbb{R}^m$, $\mathbf{y} \in [c,b]^l$ and $\mathbf{q} \in \mathbb{R}^l$ let

$$A(f) = \sum_{i=1}^{m} p_i f(x_i) - \sum_{i=1}^{m}\sum_{k=0}^{n-1}(c-a)^{2k}\left[p_i f^{(2k)}(a)\Lambda_k\left(\frac{c-x_i}{c-a}\right) + p_i f^{(2k)}(c)\Lambda_k\left(\frac{x_i-a}{c-a}\right)\right], \tag{28}$$

$$B(f) = \sum_{i=1}^{l} q_i f(y_i) - \sum_{i=1}^{l}\sum_{k=0}^{n-1}(b-c)^{2k}\left[q_i f^{(2k)}(c)\Lambda_k\left(\frac{b-y_i}{b-c}\right) + q_i f^{(2k)}(b)\Lambda_k\left(\frac{y_i-c}{b-c}\right)\right]. \tag{29}$$

Notice that, using the newly introduced functionals A and B, identity (17) applied to the intervals $[a,c]$ and $[c,b]$ can be written as

$$A(f) = (c-a)^{2n-1}\int_a^c \sum_{i=1}^{m} p_i G_n\left(\frac{x_i-a}{c-a}, \frac{s-a}{c-a}\right) f^{(2n)}(s)\,ds, \tag{30}$$

$$B(f) = (b-c)^{2n-1}\int_c^b \sum_{i=1}^{l} q_i G_n\left(\frac{y_i-c}{b-c}, \frac{s-c}{b-c}\right) f^{(2n)}(s)\,ds. \tag{31}$$

THEOREM 3. *Let $\mathbf{x} \in [a,c]^m$, $\mathbf{p} \in \mathbb{R}^m$, $\mathbf{y} \in [c,b]^l$ and $\mathbf{q} \in \mathbb{R}^l$ be such that*

$$\sum_{i=1}^{m} p_i G_n \left(\frac{x_i - a}{c-a}, \frac{s-a}{c-a} \right) \geqslant 0, \quad \text{for every } s \in [a,c], \tag{32}$$

$$\sum_{i=1}^{l} q_i G_n \left(\frac{y_i - c}{b-c}, \frac{s-c}{b-c} \right) \geqslant 0, \quad \text{for every } s \in [c,b], \tag{33}$$

$$\int_a^c \sum_{i=1}^{m} p_i G_n \left(\frac{x_i - a}{c-a}, \frac{s-a}{c-a} \right) ds = \left(\frac{b-c}{c-a} \right)^{2n-1} \int_c^b \sum_{i=1}^{l} q_i G_n \left(\frac{y_i - c}{b-c}, \frac{s-c}{b-c} \right) ds, \tag{34}$$

where G_n is Green's function given by (12), and let A and B be the linear functionals given by (28) and (29). If $f : [a,b] \to \mathbb{R}$ is $(2n+1)$-convex at point c, then

$$A(f) \leqslant B(f). \tag{35}$$

If the inequalities in (32) and (33) are reversed, then (35) holds with the reversed sign of inequality.

Proof. Let $F = f - \frac{K}{(2n)!} e_{2n}$ be as in Definition 2, *i. e.*, the function F is $2n$-concave on $[a,c]$ and $2n$-convex on $[c,b]$. Applying Theorem 1 to F on the interval $[a,c]$ we have

$$0 \geqslant A(F) = A(f) - \frac{K}{(2n)!} A(e_{2n}) \tag{36}$$

and applying Theorem 1 to F on the interval $[c,b]$ we have

$$0 \leqslant B(F) = B(f) - \frac{K}{(2n)!} B(e_{2n}). \tag{37}$$

Identities (30) and (31) applied to the function e_{2n} yield

$$A(e_{2n}) = (2n)!(c-a)^{2n-1} \int_a^c \sum_{i=1}^{m} p_i G_n \left(\frac{x_i - a}{c-a}, \frac{s-a}{c-a} \right) ds,$$
$$B(e_{2n}) = (2n)!(b-c)^{2n-1} \int_c^b \sum_{i=1}^{l} q_i G_n \left(\frac{y_i - c}{b-c}, \frac{s-c}{b-c} \right) ds.$$

Therefore, assumption (34) is equivalent to $A(e_{2n}) = B(e_{2n})$. Now, from (36) and (37) we obtain the stated inequality. $\square$

REMARK 2. In the proof of Theorem 3 we have, actually, shown that

$$A(f) \leqslant \frac{K}{(2n)!} A(e_{2n}) = \frac{K}{(2n)!} B(e_{2n}) \leqslant B(f).$$

In fact, inequality (35) still holds if we replace assumption (34) with the weaker assumption that $K\left(B(e_{2n}) - A(e_{2n})\right) \geqslant 0$.

COROLLARY 5. *Let* $j_1, j_2, n \in \mathbb{N}$, $1 \leqslant j_1, j_2 \leqslant n$, *let* $f : [a,b] \to \mathbb{R}$ *be* $(2n+1)$-*convex at point* c, *let* m-*tuples* $\mathbf{x} \in [a,c]^m$ *and* $\mathbf{p} \in \mathbb{R}^m$ *satisfy* (5) *and* (6) *with* n *replaced by* $2j_1$, *let* l-*tuples* $\mathbf{y} \in [c,b]^l$ *and* $\mathbf{q} \in \mathbb{R}^l$ *satisfy*

$$\sum_{i=1}^{l} q_i y_i^k = 0, \quad \text{for all } k = 0,1,\ldots,2j_2-1$$

$$\sum_{i=1}^{l} q_i (y_i - t)_+^{2j_2-1} \geqslant 0, \quad \text{for every } t \in [y_{(1)}, y_{(l-n+1)}]$$

and let (34) *holds. If* $n - j_1$ *and* $n - j_2$ *are even, then*

$$A(f) \leqslant B(f),$$

while the reversed inequality holds if $n - j_1$ *and* $n - j_2$ *are odd.*

Proof. See the proof of Corollary 1. □

4. Bounds for identities related to the Popoviciu-type inequalities

Let $f, h : [a,b] \to \mathbb{R}$ be two Lebesgue integrable functions. We consider the Čebyšev functional

$$T(f,h) = \frac{1}{b-a}\int_a^b f(x)h(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b h(x)\,dx\right). \tag{38}$$

The following results can be found in [3].

PROPOSITION 4. *Let* $f : [a,b] \to \mathbb{R}$ *be a Lebesgue integrable function and* $h : [a,b] \to \mathbb{R}$ *be an absolutely continuous function with* $(\cdot - a)(b - \cdot)[h']^2 \in L[a,b]$. *Then we have the inequality*

$$|T(f,h)| \leqslant \frac{1}{\sqrt{2}}\left(\frac{1}{b-a}|T(f,f)|\int_a^b (x-a)(b-x)[h'(x)]^2\,dx\right)^{\frac{1}{2}}. \tag{39}$$

The constant $\frac{1}{\sqrt{2}}$ *in* (39) *is the best possible.*

PROPOSITION 5. *Let* $h : [a,b] \to \mathbb{R}$ *be a monotonic nondecreasing function and let* $f : [a,b] \to \mathbb{R}$ *be an absolutely continuous function such that* $f' \in L_\infty[a,b]$. *Then we have the inequality*

$$|T(f,h)| \leqslant \frac{1}{2(b-a)}\|f'\|_\infty \int_a^b (x-a)(b-x)dh(x). \tag{40}$$

The constant $\frac{1}{2}$ *in* (40) *is the best possible.*

For $m-$tuples $\mathbf{p}=(p_1,\dots,p_m)\in\mathbb{R}^m$, $\mathbf{x}=(x_1,\dots,x_m)\in[a,b]^m$ and the function G_n given be (12), denote

$$\delta(s)=\sum_{i=1}^{m}p_iG_n\left(\frac{x_i-a}{b-a},\frac{s-a}{b-a}\right),\qquad \text{for } s\in[a,b]. \tag{41}$$

Similarly, for functions $g:[\alpha,\beta]\to[a,b]$ and $p:[\alpha,\beta]\to\mathbb{R}$ denote

$$\Delta(s)=\int_{\alpha}^{\beta}p(x)G_n\left(\frac{g(x)-a}{b-a},\frac{s-a}{b-a}\right)dx,\qquad \text{for } s\in[a,b]. \tag{42}$$

Now, we are ready to state the main results of this section.

THEOREM 4. *Let $n\in\mathbb{N}$, $f:[a,b]\to\mathbb{R}$ be such that $f^{(2n)}$ is an absolutely continuous function with $(\cdot-a)(b-\cdot)[f^{(2n+1)}]^2\in L[a,b]$ and let G_n, T and δ be given by (12), (38) and (41) respectively. Then*

$$\begin{aligned}\sum_{i=1}^{m}p_if(x_i)=&\sum_{i=1}^{m}\sum_{k=0}^{n-1}(b-a)^{2k}\left[p_if^{(2k)}(a)\Lambda_k\left(\frac{b-x_i}{b-a}\right)+p_if^{(2k)}(b)\Lambda_k\left(\frac{x_i-a}{b-a}\right)\right]\\&+(b-a)^{2n-2}\left(f^{(2n-1)}(b)-f^{(2n-1)}(a)\right)\int_a^b\delta(s)\,ds+R_n^1(f;a,b),\end{aligned} \tag{43}$$

where the remainder $R_n^1(f;a,b)$ satisfies the estimation

$$|R_n^1(f;a,b)|\leqslant\frac{(b-a)^{2n-\frac12}}{\sqrt{2}}\,|T(\delta,\delta)|^{\frac12}\left(\int_a^b(s-a)(b-s)[f^{(2n+1)}(s)]^2\,ds\right)^{\frac12}. \tag{44}$$

Proof. If we apply Proposition 4 for $f\to\delta$ and $h\to f^{(2n)}$, then we obtain

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b\delta(s)f^{(2n)}(s)\,ds-\left(\frac{1}{b-a}\int_a^b\delta(s)\,ds\right)\left(\frac{1}{b-a}\int_a^bf^{(2n)}(s)\,ds\right)\right|\\&\qquad\leqslant\frac{1}{\sqrt{2}}\left(\frac{1}{b-a}|T(\delta,\delta)|\int_a^b(s-a)(b-s)[f^{(2n+1)}(s)]^2\,ds\right)^{\frac12}.\end{aligned} \tag{45}$$

From (17) and (43) we obtain

$$\begin{aligned}&(b-a)^{2n-1}\int_a^b\delta(s)f^{(2n)}(s)\,ds\\&\qquad=(b-a)^{2n-2}\left(f^{(2n-1)}(b)-f^{(2n-1)}(a)\right)\int_a^b\delta(s)\,ds+R_n^1(f;a,b),\end{aligned}$$

where the estimate (44) follows from (45). $\square$

The following integral version of the previous theorem is proven analogously.

THEOREM 5. *Let $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}$ be such that $f^{(2n)}$ is an absolutely continuous function with $(\cdot - a)(b - \cdot)[f^{(2n+1)}]^2 \in L[a,b]$ and let G_n, T and Δ be given by (12), (38) and (42) respectively. Then*

$$\begin{aligned}
&\int_\alpha^\beta p(x) f(g(x))\, dx \\
&= \int_\alpha^\beta p(x) \sum_{k=j}^{n-1} (b-a)^{2k} \left[f^{(2k)}(a) \Lambda_k \left(\frac{b-g(x)}{b-a} \right) + f^{(2k)}(b) \Lambda_k \left(\frac{g(x)-a}{b-a} \right) \right] dx \\
&\qquad + (b-a)^{2n-2} \left(f^{(2n-1)}(b) - f^{(2n-1)}(a) \right) \int_a^b \Delta(s)\, ds + R_n^2(f;a,b), \quad (46)
\end{aligned}$$

where the remainder $R_n^2(f;a,b)$ satisfies the estimation

$$|R_n^2(f;a,b)| \leqslant \frac{(b-a)^{2n-\frac{1}{2}}}{\sqrt{2}} |T(\Delta,\Delta)|^{\frac{1}{2}} \left(\int_a^b (s-a)(b-s)[f^{(2n+1)}(s)]^2\, ds \right)^{\frac{1}{2}}.$$

By using Proposition 5 we obtain the following Grüss type inequality.

THEOREM 6. *Let $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}$ be such that $f^{(2n)}$ is an absolutely continuous function with $f^{(2n+1)} \geqslant 0$ and let δ be given by (41). Then we have the representation (43) and the remainder $R_n^1(f;a,b)$ satisfies the bound*

$$\begin{aligned}
|R_n^1(f;a,b)| \leqslant (b-a)^{2n} \|\delta'\|_\infty \Bigg[& \frac{f^{(2n-1)}(b) + f^{(2n-1)}(a)}{2} \\
& - \frac{f^{(2n-2)}(b) - f^{(2n-2)}(a)}{b-a} \Bigg]. \quad (47)
\end{aligned}$$

Proof. If we apply Proposition 5 for $f \to \delta$ and $h \to f^{(2n)}$ we obtain

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b \delta(s) f^{(2n)}(s)\, ds - \left(\frac{1}{b-a} \int_a^b \delta(s)\, ds \right) \left(\frac{1}{b-a} \int_a^b f^{(2n)}(s)\, ds \right) \right| \\
&\qquad \leqslant \frac{1}{2(b-a)} \|\delta'\|_\infty \int_a^b (s-a)(b-s) f^{(2n+1)}(s)\, ds.
\end{aligned}$$

Since

$$\begin{aligned}
&\int_a^b (s-a)(b-s) f^{(2n+1)}(s)\, ds = \int_a^b (2s-a-b) f^{(2n)}(s)\, ds \\
&\qquad = (b-a) \left[f^{(2n-1)}(b) + f^{(2n-1)}(a) \right] - 2 \left[f^{(2n-2)}(b) - f^{(2n-2)}(a) \right], \quad (48)
\end{aligned}$$

using identities (17) and (48) we deduce (47). □

Again, we only state the integral version of the previous result.

THEOREM 7. *Let $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}$ be such that $f^{(2n)}$ is an absolutely continuous function with $f^{(2n+1)} \geqslant 0$ and let Δ be given by (42). Then we have the representation (46) and the remainder $R_n^2(f;a,b)$ satisfies the bound*

$$|R_n^2(f;a,b)| \leqslant (b-a)^{2n} \|\Delta'\|_\infty \left[\frac{f^{(2n-1)}(b) + f^{(2n-1)}(a)}{2} - \frac{f^{(2n-2)}(b) - f^{(2n-2)}(a)}{b-a} \right].$$

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(Received September 28, 2018)

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